Katholieke Universiteit Leuven

Departement Elektrotechniek

ESAT-SISTA/TR 97-13

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Book of extended abstracts of the Hybrid Systems V workshop (HS'97),
Notre Dame, Indiana, USA, September 11-13, 1997, pp. 15–20.

[1]This paper is also available by anonymous ftp from `ftp.esat.kuleuven.ac.be` as file `97-13.ps.gz` in the directory `pub/SISTA/deschutter/reports`.

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This research was sponsored by the Concerted Action Project of the Flemish Community, entitled "Model-based Information Processing Systems", by the Belgian program on interuniversity attraction poles (IUAP P4-02 and IUAP P4-24) and by the ALAPEDES project of the European Community Training and Mobility of Researchers Program.

The Extended Linear Complementarity Problem and the Modeling and Analysis of Hybrid Systems

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Keywords: hybrid systems, traffic light control, extended linear complementarity problems, complementary slackness systems

Abstract

First we give a short description of the Extended Linear Complementarity Problem (which is a mathematical programming problem) and its use in the analysis of discrete event systems and continuous variable systems. Next we consider a traffic-light-controlled intersection, which can be considered as a hybrid system, and we construct a model that describes the evolution of the queue lengths in the various lanes (as continuous variables) as a function of time and we show that this leads to an Extended Linear Complementarity Problem. Furthermore, it can also be shown that some problems in the analysis of another class of hybrid systems, the "complementary-slackness systems", also lead to an Extended Linear Complementarity Problem.

1 The Extended Linear Complementarity Problem

The Extended Linear Complementarity Problem (ELCP) is an extension of the Linear Complementarity Problem, which is one of the fundamental problems in mathematical programming [2]. The ELCP is defined as follows:

> Given $A \in \mathbb{R}^{p\times n}$, $B \in \mathbb{R}^{q\times n}$, $c \in \mathbb{R}^p$, $d \in \mathbb{R}^q$ and m subsets ϕ_1, ϕ_2, ..., ϕ_m of $\{1, 2, \ldots, p\}$, find $x \in \mathbb{R}^n$ such that
>
> $$\sum_{j=1}^{m} \prod_{i\in\phi_j} (Ax - c)_i = 0$$
>
> subject to $Ax \geq c$ and $Bx = d$, or show that no such x exists.

This problem can be considered as a system of linear equations and inequalities ($Ax \geq c$, $Bx = d$), where there are m groups of linear inequalities (one group for each index set ϕ_j) such that in each group at least one inequality should hold with equality.

In [4, 5] we have developed an algorithm to compute the complete solution set of an ELCP. Our algorithm yields a description of the solution set of an ELCP by vertices, extreme rays and a basis of the linear subspace corresponding to the largest affine subspace of the solution

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set. In that way it provides a geometrical insight in the entire solution set of the ELCP and related problems. In [4, 5] we have also shown that the general ELCP is NP-hard.

The formulation of the ELCP arose from our work in the study of discrete event systems (DESs), typical examples of which are flexible manufacturing systems, subway traffic networks, parallel processing systems and telecommunication networks. In general the description of the behavior of a DES leads to a model that is nonlinear in conventional algebra. However, there exists a class of systems — the "max-linear DESs" — for which the model is "linear" when we express it in the max-plus algebra [1], which has maximization and addition as basic operations. In [4, 6] we have shown that many problems in the system theory for max-linear DESs (such as computing state space transformations, minimal state space realizations, max-algebraic matrix factorizations, ...) lead to an ELCP.

In [3, 17] it has been shown that the problem of finding all possible stationary operating points of electrical circuits that contain resistors and controlled sources with piecewise-linear characteristics can be reformulated as a Generalized LCP, which is a special case of the ELCP [4, 5] (see also [7]). These electrical circuits can be considered as examples of continuous variable systems (CVSs), i.e., systems that can be modeled using difference or differential equations.

