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# An Algebraic Technique for Blind MIMO Deconvolution of Constant Modulus Signals [1]

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**Abstract**

One of the most important algorithms for blind signal separation is the Analytic Constant Modulus Algorithm (ACMA) by van der Veen and Paulraj. The solution is obtained by means of a simultaneous diagonalization of a set of Hermitean matrices. In this paper we will generalize this result to convolutive mixtures. We will show that the convolutive case can be reduced to the instantaneous case by means of the calculation of the simultaneous dominant subspace of a set of Hermitean matrices. For the latter problem we will propose an algebraic algorithm.

# AN ALGEBRAIC TECHNIQUE FOR BLIND MIMO DECONVOLUTION OF CONSTANT MODULUS SIGNALS

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*One of the most important algorithms for blind signal separation is the Analytic Constant Modulus Algorithm (ACMA) by van der Veen and Paulraj. The solution is obtained by means of a simultaneous diagonalization of a set of Hermitean matrices. In this paper we will generalize this result to convolutive mixtures. We will show that the convolutive case can be reduced to the instantaneous case by means of the calculation of the simultaneous dominant subspace of a set of Hermitean matrices. For the latter problem we will propose an algebraic algorithm.*

## 1. INTRODUCTION

In this paper we consider the following data model:

$$\tilde{Y}(n) = \sum_{l=0}^{L-1} \mathbf{A}(l) X(n-l),$$

in which $X(n) \in \mathbb{C}^R$ are unkown constant-modulus inputs, $\tilde{Y}(n) \in \mathbb{C}^R$ are the observed outputs and $\mathbf{A}(l) \in \mathbb{C}^{R\times R}$, $l = 0, \ldots, L-1$ are unknown Markov parameters. We assume that it is possible to equalize the system as follows:

$$\hat{X}(n) = \sum_{l=0}^{L-1} \mathbf{B}(l) \tilde{Y}(n-l), \tag{1}$$

in which $\mathbf{B}(l) \in \mathbb{C}^{R\times R}$, $l = 0, \ldots, L-1$ have to determined such that

$$\hat{X}(n) = X(n-2L+2) \cdot \mathbf{P} \cdot \mathbf{\Lambda},$$

with $P \in \mathbb{R}^{R\times R}$ a permutation matrix and $\mathbf{\Lambda} \in \mathbb{C}^{R\times R}$ a diagonal matrix containing unit-modulus entries. For the determination of $\{\mathbf{B}(l)\}$, we will work in a deterministic framework, i.e., our technique only exploits the algebraic properties of the data and is not based on the computation of statistics.

The equivalent of this problem for instantaneous mixtures has been solved in [3]. The solution was obtained by means of a simultaneous matrix diagonalization.

In the next section we will reformulate the convolutive problem as an instantaneous problem involving $LR$-dimensional data. We will proceed as in [3] to formulate the problem as a simultaneous matrix diagonalization in $\mathbb{C}^{LR\times LR}$. In section 3 we will reduce this problem to a simultaneous matrix diagonalization in $\mathbb{C}^{R\times R}$ by means of the calculation of the simultaneous dominant subspace of the $(LR \times LR)$ matrices that originally had to be diagonalized. In section 4 we explain how the length of the equalizer can be estimated.

## 2. ALGEBRAIC FORMULATION

Let $\mathbf{B} \in \mathbb{C}^{R\times LR}$ be the concatenation of the matrices $\{\mathbf{B}(l)\}$ and let us stack the vectors $\tilde{Y}(n)$, $\tilde{Y}(n-1)$, $\ldots$, $\tilde{Y}(n-L+1)$ in one big vector $Y(n) \in \mathbb{C}^{LR\times 1}$. Then Eq. (1) can be rewritten as

$$\hat{X}(n) = \mathbf{B}Y(n). \tag{2}$$

$\mathbf{B}$ should be determined in such a way that $\hat{X}(n)$ has unit-modulus entries.

