K.U.Leuven

Department of Electrical Engineering (ESAT) **SISTA**

Technical report 93-71

A method to find all solutions of a system of multivariate polynomial equalities and inequalities in the max algebra*

B. De Schutter and B. De Moor

If you want to cite this report, please use the following reference instead:
B. De Schutter and B. De Moor, "A method to find all solutions of a system of multivariate polynomial equalities and inequalities in the max algebra," *Discrete Event Dynamic Systems: Theory and Applications*, vol. 6, no. 2, pp. 115–138, Mar. 1996.

ESAT-SISTA
K.U. Leuven
Kardinaal Mercierlaan 94
B-3001 Leuven (Heverlee)
Belgium
phone: +32-16-32.17.09 (secretary)
fax: +32-16-32.19.70
URL: `http://www.esat.kuleuven.ac.be/sista`

*This report can also be downloaded via `http://lcewww.et.tudelft.nl/~deschutt/pub/abstracts/93_71_abstr.html`

Discrete Event Dynamic Systems: Theory and Applications, 6, 115–138 (1996)

A Method to Find All Solutions of a System of Multivariate Polynomial Equalities and Inequalities in the Max Algebra*

BART DE SCHUTTER** bart.deschutter@esat.kuleuven.ac.be
BART DE MOOR† bart.demoor@esat.kuleuven.ac.be
ESAT/SISTA, Katholieke Universiteit Leuven, Kardinaal Mercierlaan 94, B-3001 Leuven, Belgium

Received January 28, 1994

Editor: P. Varaiya

Abstract. In this paper we show that finding solutions of a system of multivariate polynomial equalities and inequalities in the max algebra is equivalent to solving an Extended Linear Complementarity Problem. This allows us to find all solutions of such a system of multivariate polynomial equalities and inequalities and provides a geometrical insight in the structure of the solution set. We also demonstrate that this enables us to solve many important problems in the max algebra and the max-min-plus algebra such as matrix decompositions, construction of matrices with a given characteristic polynomial, state space transformations and the (minimal) state space realization problem.

Keywords: max algebra, multivariate max-algebraic polynomial equalities and inequalities, state space models, extended linear complementarity problem.

1. Introduction

1.1. Overview

There are many ways to model and to analyze discrete event dynamic systems. In this paper we concentrate on systems that can be modeled using the max algebra. The elements of the max algebra are the real numbers and $-\infty$, and the admissible operations are the maximum and the addition. This allows us to give a (max-)linear description of a subclass of discrete event systems and to develop a max-algebraic system theory analogous to conventional linear system theory (Baccelli, Cohen, Olsder and Quadrat, 1992).

In this paper we shall show that finding all solutions of a system of multivariate max-algebraic polynomial equalities and inequalities is equivalent to solving an

* This paper presents research results of the Belgian programme on interuniversity attraction poles (IUAP-50) initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture. The scientific responsibility is assumed by its authors.

** Research assistant with the N.F.W.O. (Belgian National Fund for Scientific Research).

† Senior research associate with the N.F.W.O.

Extended Linear Complementarity Problem (ELCP). The ELCP is an extension of the well-known Linear Complementarity Problem, which is one of the fundamental problems of mathematical programming. In (De Schutter and De Moor, 1993) we have developed an algorithm to find all solutions of an ELCP. We shall use this algorithm to find all solutions of a system of multivariate polynomial equalities and inequalities in the max algebra and to give a geometrical insight in the structure of the solution set of this kind of problems. We also show that many other problems in the max algebra and the max-min-plus algebra such as matrix decompositions, transformation of state space models, construction of matrices with a given characteristic polynomial, minimal state space realization and so on, can be solved using the same technique.

This paper is organized as follows: In Section 1 we introduce the notations and some of the concepts and definitions that will be used in the following sections. In Section 2 we propose the Extended Linear Complementarity Problem (ELCP). We give a short description of an algorithm to solve an ELCP and of the resulting solution set. In Section 3 we demonstrate that finding all solutions of a system of multivariate polynomial equalities and inequalities in the max algebra is equivalent to solving an ELCP. Next we give some other applications of the ELCP in the max algebra and in the max-min-plus algebra. We conclude with an illustrative example.

1.2. Notations and definitions

If $\mathbf{a}$ is a vector then a_i represents the ith component of $\mathbf{a}$. If $\mathbf{A}$ is an m by n matrix then the entry on the ith row and the jth column is denoted by a_{ij}. We use $\mathbf{A}_{.j}$ to represent the jth column of $\mathbf{A}$. The m by n zero matrix is denoted by $\mathbf{O}_{m,n}$. A zero column vector is represented by $\mathbf{0}$. To select submatrices of a matrix we use the following notation: $\mathbf{A}([i_1, i_2, \ldots, i_k], [j_1, j_2, \ldots, j_l])$ is the k by l matrix resulting from $\mathbf{A}$ by eliminating all rows except for rows $i_1, i_2, \ldots, i_k$ and all columns except for columns $j_1, j_2, \ldots, j_l$.

Definition. (Polyhedron) A polyhedron is the solution set of a finite system of linear inequalities.

Definition. (Polyhedral cone) A polyhedral cone is the solution set of a finite system of homogeneous linear inequalities.

We shall represent the set of all possible combinations of k different numbers out of the set $\{1, 2, \ldots, n\}$ as $\mathcal{C}_n^k$. The set of all possible permutations of the set $\{1, 2, \ldots, n\}$ is denoted by $\mathcal{P}_n$.

1.3. The max algebra

One of the mathematical tools used in this paper is the max algebra. In this introduction we explain the notations we use to represent the max-algebraic operations.

We also give some definitions that will be used in the remainder of this paper. A complete introduction to the max algebra can be found in (Baccelli, Cohen, Olsder and Quadrat, 1992) and (Cuninghame-Green, 1979).

1.3.1. The max-algebraic operations

In this paper we use the following notations: $a \oplus b = \max(a, b)$ and $a \otimes b = a + b$. The neutral element for $\oplus$ in $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ is $\varepsilon = -\infty$. Since we use both linear algebra and max algebra in this paper, we always write the $\otimes$ sign explicitly to avoid confusion. The max-algebraic power is defined as follows: $a^{\otimes^k} = \underbrace{a \otimes a \otimes \ldots \otimes a}_{k \text{ times}}$ and is equal to ka in linear algebra. The operations $\oplus$ and $\otimes$ are extended to matrices in the usual way. The max-algebraic matrix power is represented by $\mathbf{A}^{\otimes^k} = \underbrace{\mathbf{A} \otimes \mathbf{A} \otimes \ldots \otimes \mathbf{A}}_{k \text{ times}}$. The n by n identity matrix in $\mathbb{R}_{\max}$ is denoted as $\mathbf{E}_n : e_{ij} = 0$ if $i = j$ and $e_{ij} = \varepsilon$ if $i \neq j$.

We also use the extension of the max algebra $\mathbb{S}_{\max}$ that was introduced in (Baccelli, Cohen, Olsder and Quadrat, 1992) and (Gaubert, 1992) and which is a kind of symmetrization of $\mathbb{R}_{\max}$. We shall restrict ourselves to an intuitive introduction to the most important features of $\mathbb{S}_{\max}$. For a more formal derivation the interested reader is referred to (Gaubert, 1992).

There are three kinds of elements in $\mathbb{S}_{\max}$: the max-positive elements ($\mathbb{S}^{\oplus}_{\max}$, this corresponds to $\mathbb{R}_{\max}$), the max-negative elements ($\mathbb{S}^{\ominus}_{\max}$) and the balanced elements ($\mathbb{S}^{\bullet}_{\max}$). The max-positive and the max-negative elements are called signed ($\mathbb{S}^{\vee}_{\max} = \mathbb{S}^{\oplus}_{\max} \cup \mathbb{S}^{\ominus}_{\max}$). The $\oplus$ operation is extended to $\mathbb{S}_{\max}$ as follows:

$$
\begin{aligned}
a \oplus (\ominus b) &= a\,, && \text{if } a > b\,, \\
a \oplus (\ominus b) &= \ominus b\,, && \text{if } a < b\,, \\
a \oplus (\ominus a) &= a^{\bullet}\,,
\end{aligned}
$$

where $a,\, b \in \mathbb{R}_{\max}$. The $\ominus$ sign corresponds to the $-$ sign in linear algebra. By analogy we write $a \ominus b$ instead of $a \oplus (\ominus b)$.
If $a \in \mathbb{S}_{\max}$ then it can be written as $a = a^+ \ominus a^-$ where a^+ is the max-positive part of a, a^- is the max-negative part of a and $|a|_{\oplus} = a^+ \oplus a^-$ is the max-absolute value of a. There are three possible cases: if $a \in \mathbb{S}^{\oplus}_{\max}$ then $a^+ = a$ and $a^- = \varepsilon$, if $a \in \mathbb{S}^{\ominus}_{\max}$ then $a^+ = \varepsilon$ and $a^- = \ominus a$ and if $a \in \mathbb{S}^{\bullet}_{\max}$ then $a^+ = a^- = |a|_{\oplus}$.