Since hybrid systems arise from the interaction between DESs and CVSs and exhibit characteristics of both DESs and CVSs, this leads to the question as to whether the ELCP can also play a role in the modeling and analysis of certain classes of hybrid systems. In this paper we consider two classes of hybrid systems for which this is indeed the case: traffic-light-controlled intersections and complementary-slackness systems.

2 Traffic-light-controlled intersections

Consider a single intersection of two two-way streets with controllable traffic lights on each corner (see Figure 1). There are four lanes L_1, L_2, L_3 and L_4, and on each corner of the intersection there are traffic lights (T_1, T_2, T_3 and T_4). For sake of simplicity we assume that the traffic lights can be either green or red. The average arrival rate of cars in lane L_i is λ_i. When the traffic light is green, the average departure rate in lane L_i is μ_i. Let t_0, t_1, t_2, t_3, ... be the time instants at which the traffic lights switch from green to red or vice versa. The traffic light switching scheme is shown in Table 1. Define $\delta_k = t_{k+1} - t_k$. Let $l_i(t)$ be the queue length (i.e., the number of cars waiting) in lane L_i at time instant t.

Let us now write down the equations that describe the relation between the switching time instants and the queue lengths.

Consider lane L_1. When the traffic light T_1 is red, there are arrivals at lane L_1 and no

Period	T_1	T_2	T_3	T_4
$t_0 - t_1$	red	green	red	green
$t_1 - t_2$	green	red	green	red
$t_2 - t_3$	red	green	red	green
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 1: The traffic light switching scheme.

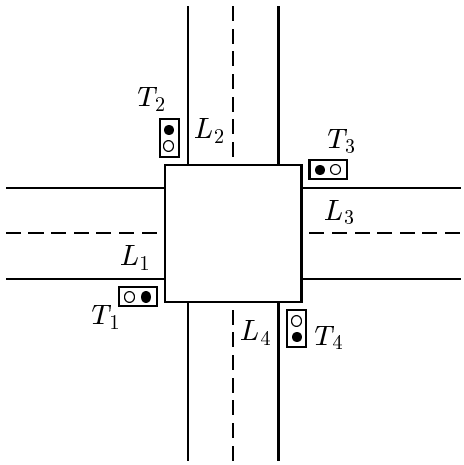

Figure 1: A traffic-light-controlled intersection of two two-way streets.

departures. As a consequence we have

$$\frac{dl_1(t)}{dt} = \lambda_1 \tag{1}$$

for $t \in (t_{2k}, t_{2k+1})$ with $k \in \mathbb{N}$, and

$$l_1(t_{2k+1}) = l_1(t_{2k}) + \lambda_1 \delta_{2k}$$

for $k = 0, 1, 2, \ldots$ When the traffic light T_1 is green, there are arrivals and departures at lane L_1. Since the net arrival rate is $\lambda_1 - \mu_1$ and since the queue length $l_1(t)$ cannot be negative, we have:

$$\frac{dl_1(t)}{dt} = \begin{cases} \lambda_1 - \mu_1 & \text{if } l_1(t) > 0 \\ 0 & \text{if } l_1(t) = 0 \end{cases} \tag{2}$$

for $t \in (t_{2k+1}, t_{2k+2})$ with $k \in \mathbb{N}$. So

$$l_1(t_{2k+2}) = \max(l_1(t_{2k+1}) + (\lambda_1 - \mu_1)\delta_{2k+1}, 0)$$

for $k = 0, 1, 2, \ldots$ Note that we also have

$$l_1(t_{2k+1}) = \max(l_1(t_{2k}) + \lambda_1 \delta_{2k}, 0)$$

for $k = 0, 1, 2, \ldots$ since $l_1(t) \geq 0$ for all t.
We can write down similar equations for $l_2(t_k)$, $l_3(t_k)$ and $l_4(t_k)$. So if we define

$$\begin{aligned} x_k &= \begin{bmatrix} l_1(t_k) & l_2(t_k) & l_3(t_k) & l_4(t_k) \end{bmatrix}^T \\ b_1 &= \begin{bmatrix} \lambda_1 & \lambda_2 - \mu_2 & \lambda_3 & \lambda_4 - \mu_4 \end{bmatrix}^T \\ b_2 &= \begin{bmatrix} \lambda_1 - \mu_1 & \lambda_2 & \lambda_3 - \mu_3 & \lambda_4 \end{bmatrix}^T , \end{aligned}$$