Initially, we may work as in [3]. Let $B \in \mathbb{C}^{1\times LR}$ be a row of $\mathbf{B}$. The signal

$$\hat{x}(n) = BY(n) \tag{3}$$

is unit-modulus if and only if

$$\begin{aligned} |\hat{x}(1)|^2 &= B\,Y(1)\,Y(1)^H\,B^H = 1 \\ &\vdots \\ |\hat{x}(N)|^2 &= B\,Y(T)\,Y(T)^H\,B^H = 1, \end{aligned} \tag{4}$$

in which $N$ is the number of samples $Y(n)$. (4) can be rewritten as

$$\begin{pmatrix} Y(1)^T \otimes Y(1)^H \\ Y(2)^T \otimes Y(2)^H \\ \vdots \\ Y(N)^T \otimes Y(N)^H \end{pmatrix} (B^T \otimes B^H) = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \tag{5}$$

in which $\otimes$ denotes the Kronecker product. By multiplying both sides of Eq. (5) with a Householder or a discrete Fourier transform matrix, we obtain a set of equations of the form

$$\begin{pmatrix} \tilde{M}_1 \\ \mathbf{M} \end{pmatrix} (B^T \otimes B^H) = \begin{pmatrix} \sqrt{N} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{6}$$

in which $\tilde{M}_1 \in \mathbb{C}^{1\times(LR)^2}$ and $\mathbf{M} \in \mathbb{C}^{(N-1)\times(LR)^2}$. The first equation is only a normalization constraint. Every row $B_r$ of $\mathbf{B}$ leads to a vector in the kernel of $\mathbf{M}$. Generically, for $N$ big enough, the kernel of $\mathbf{M}$ is of dimension $R$ (for convenience, we assume at this point that all sources have the same equalizer length; we will come back to this issue in Section 4). Hence, the problem consists of (i) computing the kernel of $\mathbf{M}$ (by means of an SVD), and (ii) looking for vectors in the kernel subspace that have a Kronecker structure.

Note that in the Single Input Single Output (SISO) case, we have a relatively simple problem. The kernel of $\mathbf{M}$ is theoretically one-dimensional, and the kernel vector theoretically corresponds to a rank-1 matrix. Hence the equalizer may be computed as the dominant eigenvector of the matrix that is represented by the kernel vector of $\mathbf{M}$.

### 3. REDUCTION TO INSTANTANEOUS MIXTURE

The fact that the kernel of $\mathbf{M}$ is spanned by the vectors $\{B_r^T \otimes B_r^H\}$, can be reformulated as

$$\begin{aligned} \mathbf{F_1} &= \mathbf{B}^T \cdot \mathbf{D_1} \cdot \mathbf{B}^* \\ &\vdots \\ \mathbf{F_R} &= \mathbf{B}^T \cdot \mathbf{D_R} \cdot \mathbf{B}^*, \end{aligned} \tag{7}$$

in which $\mathbf{F_1}, \ldots, \mathbf{F_R}$ are $(LR \times LR)$ matrix representations of $R$ basis vectors of the kernel, and in which $\mathbf{D_1}, \ldots, \mathbf{D_R} \in \mathbb{R}^{R\times R}$ are diagonal.

We can reduce the dimensionality of the problem by noticing that $\mathbf{B}^T$ spans the column space of every matrix $\mathbf{F_r}$. Hence, in the absence of noise, the column space of $\mathbf{B}^T$ can be determined as the subspace corresponding to the $R$ dominant left singular vectors of the matrix $\mathbf{F} = [\mathbf{F_1} \ldots \mathbf{F_R}]$. Let the columns of $\mathbf{U} \in \mathbb{C}^{LR\times R}$

form an orthonormal basis of this subspace. Then the remaining problem is of the form

$$\begin{aligned}\mathbf{G_1} &= \mathbf{E} \cdot \mathbf{D_1} \cdot \mathbf{E}^H \\ &\vdots \\ \mathbf{G_R} &= \mathbf{E} \cdot \mathbf{D_R} \cdot \mathbf{E}^H, \end{aligned} \tag{8}$$

in which $\mathbf{G_r} \in \mathbb{C}^{R \times R}$, $1 \leqslant r \leqslant R$, are defined by

$$\mathbf{G_r} = \mathbf{U}^H \cdot \mathbf{F_r} \cdot \mathbf{U}$$

and in which $\mathbf{E} \in \mathbb{C}^{R \times R}$ is equal to $\mathbf{U}^H \cdot \mathbf{B}^T$. This problem has been solved in [3].