Example: Let $a = 3^{\bullet} \in \mathbb{S}^{\bullet}_{\max}$, then $a^+ = 3$, $a^- = 3$ and $|a|_{\oplus} = 3$.
For $b = \ominus 2 \in \mathbb{S}^{\ominus}_{\max}$ we have $b^+ = \varepsilon$, $b^- = 2$ and $|b|_{\oplus} = 2$. □

This symmetrization allows us to "solve" equations that have no solutions in $\mathbb{R}_{\max}$. However, since $\ominus$ is not cancellative – i.e. in general $a \ominus a \neq \varepsilon$, the zero element for $\oplus$ – we have to introduce balances (∇) instead of equalities. If $a,\, b \in \mathbb{S}_{\max}$ then we have that $a \;\nabla\; b$ if and only if $a^+ \oplus b^- = a^- \oplus b^+$. Informally an $\ominus$ sign in a balance means that the element should be at the other side. If both sides of a balance are signed (max-positive or max-negative) we can replace the balance by an equality. We shall illustrate these concepts and properties with a few examples.

Example: We have that $3 \nabla 4^\bullet$ since

$$\begin{aligned} 3 \nabla 4^\bullet &\Leftrightarrow 3 \nabla 4 \ominus 4 && (\text{definition of } 4^\bullet) \\ &\Leftrightarrow 3 \oplus 4 = 4 && (\text{definition of } \nabla) \\ &\Leftrightarrow 4 = 4 \end{aligned}$$

since $3 \oplus 4 = \max(3,4) = 4$. □

Example: We have that $3 \not\nabla \ominus 1$ since

$$\begin{aligned} 3 \nabla \ominus 1 &\Leftrightarrow 3 \oplus 1 = \varepsilon \\ &\Leftrightarrow 3 = \varepsilon \ , \end{aligned}$$

but $3 \neq \varepsilon$. □

Example: If we want to solve $x \oplus 4 \nabla 3$, we get $x \nabla 3 \ominus 4$ or $x \nabla \ominus 4$. If we want a signed solution the latter balance becomes an equality and this yields $x = \ominus 4$. □

1.3.2. Some definitions

Definition. (Determinant) Consider a matrix $\mathbf{A} \in \mathbb{S}_{\max}^{n \times n}$. The determinant of $\mathbf{A}$ is defined as

$$\det \mathbf{A} = \bigoplus_{\sigma \in \mathcal{P}_n} \operatorname{sgn}(\sigma) \otimes \bigotimes_{i=1}^{n} a_{i\sigma(i)} \ ,$$

where $\mathcal{P}_n$ is the set of all permutations of $\{1, \ldots, n\}$, and $\operatorname{sgn}(\sigma) = 0$ if the permutation σ is even and $\operatorname{sgn}(\sigma) = \ominus 0$ if the permutation is odd.

Definition. (Minor rank) Let $\mathbf{A} \in \mathbb{S}_{\max}^{m \times n}$. The minor rank of $\mathbf{A}$ is defined as the dimension of the largest square submatrix of $\mathbf{A}$ the determinant of which is not balanced and not equal to ε.

Definition. (Characteristic equation) Let $\mathbf{A} \in \mathbb{S}_{\max}^{n \times n}$. The characteristic equation of $\mathbf{A}$ is defined as $\det(\mathbf{A} \ominus \lambda \otimes \mathbf{E}_n) \nabla \varepsilon$.

If we work this out we get

$$\lambda^{\otimes^n} \oplus \bigoplus_{p=1}^{n} a_p \otimes \lambda^{\otimes^{n-p}} \ \nabla \ \varepsilon \ ,$$

which will be called a *monic* balance, since the coefficient of $\lambda^{\otimes^n}$ is equal to 0 (i.e. the identity element for $\otimes$).

2. The Extended Linear Complementarity Problem

2.1. Problem formulation

Consider the following problem:

> Given two matrices $\mathbf{A} \in \mathbb{R}^{p\times n}, \mathbf{B} \in \mathbb{R}^{q\times n}$, two column vectors $\mathbf{c} \in \mathbb{R}^p, \mathbf{d} \in \mathbb{R}^q$ and m subsets ϕ_j of $\{1, 2, \ldots, p\}$, find a column vector $\mathbf{x} \in \mathbb{R}^n$ such that
>
> $$\sum_{j=1}^{m} \prod_{i\in\phi_j} (\mathbf{Ax} - \mathbf{c})_i = 0 \tag{1}$$
>
> $$\text{subject to} \quad \begin{aligned} \mathbf{Ax} &\geqslant \mathbf{c} \\ \mathbf{Bx} &= \mathbf{d} \ , \end{aligned}$$
>
> or show that no such vector $\mathbf{x}$ exists.

In (De Schutter and De Moor, 1993) we have demonstrated that this problem is an extension of the Linear Complementarity Problem (Cottle, Pang and Stone, 1992). Therefore we call it the Extended Linear Complementarity Problem (ELCP). Equation (1) represents the *complementarity condition*. One possible interpretation of this condition is the following: since $\mathbf{Ax} \geqslant \mathbf{c}$, condition (1) is equivalent to

$$\prod_{i\in\phi_j} (\mathbf{Ax} - \mathbf{c})_i = 0 \ , \quad \forall j \in \{1, 2, \ldots, m\} \quad . \tag{2}$$

So we could say that each set ϕ_j corresponds to a subgroup of inequalities of $\mathbf{Ax} \geqslant \mathbf{c}$ and that in each group at least one inequality should hold with equality:

$$\forall j \in \{1, 2, \ldots, m\} \ : \ \exists i \in \phi_j \text{ such that } (\mathbf{Ax} - \mathbf{c})_i = 0 \ .$$

We shall use this interpretation in Section 3 to find all solutions of a system of multivariate polynomial equalities and inequalities in the max algebra.

2.2. The solution set of an ELCP

In (De Schutter and De Moor, 1993) we have made a thorough study of the solution set of an ELCP and developed an algorithm to find all its solutions. We shall briefly state the main results of that paper.

In order to solve the ELCP we homogenize it by introducing a scalar $\alpha \geqslant 0$ and defining

$$\mathbf{u} = \begin{bmatrix} \mathbf{x} \\ \alpha \end{bmatrix} \ , \ \mathbf{P} = \begin{bmatrix} \mathbf{A} & -\mathbf{c} \\ \mathbf{O}_{1,n} & 1 \end{bmatrix} \ \text{ and } \ \mathbf{Q} = \begin{bmatrix} \mathbf{B} & -\mathbf{d} \end{bmatrix} \ .$$

Then we get a homogeneous ELCP of the following form:

Given two matrices $\mathbf{P} \in \mathbb{R}^{p \times n}$, $\mathbf{Q} \in \mathbb{R}^{q \times n}$ and m subsets ϕ_j of $\{1, 2, \ldots, p\}$, find a non-trivial column vector $\mathbf{u} \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{m} \prod_{i \in \phi_j} (\mathbf{Pu})_i = 0 \tag{3}$$

subject to $\begin{aligned} \mathbf{Pu} &\geqslant \mathbf{0} \\ \mathbf{Qu} &= \mathbf{0} \ , \end{aligned}$

or show that no such vector $\mathbf{u}$ exists.

So now we have a system of homogeneous linear equalities and inequalities subject to a complementarity condition.
The solution set of the system of homogeneous linear inequalities and equalities

$$\mathbf{Pu} \geqslant \mathbf{0} \tag{4}$$

$$\mathbf{Qu} = \mathbf{0} \ , \tag{5}$$

is a polyhedral cone $\mathcal{P}$ and can be described using two sets of rays: a set of central rays $\mathcal{C}$ and a set of extremal rays $\mathcal{E}$. The set of central rays can be considered as a basis for the linear subspace of the polyhedral cone $\mathcal{P}$. If $\mathbf{c} \in \mathcal{C}$ then $\mathbf{Pc} = \mathbf{0}$, and if $\mathbf{e} \in \mathcal{E}$ then $\mathbf{Pe} \neq \mathbf{0}$.
We have that $\mathbf{u}$ is a solution of (4) – (5) if and only if it can be written as

$$\mathbf{u} = \sum_{\mathbf{c}_i \in \mathcal{C}} \alpha_i \, \mathbf{c}_i + \sum_{\mathbf{e}_i \in \mathcal{E}} \beta_i \, \mathbf{e}_i \ , \tag{6}$$

with $\alpha_i \in \mathbb{R}$ and $\beta_i \geqslant 0$.

To calculate the sets $\mathcal{C}$ and $\mathcal{E}$ we use an iterative algorithm that is an adaptation of the double description method of Motzkin (Motzkin, Raiffa, Thompson and Thrall, 1953). During the iteration we already remove rays that do not satisfy the partial complementarity condition since such rays cannot yield solutions of the ELCP. In the kth step the partial complementarity condition is defined as follows:

$$\prod_{i \in \phi_j} (\mathbf{Pu})_i = 0, \qquad \forall j \in \{1, 2, \ldots, m\} \text{ such that } \phi_j \subset \{1, 2, \ldots, k\} \ . \tag{7}$$

So we only consider those groups of inequalities that have already been processed entirely. For $k \geqslant p$ the partial complementarity condition (7) coincides with the full complementarity condition (3). This leads to the following algorithm:

Algorithm 1 : Calculation of the central and extremal rays.

Initialization:

- $\mathcal{C}_0 := \{\mathbf{c}_i \mid \mathbf{c}_i = (\mathbf{I}_n)_{.i} \text{ for } i = 1, 2, \ldots, n\}$
- $\mathcal{E}_0 := \emptyset$

Iteration:

for $k := 1, 2, \ldots, p+q$,

- Calculate the intersection of the current polyhedral cone (described by $\mathcal{C}_{k-1}$ and $\mathcal{E}_{k-1}$) with the half-space or hyperplane determined by the kth inequality or equality. This yields a new polyhedral cone described by $\mathcal{C}_k$ and $\mathcal{E}_k$.
- Remove the rays that do not satisfy the partial complementarity condition.

Result: $\mathcal{C} := \mathcal{C}_{p+q}$ and $\mathcal{E} := \mathcal{E}_{p+q}$

Not every combination of the form (6) satisfies the complementarity condition. Although every linear combination of the central rays satisfies the complementarity condition, not every positive combination of the extremal rays satisfies the complementarity condition. Therefore we introduce the concept of cross-complementarity:

Definition. (Cross-complementarity) Let $\mathcal{E}$ be the set of extremal rays of an homogeneous ELCP. A subset $\mathcal{E}_s$ of $\mathcal{E}$ is cross-complementary if every combination of the form

$$\mathbf{u} = \sum_{\mathbf{e}_i \in \mathcal{E}_s} \beta_i \, \mathbf{e}_i \quad ,$$

with $\beta_i \geqslant 0$, satisfies the complementarity condition.

In (De Schutter and De Moor, 1993) we have proven that in order to check whether a set $\mathcal{E}_s$ is cross-complementary it suffices to test only one strictly positive combination of the rays in $\mathcal{E}_s$, e.g. the combination with $\forall i : \beta_i = 1$. Now we can determine Γ, the set of cross-complementary sets of extremal rays: $\Gamma = \{ \mathcal{E}_s \mid \mathcal{E}_s \text{ is cross-complementary } \}$.

Algorithm 2 : Determination of the cross-complementary sets of extremal rays.

Initialization:

- $\Gamma := \emptyset$
- Construct the cross-complementarity graph $\mathcal{G}$ with a node e_i for each ray $\mathbf{e}_i \in \mathcal{E}$ and an edge between nodes e_k and e_l if the set $\{\mathbf{e}_k, \mathbf{e}_l\}$ is cross-complementary.
- $\mathcal{S} := \{\mathbf{e}_1\}$

Depth-first search in $\mathcal{G}$:

- Select a new node e^{new} that is connected by an edge to all nodes of the set $\mathcal{S}$ and add the corresponding ray to the test set: $\mathcal{S}^{\text{new}} := \mathcal{S} \cup \{\mathbf{e}^{\text{new}}\}$.

- **if** $\mathcal{S}^{\text{new}}$ is cross-complementary
  then Select a new node and add it to the test set.
  else Add $\mathcal{S}$ to Γ: $\Gamma := \Gamma \cup \{\mathcal{S}\}$, and go back to the last point where a choice was made.
  Continue until all possible choices have been considered.

Result: Γ

Once $\mathcal{C}$, $\mathcal{E}$ and Γ have been determined, the solution set of the homogeneous ELCP is given by the following theorem:

THEOREM 1 *When the set of central rays $\mathcal{C}$, the set of extremal rays $\mathcal{E}$ and the set of cross-complementary sets of extremal rays Γ are given then $\mathbf{u}$ is a solution of the homogeneous ELCP if and only if there exists a set $\mathcal{E}_s \in \Gamma$ such that $\mathbf{u}$ can be written as*

$$\mathbf{u} = \sum_{\mathbf{c}_i \in \mathcal{C}} \alpha_i \mathbf{c}_i + \sum_{\mathbf{e}_i \in \mathcal{E}_s} \beta_i \mathbf{e}_i \quad , \tag{8}$$

with $\alpha_i \in \mathbb{R}$ and $\beta_i \geqslant 0$.

Finally we have to extract the solutions of the original ELCP, i.e. we have to retain solutions of the form (8) that have an α component equal to 1 ($\mathbf{u}_\alpha = 1$). We transform the sets $\mathcal{C}$, $\mathcal{E}$ and Γ as follows:

- If $\mathbf{c} \in \mathcal{C}$ then $\mathbf{c}_\alpha = 0$. We drop the α component and put the result in the set $\mathcal{X}^{\text{cen}}$, the set of central rays.
- If $\mathbf{e} \in \mathcal{E}$ then there are two possibilities:
  - If $\mathbf{e}_\alpha = 0$ then we drop the α component and put the result in the set $\mathcal{X}^{\text{inf}}$, the set of infinite rays.
  - If $\mathbf{e}_\alpha > 0$ then we normalize $\mathbf{e}$ such that $\mathbf{e}_\alpha = 1$. Next we drop the α component and put the result in the set $\mathcal{X}^{\text{fin}}$, the set of finite rays.
- For each set $\mathcal{E}_s \in \Gamma$ we construct the set of corresponding infinite rays $\mathcal{X}_s^{\text{inf}}$ and the set of corresponding finite rays $\mathcal{X}_s^{\text{fin}}$. If $\mathcal{X}_s^{\text{fin}} \neq \emptyset$ then we add the pair $\{\mathcal{X}_s^{\text{inf}}, \mathcal{X}_s^{\text{fin}}\}$ to Λ, the set of pairs of cross-complementary sets of infinite and finite rays.

Now we can characterize the solution set of the ELCP:

THEOREM 2 *When $\mathcal{X}^{\text{cen}}$, $\mathcal{X}^{\text{inf}}$, $\mathcal{X}^{\text{fin}}$ and Λ are given, then $\mathbf{x}$ is a solution of the ELCP if and only if there exists a pair $\{\mathcal{X}_s^{\text{inf}}, \mathcal{X}_s^{\text{fin}}\} \in \Lambda$ such that*

$$\mathbf{x} = \sum_{\mathbf{x}_k \in \mathcal{X}^{\text{cen}}} \lambda_k \mathbf{x}_k + \sum_{\mathbf{x}_k \in \mathcal{X}_s^{\text{inf}}} \kappa_k \mathbf{x}_k + \sum_{\mathbf{x}_k \in \mathcal{X}_s^{\text{fin}}} \mu_k \mathbf{x}_k \quad , \tag{9}$$

with $\lambda_k \in \mathbb{R}$, $\kappa_k \geqslant 0$, $\mu_k \geqslant 0$ *and* $\sum_k \mu_k = 1$.

This leads to:

THEOREM 3 *The general solution set of an ELCP consists of the union of (bounded and unbounded) polyhedra.*

When we are not interested in obtaining all solutions of the ELCP we can skip the calculation of the pairs $\{\mathcal{X}_s^{\text{inf}}, \mathcal{X}_s^{\text{fin}}\}$. Each element of $\mathcal{X}^{\text{fin}}$ will then be a solution of the ELCP.

For a more detailed and precise description of the algorithms and for the proofs of the theorems of this subsection the interested reader is referred to (De Schutter and De Moor, 1993), where also a worked example can be found.

3. The solution of a system of multivariate polynomial equalities and inequalities in the max algebra

In this section we consider a system of multivariate polynomial equalities and inequalities in the max algebra, which can be seen as a generalized framework for many important max-algebraic problems such as matrix decompositions, transformation of state space models, state space realization of impulse responses, construction of matrices with a given characteristic polynomial and so on. These applications will be treated in detail in Section 4.

3.1. Problem formulation

Consider the following problem:

> Given a set of integers $\{m_k\}$ and three sets of coefficients $\{a_{ki}\}$, $\{b_k\}$ and $\{c_{kij}\}$ with $i \in \{1, \ldots, m_k\}$, $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, p_1, p_1+1, \ldots, p_1+p_2\}$, find a vector $\mathbf{x} \in \mathbb{R}^n$ that satisfies
>
> $$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j{}^{\otimes^{c_{kij}}} = b_k\,, \qquad \text{for } k = 1, 2, \ldots, p_1\,, \tag{10}$$
>
> $$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j{}^{\otimes^{c_{kij}}} \leqslant b_k\,, \qquad \text{for } k = p_1+1, p_1+2, \ldots, p_1+p_2\,, \tag{11}$$
>
> or show that no such vector $\mathbf{x}$ exists.