Let us associate to $\mathbf{U}^T$ a filter $\mathbf{U}^T(z)$ of which the Markov parameters are the subsequent $(R \times R)$ submatrices of $\mathbf{U}^T$. Passing $\tilde{Y}(n)$ through $\mathbf{U}^T(z)$ in fact reduces the equalization problem to the blind separation of a linear mixture, since we have that $\mathbf{B} = \mathbf{E}^T \cdot \mathbf{U}^T$.

If the data are noisy, an appropriate $\mathbf{U}$ can be determined in a least-squares sense as a column-wise orthonormal matrix that maximizes the function

$$f(\mathbf{U}) = \sum_{r=1}^{R} \|\mathbf{U}^H \cdot \mathbf{F_r} \cdot \mathbf{U}\|^2. \tag{9}$$

In general the global maximum is not obtained by computing the subspace of the dominant left singular vectors of the matrix $\mathbf{F}$, although this is usually a good estimate of the global maximum. Actually this problem corresponds to the best rank-$(R \times R \times R)$ approximation of the $(LR \times LR \times R)$-tensor obtained by stacking the matrices $\{\mathbf{F_r}\}$ [2]. Computation of the dominant left singular subspace of $\mathbf{F}$ corresponds to truncation of the "Higher-Order Singular Value Decomposition" (HOSVD), discussed in [1]. In contrast to the situation for matrices, the best rank-$(R_1 \times R_2 \times R_3)$ approximation of a third-order tensor cannot be obtained in general by simple truncation of the HOSVD.

However, the HOSVD-truncate may be used as a good initial value for the optimization of (9); the optimization itself may take the form of an Alternating Least Squares (ALS) iteration [2]. The idea is to compute an estimate $\mathbf{U}_{k+1}$ at iteration step $k+1$, given an estimate $\mathbf{U}_k$ at iteration step $k$, as a matrix that maximizes

$$f(\mathbf{U}_{k+1}) = \sum_{r=1}^{R} \|\mathbf{U}_{k+1}^H \cdot \mathbf{F_r} \cdot \mathbf{U}_k\|^2. \qquad (10)$$

The columns of $\mathbf{U}_{k+1}$ can be determined as orthonormal basis vectors of the subspace of the $R$ dominant left singular vectors of the matrix $[\mathbf{F_1}\cdot\mathbf{U}_k \ldots \mathbf{F_R}\cdot\mathbf{U}_k]$.

## 4. DETERMINATION OF THE EQUALIZER LENGTH

Let us first assume that each source has the same equalizer length $L$. Let $\hat{L}$ be an estimate of $L$. Generically, for $N$ big enough, $\mathbf{M} \in \mathbb{C}^{(N-1)\times(\hat{L}R)^2}$ will be nonsingular if $\hat{L} < L$. For $\hat{L} = L$, $\mathbf{M}$ has a kernel of dimension $R$. For $\hat{L} = L+1$, the kernel is of dimension $2R$: it is generated by the columns of $\mathbf{B}_l^T \odot \mathbf{B}_l^H$ and $\mathbf{B}_r^T \odot \mathbf{B}_r^H$, in which $\mathbf{B}_l, \mathbf{B}_r \in \mathbb{C}^{R\times(L+1)R}$ are defined by

$$\begin{aligned} \mathbf{B}_l &= [\mathbf{0} \;\; \mathbf{B}] \\ \mathbf{B}_r &= [\mathbf{B} \;\; \mathbf{0}] \end{aligned}$$

and in which $\odot$ denotes the Kathri-Rao or column-wise Kronecker product. So $L$ may be determined by inspection of the singular values of $\mathbf{M}$, for different values of $\hat{L}$.