We call (10) – (11) a system of multivariate polynomial equalities and inequalities in the max algebra. Note that the exponents can be negative or real. In the next subsection we shall show that we can use the ELCP algorithm of Section 2 to solve this problem.

3.2. Translation to linear algebra

First we consider one equation of the form (10):

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j{}^{\otimes^{c_{kij}}} = b_k \quad .$$

In linear algebra this is equivalent to the system of linear inequalities

$$a_{ki} + c_{ki1}x_1 + c_{ki2}x_2 + \ldots + c_{kin}x_n \leqslant b_k \, , \qquad \text{for } i = 1, 2, \ldots, m_k \quad ,$$

where at least one inequality should hold with equality.
If we transfer the a_{ki}'s to the right hand side and if we define $d_{ki} = b_k - a_{ki}$, we get the following system of linear inequalities:

$$c_{ki1}x_1 + c_{ki2}x_2 + \ldots + c_{kin}x_n \leqslant d_{ki} \, , \qquad \text{for } i = 1, 2, \ldots, m_k \quad ,$$

where at least one inequality should hold with equality. So equation (10) will lead to p_1 groups of linear inequalities, where in each group at least one inequality should hold with equality.
Using the same reasoning equations of the form (11) can also be transformed into a system of linear inequalities, but without an extra condition.
If we define $p_1 + p_2$ matrices $\mathbf{C}_k$ and $p_1 + p_2$ column vectors $\mathbf{d}_k$ such that $(\mathbf{C}_k)_{ij} = c_{kij}$ and $(\mathbf{d}_k)_i = d_{ki}$, then our original problem is equivalent to $p_1 + p_2$ groups of linear inequalities

$$\mathbf{C}_k \mathbf{x} \leqslant \mathbf{d}_k \quad ,$$

where there has to be at least one inequality that holds with equality in each group $\mathbf{C}_k \mathbf{x} \leqslant \mathbf{d}_k$ for $k = 1, \ldots, p_1$.

Now we put all $\mathbf{C}_k$'s in one large matrix $\mathbf{A} = \begin{bmatrix} -\mathbf{C}_1 \\ -\mathbf{C}_2 \\ \vdots \\ -\mathbf{C}_{p_1+p_2} \end{bmatrix}$ and all $\mathbf{d}_k$'s in one large vector $\mathbf{c} = \begin{bmatrix} -\mathbf{d}_1 \\ -\mathbf{d}_2 \\ \vdots \\ -\mathbf{d}_{p_1+p_2} \end{bmatrix}$. We also define p_1 sets ϕ_j such that $\phi_j = \{s_j + 1, s_j + 2, \ldots, s_j + m_j\}$, for $j = 1, 2, \ldots, p_1$, where $s_1 = 0$ and $s_{j+1} = s_j + m_j$ for $j = 1, 2, \ldots, p_1 - 1$. Our original problem (10)–(11) is then equivalent to the following ELCP:

Find a column vector $\mathbf{x} \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{p_1} \prod_{i \in \phi_j} (\mathbf{A}\mathbf{x} - \mathbf{c})_i = 0 \tag{12}$$

subject to $\mathbf{Ax} \geqslant \mathbf{c}$,
or show that no such vector $\mathbf{x}$ exists.

Remark. If any of the m_k's in the set of max-algebraic equalities (10) equals 1 then we get a linear equality instead of a system of linear inequalities. All these equalities can be extracted from $\mathbf{Ax} \geqslant \mathbf{c}$ and from the complementarity condition and be put in $\mathbf{Bx} = \mathbf{d}$. This is not necessary but it will certainly enhance the efficiency of the ELCP algorithm.

Since some exponents may be negative and to avoid problems arising from $0 \cdot \varepsilon$ we have assumed that all the coefficients and all the components of $\mathbf{x}$ are finite. However, in some cases we can allow b_k's that are equal to ε. Then we have to introduce a negative number ξ that is large enough in absolute value and transform equations of the form $\bigoplus_i t_i = \varepsilon$ into $\bigoplus_i t_i \leqslant \xi$. Afterwards we replace every negative component of $\mathbf{x}$ that has the same order of magnitude as ξ by ε provided that

- this does not cause any problems arising from taking negative powers of ε,
- $\mathbf{x}$ has no positive components of the same order of magnitude as ξ. Positive components of the same order of magnitude as ξ would have to be replaced by $+\infty$, but $+\infty$ does not belong to $\mathbb{R}_{\max}$.

Another way to obtain solutions with components equal to ε is to allow some of the λ_k's or κ_k's in (9) to become infinite, but in a controlled way, since we only allow infinite components that are equal to ε and since negative powers of ε are not defined. The max operation hides small numbers from larger numbers. Therefore it suffices in practice to replace negative components that are large enough in absolute value by ε provided that there are no positive components of the same order of magnitude.

THEOREM 4 *A system of multivariate polynomial equalities and inequalities in the max algebra is equivalent to an extended linear complementarity problem.*

Proof: We have already demonstrated how a system of multivariate polynomial equalities and inequalities in the max algebra can be transformed into an ELCP.

Now we prove that an ELCP can also be transformed into a system of multivariate polynomial equalities and inequalities in the max algebra.
Consider the following ELCP:

Given two matrices $\mathbf{A} \in \mathbb{R}^{p \times n}, \mathbf{B} \in \mathbb{R}^{q \times n}$, two vectors $\mathbf{c} \in \mathbb{R}^p, \mathbf{d} \in \mathbb{R}^q$ and m subsets ϕ_j of $\{1, 2, \ldots, p\}$, find a column vector $\mathbf{x} \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{m} \prod_{i \in \phi_j} (\mathbf{Ax} - \mathbf{c})_i = 0 \qquad (13)$$

$$\text{subject to} \quad \begin{aligned} \mathbf{Ax} &\geqslant \mathbf{c} \\ \mathbf{Bx} &= \mathbf{d} \ . \end{aligned}$$

First we define $\Phi = \bigcup_{j=1}^{m} \phi_j$ and $\Phi^c = \{1, 2, \ldots, p\} \setminus \Phi$. There are three possible cases to consider:

1. Groups of linear inequalities where at least one inequality should hold with equality:

$$\sum_{k=1}^{n} a_{ik} x_k \geqslant c_i \,, \qquad \forall i \in \Phi \ , \tag{14}$$

   with

$$\prod_{i \in \phi_j} (\mathbf{Ax} - \mathbf{c})_i = 0 \,, \qquad \forall j \in \{1, 2, \ldots, m\} \ , \tag{15}$$

   where we use the alternative formulation of the complementarity condition, which means that at least one inequality should hold with equality. Equation (14) is equivalent to

$$c_i + \sum_{k=1}^{n} (-a_{ik}) x_k \leqslant 0 \,, \qquad \forall i \in \Phi \ .$$

   If we translate everything into max algebra and if we also take condition (15) into account, we get m multivariate max-algebraic polynomial equalities:

$$\bigoplus_{i \in \phi_j} c_i \otimes \bigotimes_{k=1}^{n} {x_k}^{\otimes^{(-a_{ik})}} = 0 \,, \qquad \forall j \in \{1, 2, \ldots, m\} \ .$$

2. Linear equalities:

$$\sum_{k=1}^{n} b_{ik} x_k = d_i \,, \qquad \forall i \in \{1, 2, \ldots, q\} \ .$$

   These can be transformed into q multivariate max-algebraic polynomial equalities:

$$d_i \otimes \bigotimes_{k=1}^{n} {x_k}^{\otimes^{(-b_{ik})}} = 0 \,, \qquad \forall i \in \{1, 2, \ldots, q\} \ .$$

3. The remaining linear inequalities:

$$\sum_{k=1}^{n} a_{ik} x_k \geqslant c_i \, , \qquad \forall i \in \Phi^c \ ,$$

can be transformed into one multivariate polynomial inequality in the max algebra:

$$\bigoplus_{i \in \Phi^c} c_i \otimes \bigotimes_{k=1}^{n} {x_k}^{\otimes^{(-a_{ik})}} \leqslant 0 \ .$$

So an ELCP can be transformed into a system of multivariate polynomial equalities and inequalities in the max algebra. ■

THEOREM 5 *The general solution set of a system of multivariate max-algebraic polynomial equalities and inequalities is the union of a set of bounded and unbounded polyhedra (some of which may be degenerate).*

Proof: This is a direct consequence of Theorem 3 and Theorem 4. ■

4. Applications

In this section we treat some important problems in the max algebra that can be reformulated as a system of multivariate max-algebraic polynomial equalities and inequalities. These problems can thus be solved using the ELCP algorithm. In general their solution set consists of the union of a set of polyhedra.

4.1. Matrix decompositions

Consider the following problem:

Given a matrix $\mathbf{A} \in \mathbb{R}_{\max}^{m \times n}$ and an integer $p > 0$, find $\mathbf{B} \in \mathbb{R}_{\max}^{m \times p}$ and $\mathbf{C} \in \mathbb{R}_{\max}^{p \times n}$ such that

$$\mathbf{A} = \mathbf{B} \otimes \mathbf{C} \ ,$$

or show that no such decomposition exists.

So we have to find b_{ik} and c_{kj} for $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, p\}$ such that

$$\bigoplus_{k=1}^{p} b_{ik} \otimes c_{kj} = a_{ij} \, , \qquad \forall i \in \{1, \ldots, m\} \ , \forall j \in \{1, \ldots, n\} \ ,$$

and this can clearly be considered as a system of multivariate polynomial equations in b_{ik} and c_{kj}. It is obvious that if we take p too small, this problem will not have any solutions.
This technique can easily be extended to the decomposition of $\mathbf{A}$ into the product of any number of matrices of a specified size:

> Given a matrix $\mathbf{A} \in \mathbb{R}_{\max}^{m \times n}$ and a set of $p+1$ strictly positive integers $\{m_i\}$ with $m_1 = m$ and $m_{p+1} = n$, find p matrices $\mathbf{P}_i \in \mathbb{R}_{\max}^{m_i \times m_{i+1}}$ such that
>
> $$\mathbf{A} = \bigotimes_{i=1}^{p} \mathbf{P}_i \ .$$

It is also possible to impose a certain structure on the composing matrices (e.g. triangular, diagonal, Hessenberg, ...).

4.2. Transformation of state space models

Consider a discrete event system that can be described by the following nth order state space model:

$$\mathbf{x}(k+1) = \mathbf{A} \otimes \mathbf{x}(k) \ \oplus \ \mathbf{B} \otimes \mathbf{u}(k) \tag{16}$$

$$\mathbf{y}(k) = \mathbf{C} \otimes \mathbf{x}(k) \ , \tag{17}$$

where $\mathbf{A} \in \mathbb{R}_{\max}^{n \times n}$, $\mathbf{B} \in \mathbb{R}_{\max}^{n \times l}$ and $\mathbf{C} \in \mathbb{R}_{\max}^{p \times n}$. The vector $\mathbf{x}$ represents the state, $\mathbf{u}$ is the input and $\mathbf{y}$ is the output of the system.

In contrast to linear algebra, where we can use similarity transformations to change the basis of the state space, this is not always possible in the max algebra. Moreover, since only permuted diagonal matrices are invertible, max-algebraic similarity transformations have a limited scope.

We could transfer our problem from $\mathbb{R}_{\max}$ to $\mathbb{S}_{\max}$ since it is possible to define a similarity transformation in $\mathbb{S}_{\max}$. But this approach has two major drawbacks: first of all we get balances instead of equalities in $\mathbb{S}_{\max}$. Moreover, it is not clear how to find a similarity transformation such that the resulting matrices are max-positive (i.e. have entries in $\mathbb{S}_{\max}^{\oplus} = \mathbb{R}_{\max}$). This means that in general we cannot transfer the results back to $\mathbb{R}_{\max}$.

Therefore we propose an approach that is entirely based on $\mathbb{R}_{\max}$. We transform a given state space model into another state space model that has the same input-output behavior. This approach was hinted at, but not proven, in (Moller, 1986) and we extend it such that the dimension of the state space vector can also change.

Suppose that our system is described by the state space model (16)–(17). If we can find a common factor $\mathbf{L} \in \mathbb{R}_{\max}^{r \times n}$ of $\mathbf{A}$ and $\mathbf{C}$ with $\mathbf{A} = \hat{\mathbf{A}} \otimes \mathbf{L}$ and $\mathbf{C} = \hat{\mathbf{C}} \otimes \mathbf{L}$ then we can transform the state space model into

$$\mathbf{L} \otimes \mathbf{x}(k+1) = \mathbf{L} \otimes \hat{\mathbf{A}} \otimes \mathbf{L} \otimes \mathbf{x}(k) \ \oplus \ \mathbf{L} \otimes \mathbf{B} \otimes \mathbf{u}(k)$$

$$\mathbf{y}(k) = \hat{\mathbf{C}} \otimes \mathbf{L} \otimes \mathbf{x}(k)$$

or

$$\begin{aligned}\tilde{\mathbf{x}}(k+1) &= \tilde{\mathbf{A}} \otimes \tilde{\mathbf{x}}(k) \quad \oplus \quad \tilde{\mathbf{B}} \otimes \mathbf{u}(k) \\ \mathbf{y}(k) &= \tilde{\mathbf{C}} \otimes \tilde{\mathbf{x}}(k)\end{aligned}$$

with

$$\tilde{\mathbf{A}} = \mathbf{L} \otimes \hat{\mathbf{A}} \tag{18}$$
$$\tilde{\mathbf{B}} = \mathbf{L} \otimes \mathbf{B} \tag{19}$$
$$\tilde{\mathbf{C}} = \hat{\mathbf{C}} \tag{20}$$
$$\tilde{\mathbf{x}}(k) = \mathbf{L} \otimes \mathbf{x}(k) \ . \tag{21}$$

Now we prove that this system has the same input-output behavior as the first system provided that we also adapt the initial state by setting $\tilde{\mathbf{x}}(0) = \mathbf{L} \otimes \mathbf{x}(0)$. The output of system (16)–(17) is given by

$$\mathbf{y}(k) = \mathbf{C} \otimes \mathbf{A}^{\otimes^k} \otimes \mathbf{x}(0) \quad \oplus \quad \bigoplus_{i=0}^{k-1} \mathbf{G}_i \otimes \mathbf{u}(k-1-i) \,, \qquad \text{for } k > 0 \ ,$$

with

$$\begin{aligned}\mathbf{G}_i &= \mathbf{C} \otimes \mathbf{A}^{\otimes^i} \otimes \mathbf{B} \\ &= \hat{\mathbf{C}} \otimes \mathbf{L} \otimes \left(\hat{\mathbf{A}} \otimes \mathbf{L}\right)^{\otimes^i} \otimes \mathbf{B} \\ &= \hat{\mathbf{C}} \otimes \left(\mathbf{L} \otimes \hat{\mathbf{A}}\right)^{\otimes^i} \otimes \mathbf{L} \otimes \mathbf{B} \\ &= \tilde{\mathbf{C}} \otimes \tilde{\mathbf{A}}^{\otimes^i} \otimes \tilde{\mathbf{B}} \\ &= \tilde{\mathbf{G}}_i\end{aligned}$$

and

$$\begin{aligned}\mathbf{C} \otimes \mathbf{A}^{\otimes^k} \otimes \mathbf{x}(0) &= \hat{\mathbf{C}} \otimes \mathbf{L} \otimes \left(\hat{\mathbf{A}} \otimes \mathbf{L}\right)^{\otimes^k} \otimes \mathbf{x}(0) \\ &= \hat{\mathbf{C}} \otimes \left(\mathbf{L} \otimes \hat{\mathbf{A}}\right)^{\otimes^k} \otimes \mathbf{L} \otimes \mathbf{x}(0) \\ &= \tilde{\mathbf{C}} \otimes \tilde{\mathbf{A}}^{\otimes^k} \otimes \tilde{\mathbf{x}}(0) \ .\end{aligned}$$

So both systems have indeed the same input-output behavior.

We see that $\mathbf{L}$ is not necessarily invertible (at least not in $\mathbb{R}_{\max}$) even if $r = n$, so this transformation is not a similarity transformation. But if we do the same operations in $\mathbb{S}_{\max}$, then $\mathbf{L}$ is invertible (provided that $\mathbf{L}$ is square and $\det \mathbf{L} \not\nabla \varepsilon$) and then we have a similarity transformation since equations (18)–(20) can be transformed into

$$\begin{aligned}\tilde{\mathbf{A}} &\ \nabla\ \mathbf{L} \otimes \mathbf{A} \otimes \mathbf{L}^{\otimes^{-1}} \\ \tilde{\mathbf{B}} &\ \nabla\ \mathbf{L} \otimes \mathbf{B} \\ \tilde{\mathbf{C}} &\ \nabla\ \mathbf{C} \otimes \mathbf{L}^{\otimes^{-1}} \ .\end{aligned}$$

So to get another equivalent state space model of the system described by equations (16) – (17) all we have to do is find a decomposition

$$\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{A}} \\ \hat{\mathbf{C}} \end{bmatrix} \otimes \mathbf{L} ,$$

with $\mathbf{L} \in \mathbb{R}_{\max}^{r \times n}$, $\hat{\mathbf{A}} \in \mathbb{R}_{\max}^{n \times r}$ and $\hat{\mathbf{C}} \in \mathbb{R}_{\max}^{p \times r}$. This is a matrix decomposition that can be considered as a system of multivariate max-algebraic equalities as was shown in the previous subsection. If $r = n$ then $\mathbf{L}$ will be square and the new model will also be an nth order model. If we take a rectangular $\mathbf{L}$ matrix, we can change the dimension of the state space vector and get an rth order state space model instead of an nth order state space model.