Now assume that for $R' < R$ sources the equalizer has length $L-1$. Then, for $\hat{L} = L-1$, the kernel of $\mathbf{M}$ will already be $R'$-dimensional. By proceeding as in the previous section, the corresponding sources may be equalized. In analogy with the previous paragraph, we obtain that the kernel of $\mathbf{M}$, for $\hat{L} = L$ will have dimension $(R-R') + 2R'$. In the derivation of the previous section, we already know $2R'$ columns of $\mathbf{U}$: they form an orthonormal basis for the columns of

$$\begin{aligned} \mathbf{U'}_l &= [\mathbf{0} \;\; \mathbf{U'}^T]^T \\ \mathbf{U'}_r &= [\mathbf{U'}^T \;\; \mathbf{0}]^T, \end{aligned}$$

in which $\mathbf{U}'$ is the solution of Eq. (9) for $\hat{L} = L-1$. After computation of the remaining $R-R'$ columns of $\mathbf{U}$, and a reduction of the dimensionality, we obtain a simultaneous diagonalization in $\mathbb{C}^{(R+R')\times(R+R')}$. Here too, we already know $2R'$ columns of the transformation matrix: for instance, $\mathbf{U'}_l$ should be multiplied by a matrix $\mathbf{E}' \in \mathbb{C}^{R'\times R'}$, such that

$$\mathbf{U'}_l \cdot \mathbf{E}' = [\mathbf{0} \;\; \mathbf{B}']^T,$$

in which $\mathbf{B}'$ is the equalization matrix obtained for $\hat{L} = L - 1$. Eq. (8) now takes the following form:

$$\begin{pmatrix} \mathbf{G}_{\mathbf{r},11} & \mathbf{G}_{\mathbf{r},12} \\ \mathbf{G}_{\mathbf{r},12}^H & \mathbf{G}_{\mathbf{r},22} \end{pmatrix} = \begin{pmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} \\ \mathbf{0} & \mathbf{E}_{22} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{D}_{\mathbf{r},1} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{\mathbf{r},2} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} \\ \mathbf{0} & \mathbf{E}_{22} \end{pmatrix}^H, \quad (11)$$

in which $\mathbf{D}_{\mathbf{r},1}$ and $\mathbf{D}_{\mathbf{r},2}$ are diagonal and $\mathbf{E}_{11}$ is known (this is the part that corresponds to $\mathbf{U}'_l$ and $\mathbf{U}'_r$) $(1 \leqslant r \leqslant R)$. This equation shows that $\mathbf{E}_{22}$ may be obtained from a simultaneous diagonalization in $\mathbb{C}^{(R-R')\times(R-R')}$:

$$\mathbf{G}_{\mathbf{r},22} = \mathbf{E}_{22} \cdot \mathbf{D}_{\mathbf{r},2} \cdot \mathbf{E}_{22}^H \quad 1 \leqslant r \leqslant R. \quad (12)$$

The matrix $\mathbf{E}_{12}$ may subsequently be estimated from an overdetermined set of linear equations:

$$\mathbf{G}_{\mathbf{r},12} = \mathbf{E}_{12} \cdot \mathbf{D}_{\mathbf{r},2} \cdot \mathbf{E}_{22}^H \quad 1 \leqslant r \leqslant R. \quad (13)$$

The error accumulation that is inherent to a deflation approach may be avoided by (i) computation of the optimum of (9), (ii) determination of the directions in the dominant subspace that have a shift structure (this can be done by solving a set of linear equations), (iii) simultaneous diagonalization in $\mathbb{C}^{(R+R')\times(R+R')}$, and (iv) retaining the new components.

The technique described in this section may be generalized to handle arbitrary equalizer lengths.

## 5. CONCLUSION

In this paper we have generalized the constant modulus algorithm derived in [3] to convolutive mixtures.

The technique involves (i) the computation of the $R$-dimensional kernel of an $((N-1) \times (LR)^2)$ matrix, (ii) the best rank-$(R, R, R)$ approximation of an $(LR \times LR \times R)$ tensor, and (iii) the simultaneous diagonalization of $R$ $(R \times R)$ matrices ($N$ is the number of samples, $L$ the length of the equalizer, and $R$ the number of sources).

Our approach is deterministic. As opposed to statistical techniques, in which moments up to order 4 have to be estimated, only short datasets are needed.

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