It is obvious that the first dimension of $\mathbf{L}$ should be greater than or equal to the minimal system order otherwise we cannot find a common factor of $\mathbf{A}$ and $\mathbf{C}$ (see also the remark at the end of the next subsection).

In (De Schutter and De Moor, 1994b) we have shown that the decomposition technique of this subsection does not yield the entire set of all state space realizations of given input-output behavior. In the next subsection we shall demonstrate how we can use the ELCP algorithm to find the entire set of all state space realizations of a given impulse response.

4.3. Minimal state space realization

Consider a single input single output (SISO) discrete event system that can be described by the following nth order state space model:

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{A} \otimes \mathbf{x}(k) \quad \oplus \quad \mathbf{b} \otimes u(k) \\ y(k) &= \mathbf{c} \otimes \mathbf{x}(k) \ , \end{aligned}$$

with $\mathbf{A} \in \mathbb{R}_{\max}^{n \times n}$, $\mathbf{b} \in \mathbb{R}_{\max}^{n \times 1}$ and $\mathbf{c} \in \mathbb{R}_{\max}^{1 \times n}$.

If we apply a unit impulse: $\begin{aligned} e(k) &= 0 \quad \text{if } k = 0 \, , \\ &= \varepsilon \quad \text{otherwise} \ , \end{aligned}$

to the system and if we assume that the initial state $\mathbf{x}(0)$ satisfies $\mathbf{x}(0) = \varepsilon$ or $\mathbf{A} \otimes \mathbf{x}(0) \leqslant \mathbf{b}$, we get the impulse response as the output of the system:

$$y(k) = \mathbf{c} \otimes \mathbf{A}^{\otimes^{k-1}} \otimes \mathbf{b} \, , \qquad \text{for } k > 0 \ .$$

Let $g_k = \mathbf{c} \otimes \mathbf{A}^{\otimes^{k}} \otimes \mathbf{b}$. The g_k's are called the *Markov parameters.*

Suppose that $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$ are unknown, and that we only know the Markov parameters (e.g. from experiments – where we assume that the system is max-linear and time-invariant and that there is no noise present). How can we construct $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$ from the g_k's? This process is called state space realization. If we make the dimension of $\mathbf{A}$ minimal, we have a minimal state space realization.

First we need a lower bound for the minimal system order. In (De Schutter and De Moor, 1995) and (De Schutter and De Moor, 1994a) we have presented a procedure for finding such a lower bound. We could also use the following theorem of (Gaubert, 1992) which also yields a lower bound:

THEOREM 6 *Let $\{g_k\}_{k=0}^{\infty}$ be the impulse response of a SISO time-invariant max-linear system. Then the minor rank of the Hankel matrix*

$$H = \begin{bmatrix} g_0 & g_1 & g_2 & \cdots \\ g_1 & g_2 & g_3 & \cdots \\ g_2 & g_3 & g_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

is a lower bound for the minimal system order.

We start with r equal to this lower bound. Now we try to find an rth order state space realization of the given impulse response. We have to find $\mathbf{A} \in \mathbb{R}_{\max}^{r \times r}$, $\mathbf{b} \in \mathbb{R}_{\max}^{r \times 1}$ and $\mathbf{c} \in \mathbb{R}_{\max}^{1 \times r}$ such that

$$\mathbf{c} \otimes \mathbf{A}^{\otimes^k} \otimes \mathbf{b} = g_k \ , \qquad \text{for } k = 0, 1, 2, \ldots, N-1 \ , \tag{22}$$

for N large enough. If we work out the equations of the form (22) we get for $k = 0$:

$$\bigoplus_{i=1}^{r} c_i \otimes b_i = g_0 \ ,$$

and for $k > 0$:

$$\bigoplus_{i=1}^{r} \bigoplus_{j=1}^{r} t_{kij} = g_k$$

with

$$t_{kij} = \bigoplus_{i_1=1}^{r} \ldots \bigoplus_{i_{k-1}=1}^{r} c_i \otimes a_{ii_1} \otimes a_{i_1 i_2} \otimes \ldots \otimes a_{i_{k-1} j} \otimes b_j \ .$$

This can be rewritten as

$$\bigoplus_{i=1}^{r} \bigoplus_{j=1}^{r} \bigoplus_{l=1}^{r^{k-1}} c_i \otimes \bigotimes_{u=1}^{r} \bigotimes_{v=1}^{r} a_{uv}{}^{\otimes^{\gamma_{kijluv}}} \otimes b_j = g_k \ ,$$

where γ_{kijluv} is the number of times that a_{uv} appears in the lth subterm of term t_{kij}. If a_{uv} does not appear in that subterm we take $\gamma_{kijluv} = 0$, since we have that $a^{\otimes^0} = 0 \cdot a = 0$, the identity element for $\otimes$. At first sight one could think that we are then left with r^{k+1} terms. However, some of these are the same and can thus be left out. If we use the fact that $\forall x, y \in \mathbb{R}_{\max} : x \otimes y \leqslant x \otimes x \oplus y \otimes y$, we can again remove many redundant terms. Then we are left with, say, w_k terms where

$w_k \leqslant r^{k+1}$.

If we put all unknowns in one large vector $\mathbf{x}$ of length $r(r+2)$ we have to solve a system of multivariate max-algebraic polynomial equations of the following form:

$$\bigoplus_{i=1}^{r} \bigotimes_{j=1}^{r(r+2)} x_j{}^{\otimes^{\kappa_{0ij}}} = g_0 \tag{23}$$

$$\bigoplus_{i=1}^{w_k} \bigotimes_{j=1}^{r(r+2)} x_j{}^{\otimes^{\kappa_{kij}}} = g_k \,, \qquad \text{for } k = 1, 2, \ldots, N-1 \ . \tag{24}$$

If we do not get any solutions, this means that r is less than the minimal system order, i.e. the lower bound is not tight. Then we have to augment our estimate of the minimal system order and repeat the above procedure but with $r+1$ instead of r. We continue until we find a solution of (23)–(24). This will then yield a minimal state space realization of the given impulse response.

For a detailed description of how to find a lower bound of the system order and for an example the interested reader is referred to (De Schutter and De Moor, 1995).

Remark. If we already have a state space realization of the given impulse response, we could try to use the state space transformation technique of Section 4.2 with the lower bound r as the number of rows of $\mathbf{L}$ to get a minimal realization. If we do not get any solutions, we augment r and repeat the procedure until we get a solution. However, in (De Schutter and De Moor, 1994b) we have shown that it is not always possible to obtain a minimal state space realization in this way, i.e. it is possible that the system order of the final solution obtained with the state space transformation technique of Section 4.2 is larger than the minimal system order.

4.4. Construction of matrices with a given characteristic polynomial

Consider the following problem:

> Given a monic polynomial in $\mathbb{S}_{\max}$
>
> $$\lambda^{\otimes^n} \oplus \bigotimes_{p=1}^{n} b_p \otimes \lambda^{\otimes^{n-p}} \ ,$$
>
> find a matrix $\mathbf{A} \in \mathbb{R}_{\max}^{n \times n}$ such that the characteristic polynomial of $\mathbf{A}$ is equal to the given polynomial.

The coefficients of the characteristic polynomial of a matrix $\mathbf{A}$ are given by

$$a_p = (\ominus 0)^{\otimes^p} \otimes \bigoplus_{\varphi \in \mathcal{C}_n^p} \det \mathbf{A}([i_1, i_2, \ldots, i_p], [i_1, i_2, \ldots, i_p]) \ ,$$

where $\mathcal{C}_n^p$ is the set of all combinations of p different numbers out of the set $\{1, \ldots, n\}$ and $\varphi = \{i_1, i_2, \ldots, i_p\}$. If we consider the max-positive and the max-negative parts of the coefficients of the characteristic polynomial of a matrix $\mathbf{A} \in \mathbb{R}_{\max}^{n \times n}$ (without simplifying $\ominus$) we have for $k > 0$:

$$a_{2k}^{+} = \bigoplus_{\varphi \in \mathcal{C}_n^{2k}} \bigoplus_{\sigma \in \mathcal{P}_{2k,\mathrm{even}}} a_{i_1 i_{\sigma(1)}} \otimes a_{i_2 i_{\sigma(2)}} \otimes \ldots \otimes a_{i_{2k} i_{\sigma(2k)}} \tag{25}$$

$$a_{2k}^{-} = \bigoplus_{\varphi \in \mathcal{C}_n^{2k}} \bigoplus_{\sigma \in \mathcal{P}_{2k,\mathrm{odd}}} a_{i_1 i_{\sigma(1)}} \otimes a_{i_2 i_{\sigma(2)}} \otimes \ldots \otimes a_{i_{2k} i_{\sigma(2k)}} \tag{26}$$

$$a_{2k+1}^{+} = \bigoplus_{\varphi \in \mathcal{C}_n^{2k+1}} \bigoplus_{\sigma \in \mathcal{P}_{2k+1,\mathrm{odd}}} a_{i_1 i_{\sigma(1)}} \otimes a_{i_2 i_{\sigma(2)}} \otimes \ldots \otimes a_{i_{2k+1} i_{\sigma(2k+1)}} \tag{27}$$

$$a_{2k+1}^{-} = \bigoplus_{\varphi \in \mathcal{C}_n^{2k+1}} \bigoplus_{\sigma \in \mathcal{P}_{2k+1,\mathrm{even}}} a_{i_1 i_{\sigma(1)}} \otimes a_{i_2 i_{\sigma(2)}} \otimes \ldots \otimes a_{i_{2k+1} i_{\sigma(2k+1)}} \ , \tag{28}$$

where $\mathcal{P}_{k,\mathrm{even}}$ is the set of all permutations of $\{1, 2, \ldots, k\}$ with even parity and $\mathcal{P}_{k,\mathrm{odd}}$ is the set of all permutations of $\{1, 2, \ldots, k\}$ with odd parity.
Now we have to find the elements of the matrix $\mathbf{A}$ such that $a_p^{+} \ominus a_p^{-} = b_p$.
There are three possible cases for each $p > 1$:

1. if $b_p \in \mathbb{S}_{\max}^{\oplus}$ we should have that $\begin{cases} a_p^{+} = |b_p|_{\oplus} \\ a_p^{-} < |b_p|_{\oplus} \end{cases}$,

2. if $b_p \in \mathbb{S}_{\max}^{\ominus}$ we should have that $\begin{cases} a_p^{+} < |b_p|_{\oplus} \\ a_p^{-} = |b_p|_{\oplus} \end{cases}$,

3. if $b_p \in \mathbb{S}_{\max}^{\bullet}$ we should have that $\begin{cases} a_p^{+} = |b_p|_{\oplus} \\ a_p^{-} = |b_p|_{\oplus} \end{cases}$.

Remark. For a matrix $A \in \mathbb{R}_{\max}^{n \times n}$ we always have that $a_1 \in \mathbb{S}_{\max}^{\ominus}$.

It is always possible to transform the strict inequalities into non-strict inequalities by subtracting a small positive number of the right hand sides. This leads to a combination of multivariate polynomial equalities and inequalities in the max algebra.

Since not every nth order monic polynomial in $\mathbb{S}_{\max}$ corresponds to a characteristic equation of a matrix $\mathbf{A} \in \mathbb{R}_{\max}^{n \times n}$, it is useful to be able to determine whether a solution exists or not before starting the algorithm. In (De Schutter and De Moor, 1994a) we have presented some necessary conditions for a monic polynomial in $\mathbb{S}_{\max}$ to be the characteristic equation of a matrix with elements in $\mathbb{R}_{\max}$. In (De Schutter and De Moor, 1994a) we have also outlined a heuristic algorithm to find such a matrix.

4.5. Mixed max-min problems

We can also use the technique of Section 3 to solve mixed max-min problems. First we introduce the $\oplus'$ operation: $a \oplus' b = \min(a, b)$. The neutral element for $\oplus'$ is $+\infty$. We have to extend $\mathbb{R}_{\max}$ to $\overline{\mathbb{R}} = (\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}, \oplus, \oplus', \otimes)$ and define the $\otimes$ operation for all elements in $\overline{\mathbb{R}}$. For more information about this max-min-plus algebra, the interested reader is referred to (Cuninghame-Green, 1979) or (Olsder, 1991).

Now we consider the following problem:

> Given two sets of integers $\{m_k\}$ and $\{m_{kl_1}\}$ and three sets of coefficients $\{a_{kl_1l_2}\}$, $\{b_k\}$ and $\{c_{kl_1l_2j}\}$ with $k \in \{1, \ldots, m\}$, $l_1 \in \{1, \ldots, m_k\}$, $l_2 \in \{1, \ldots, m_{kl_1}\}$ and $j \in \{1, \ldots, n\}$, find a vector $\mathbf{x} \in \overline{\mathbb{R}}^n$ that satisfies
>
> $$\bigoplus_{l_1=1}^{m_k}{}' \bigoplus_{l_2=1}^{m_{kl_1}} a_{kl_1l_2} \otimes \bigotimes_{j=1}^{n} x_j{}^{\otimes^{c_{kl_1l_2j}}} = b_k \,, \qquad \text{for } k = 1, 2, \ldots, m \,, \tag{29}$$
>
> or show that no such vector $\mathbf{x}$ exists.

If we define

$$t_{kl_1} = \bigoplus_{l_2=1}^{m_{kl_1}} a_{kl_1l_2} \otimes \bigotimes_{j=1}^{n} x_j{}^{\otimes^{c_{kl_1l_2j}}} \quad , \tag{30}$$

we get

$$\bigoplus_{l_1=1}^{m_k}{}' t_{kl_1} = b_k \,, \qquad \text{for } k = 1, 2, \ldots, m \ . \tag{31}$$

If we assume that the b_k's are finite, then also the t_{kl_1}'s are finite. Therefore their inverses exist and (30) becomes

$$\bigoplus_{l_2=1}^{m_{kl_1}} a_{kl_1l_2} \otimes \bigotimes_{j=1}^{n} x_j{}^{\otimes^{c_{kl_1l_2j}}} \otimes t_{kl_1}{}^{\otimes^{-1}} = 0 \,, \qquad \begin{array}{l} \text{for } \ k = 1, 2, \ldots, m \\ \text{and } \ l_1 = 1, 2, \ldots, m_k \ . \end{array} \tag{32}$$

Now we consider an equation of the form (31). This is equivalent to

$$t_{kl_1} \geqslant b_k \,, \qquad \text{for } l_1 = 1, 2, \ldots, m_k \ ,$$

where at least one inequality should hold with equality. So the min equations will yield m groups of inequalities where in each group at least one inequality should hold with equality.

Equations of the form (32) are multivariate polynomial equations in the max algebra and can thus also be written as groups of linear inequalities with in each

group at least one inequality that should hold with equality.
This means that the combined max-min problem (29) can be transformed into an ELCP.
We can also use this technique for systems of combined max-min equations of the form

$$\bigoplus_{l_1}{}' \bigoplus_{l_2} \bigoplus_{l_3}{}' \ldots \bigoplus_{l_q} a_{kl_1l_2\ldots l_q} \otimes \bigotimes_{j=1}^{n} x_i{}^{\otimes^{c_{kl_1l_2\ldots l_qj}}} = b_k \,, \qquad \text{for } k = 1, \ldots, m \;,$$

or for analogous equations but with $\oplus$ replaced by $\oplus'$ and vice versa or when some of the equalities are replaced by inequalities.

4.6. Max-max and max-min problems

In this section we consider systems of equations where the right hand sides are also multivariate max-algebraic polynomials. Since we are working in $\mathbb{R}_{\max}$ we cannot simply transfer terms from the right hand side to the left hand side as we would do in linear algebra. However, these problems can also be solved using a technique similar to that of Section 4.5.
We consider the following problem:

> Given two sets of integers $\{m_k\}$ and $\{p_k\}$ and four sets of coefficients $\{a_{ki}\}$, $\{b_{kij}\}$, $\{c_{kl}\}$ and $\{d_{klj}\}$ with $k \in \{1, \ldots, q\}$, $i \in \{1, \ldots, m_k\}$, $l \in \{1, \ldots, p_k\}$ and $j \in \{1, \ldots, n\}$, find a vector $\mathbf{x} \in \mathbb{R}^n$ that satisfies
>
> $$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j{}^{\otimes^{b_{kij}}} = \bigoplus_{l=1}^{p_k} c_{kl} \otimes \bigotimes_{j=1}^{n} x_j{}^{\otimes^{d_{klj}}} \;, \qquad \text{for } k = 1, \ldots, q \,. \qquad (33)$$

We define q dummy variables t_k such that

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j{}^{\otimes^{b_{kij}}} = t_k \;, \qquad \text{for } k = 1, 2, \ldots q \;.$$

Since we know that $\mathbf{x}$ is finite the t_k's will also be finite and their inverse will exist. So problem (33) is equivalent to

$$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j{}^{\otimes^{b_{kij}}} \otimes t_k{}^{\otimes^{-1}} = 0 \,, \qquad \text{for } k = 1, 2, \ldots q \;,$$

$$\bigoplus_{l=1}^{p_k} c_{kl} \otimes \bigotimes_{j=1}^{n} x_j{}^{\otimes^{d_{klj}}} \otimes t_k{}^{\otimes^{-1}} = 0 \,, \qquad \text{for } k = 1, 2, \ldots q \;,$$

which is again a system of multivariate max-algebraic polynomial equalities and can thus be transformed into an ELCP.

Using an analogous reasoning we can also solve problems that contain a mixture of equations of the following forms:

- $\displaystyle\bigoplus_i{}' \, l_i(\mathbf{x}) \;=\; \bigoplus_i r_i(\mathbf{x})$
- $\displaystyle\bigoplus_i{}' \, l_i(\mathbf{x}) \;=\; \bigoplus_i{}' \, r_i(\mathbf{x})$
- $\displaystyle\bigoplus_i{}' \, l_i(\mathbf{x}) \;\leqslant\; \bigoplus_i r_i(\mathbf{x})$
- $\displaystyle\bigoplus_i{}' \, l_i(\mathbf{x}) \;\geqslant\; \bigoplus_i r_i(\mathbf{x})$
- $\displaystyle\bigoplus_i l_i(\mathbf{x}) \;\leqslant\; \bigoplus_i r_i(\mathbf{x})$
- $\displaystyle\bigoplus_i{}' \, l_i(\mathbf{x}) \;\leqslant\; \bigoplus_i{}' \, r_i(\mathbf{x})$,

where $l_i(\mathbf{x})$ and $r_i(\mathbf{x})$ are max-algebraic monomials of the form $a_i \otimes \displaystyle\bigotimes_{j=1}^{n} x_j{}^{\otimes^{b_{ij}}}$.

Remark. It is obvious that e.g. systems of max-linear equations and eigenvalue problems in the max algebra can also be transformed into an ELCP, but for these problems there are other algorithms that are more efficient, especially if we only want one solution (Baccelli, Cohen, Olsder and Quadrat, 1992), (Braker, 1993), (Cuninghame-Green, 1979).

5. An example

Consider the following system of multivariate polynomial equalities and inequalities:

$$\left\{ \begin{array}{rcl} 4 \otimes x_1 \otimes x_3 \otimes x_4{}^{\otimes^{-2}} \;\oplus\; 3 \otimes x_1{}^{\otimes^{2}} \otimes x_4 \;\oplus\; x_2{}^{\otimes^{3}} \otimes x_3{}^{\otimes^{-1}} \otimes x_5{}^{\otimes^{-3}} & = & 1 \\ 2 \otimes x_1 \otimes x_3{}^{\otimes^{2}} \;\oplus\; 1 \otimes x_2{}^{\otimes^{-1}} \otimes x_3 \otimes x_4{}^{\otimes^{2}} \otimes x_5 & = & 0 \\ x_1{}^{\otimes^{2}} \otimes x_3{}^{\otimes^{-3}} \otimes x_4 & \leqslant & 2 \end{array} \right.$$

with $\mathbf{x} \in \mathbb{R}^5$.
This can be transformed into the following ELCP:

Given

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & -1 & 2 & 0 \\ -2 & 0 & 0 & -1 & 0 \\ 0 & -3 & 1 & 0 & 3 \\ -1 & 0 & -2 & 0 & 0 \\ 0 & 1 & -1 & -2 & -1 \\ -2 & 0 & 3 & -1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ -1 \\ 2 \\ 1 \\ -2 \end{bmatrix} ,$$

Table 1. The rays of the solution set.

Set	$\mathcal{X}^{\text{cen}}$			$\mathcal{X}^{\text{inf}}$				$\mathcal{X}^{\text{fin}}$
Ray	$\mathbf{x}_1^{\text{c}}$	$\mathbf{x}_1^{\text{i}}$	$\mathbf{x}_2^{\text{i}}$	$\mathbf{x}_3^{\text{i}}$	$\mathbf{x}_4^{\text{i}}$	$\mathbf{x}_5^{\text{i}}$	$\mathbf{x}_1^{\text{f}}$	$\mathbf{x}_2^{\text{f}}$
x_1	0	-12	-4	-6	-1	-5	-12	-1.25
x_2	1	0	0	0	0	0	0	0
x_3	0	6	2	3	-1	-3	5	-1.45
x_4	0	-3	-1	-1	-1	1	-2	0.15
x_5	1	-2	0	-1	3	1	-2	0.15

Table 2. The pairs of cross-complementary sets.

s	$\mathcal{X}_s^{\text{inf}}$	$\mathcal{X}_s^{\text{fin}}$
1	$\{\mathbf{x}_1^{\text{i}}, \mathbf{x}_2^{\text{i}}\}$	$\{\mathbf{x}_1^{\text{f}}\}$
2	$\{\mathbf{x}_1^{\text{i}}, \mathbf{x}_3^{\text{i}}\}$	$\{\mathbf{x}_1^{\text{f}}\}$
3	$\{\mathbf{x}_2^{\text{i}}, \mathbf{x}_4^{\text{i}}\}$	$\{\mathbf{x}_1^{\text{f}}, \mathbf{x}_2^{\text{f}}\}$
4	$\{\mathbf{x}_3^{\text{i}}, \mathbf{x}_5^{\text{i}}\}$	$\{\mathbf{x}_1^{\text{f}}, \mathbf{x}_2^{\text{f}}\}$

find a column vector $\mathbf{x} \in \mathbb{R}^5$ such that

$$(\mathbf{Ax} - \mathbf{c})_1 (\mathbf{Ax} - \mathbf{c})_2 (\mathbf{Ax} - \mathbf{c})_3 + (\mathbf{Ax} - \mathbf{c})_4 (\mathbf{Ax} - \mathbf{c})_5 = 0$$

subject to $\mathbf{Ax} \geqslant \mathbf{c}$.

The ELCP algorithm yields the rays of Table 1 and the pairs of cross-complementary sets of Table 2.
Any arbitrary solution of the system of multivariate polynomial equalities and inequalities can now be expressed as

$$\mathbf{x} = \lambda_1 \mathbf{x}_1^{\text{c}} + \sum_{\mathbf{x}_k^{\text{i}} \in \mathcal{X}_s^{\text{inf}}} \kappa_k \mathbf{x}_k^{\text{i}} + \sum_{\mathbf{x}_k^{\text{f}} \in \mathcal{X}_s^{\text{fin}}} \mu_k \mathbf{x}_k^{\text{f}} \ ,$$

for some $s \in \{1, \ldots, 4\}$ with $\lambda_1 \in \mathbb{R}$, $\kappa_k \geqslant 0$, $\mu_k \geqslant 0$ and $\sum_k \mu_k = 1$.

6. Conclusions and further research

We have demonstrated that many problems in the max algebra and the max-min-plus algebra can be transformed into an ELCP. These problems can then be solved using our ELCP algorithm. One of the main characteristics of this ELCP algorithm is that it finds all solutions and that it gives a geometrical insight in the solution set of the problems we considered. On the other hand this also leads to large computation times and storage space requirements if the number of variables and equations is large. Therefore it might be interesting to develop (heuristic) algorithms that only find one solution as we have done for the minimal realization problem in (De Schutter and De Moor, 1994a).

It could also be interesting to make a more thorough study of the class of problems that can be reduced to solving a system of multivariate max-algebraic polynomial equalities and inequalities. Every instance of this class can then be solved using our ELCP algorithm.

References

Baccelli, F., Cohen, G., Olsder, G.J., and Quadrat, J.P. 1992. *Synchronization and Linearity.* New York: John Wiley & Sons.

Braker, J.G. 1993. An extended algorithm for performance evaluation of timed event graphs. *Proc. 2nd European Control Conference*, Groningen, The Netherlands, pp. 524–529.

Cottle, R.W., Pang, J.S., and Stone, R.E. 1992. *The Linear Complementarity Problem.* Boston: Academic Press.

Cuninghame-Green, R.A. 1979. *Minimax Algebra.* Berlin: Springer-Verlag.

De Schutter, B. and De Moor, B. 1993. The extended linear complementarity problem, ESAT/SISTA, Katholieke Universiteit Leuven, Leuven, Belgium, Tech. Rep. 93-69. Accepted for publication in Mathematical Programming.

De Schutter, B. and De Moor, B. 1994a. The characteristic equation and minimal state space realization of SISO systems in the max algebra. *Proc. 11th Int. Conf. on Analysis and Optimization of Systems*, Sophia-Antipolis, France, pp. 273–282.

De Schutter, B. and De Moor, B. 1994b. Minimal realization in the max algebra, ESAT/SISTA, Katholieke Universiteit Leuven, Leuven, Belgium, Tech. Rep. 94-29.

De Schutter, B. and De Moor, B. 1995. Minimal realization in the max algebra is an extended linear complementarity problem. *Systems & Control Letters* 25:2, pp. 103–111.

Gaubert, S. 1992. *Théorie des Systèmes Linéaires dans les Dioïdes.* PhD thesis, Ecole Nationale Supérieure des Mines de Paris, France.

Moller, P. 1986. Théorème de Cayley-Hamilton dans les dioïdes et application à l'étude des systèmes à événements discrets. *Proc. 7th Int. Conf. on Analysis and Optimization of Systems*, Antibes, France, pp. 215–226.

Motzkin, T.S., Raiffa, H., Thompson, G.L., and Thrall, R.M. 1953. The double description method. In *Contributions to the theory of games*, no. 28 in Annals of Math. Studies. Princeton: Princeton University Press, pp. 51–73.

Olsder, G.J. 1991. Eigenvalues of dynamic max-min systems. *Discrete Event Dynamic Systems: Theory and Applications* 1, pp. 177–207.