then we have

$$x_{2k+1} = \max(x_{2k} + b_1 \delta_{2k}, 0) \tag{3}$$

$$x_{2k+2} = \max(x_{2k+1} + b_2 \delta_{2k+1}, 0) \tag{4}$$

for $k = 0, 1, 2, \ldots$

The traffic-light-controlled intersection can be considered as a hybrid system with time and the queue lengths as states variables and that can operate in two regimes characterized by differential equations of the form (1) or (2) depending on the value of a discrete control variable that can have the value "red" or "green".

Now we show that the system (3) – (4) can be reformulated as an ELCP. First consider (3) for an arbitrary index k. This equation can be rewritten as follows:

$$\begin{gathered} x_{2k+1} \geq x_{2k} + b_1 \delta_{2k} \\ x_{2k+1} \geq 0 \\ (x_{2k+1})_i = (x_{2k} + b_1 \delta_{2k})_i \quad \text{or} \quad (x_{2k+1})_i = 0 \qquad \text{for } i = 1, 2, 3, 4\,, \end{gathered}$$

or equivalently

$$\begin{gathered} x_{2k+1} - x_{2k} - b_1 \delta_{2k} \geq 0 \\ x_{2k+1} \geq 0 \\ (x_{2k+1} - x_{2k} - b_1 \delta_{2k})_i (x_{2k+1})_i = 0 \qquad \text{for all } i\,. \end{gathered}$$

Since a sum of nonnegative numbers is equal to 0 if and only if all the numbers are equal to 0, this system of equations is equivalent to:

$$\begin{gathered} x_{2k+1} - x_{2k} - b_1 \delta_{2k} \geq 0 \\ x_{2k+1} \geq 0 \\ \sum_{i=1}^{4} (x_{2k+1} - x_{2k} - b_1 \delta_{2k})_i \, (x_{2k+1})_i = 0 \ . \end{gathered}$$

We can repeat this reasoning for (4) and for each index k.
If we consider N switching time instants and if we define

$$x^* = \begin{bmatrix} x_1^T & x_2^T & \cdots & x_N^T \end{bmatrix}^T \text{ and } \delta^* = \begin{bmatrix} \delta_0 & \delta_1 & \cdots & \delta_{N-1} \end{bmatrix}^T \ ,$$

we finally get a description of the form

$$Ax^* + B\delta^* + c \geq 0 \tag{5}$$

$$x^* \geq 0 \tag{6}$$

$$(Ax^* + B\delta^* + c)^T x^* = 0 \ . \tag{7}$$

It is easy to verify that the system (5) – (7) is a special case of an ELCP.

Now we can compute traffic light switching schemes that minimize objective functions such as average queue length, worst case queue length, average waiting time, and so on. Furthermore, we can impose extra conditions such as minimum and maximum durations for the green and the red time[1], maximum queue lengths[2], and so on. This leads to the following problem:

$$\text{minimize } J \tag{8}$$

[1] A green time that is too short is wasteful. If the red time is too long, drivers tend to believe that the signals have broken down.

[2] This could correspond to an upper bound on the available storage space due to the distance to the preceding junction or to the layout of the intersection.

subject to

$$\delta_{\min,\mathrm{r}} \le \delta_{2k} \le \delta_{\max,\mathrm{r}} \qquad \text{for } k \in \alpha(N) \tag{9}$$

$$\delta_{\min,\mathrm{g}} \le \delta_{2k+1} \le \delta_{\max,\mathrm{g}} \qquad \text{for } k \in \beta(N) \tag{10}$$

$$x_k \le x_{\max} \qquad \text{for } k = 1, 2, \ldots, N \tag{11}$$

$$x_{2k+1} = \max(x_{2k} + b_1 \delta_{2k}, 0) \qquad \text{for } k \in \alpha(N) \tag{12}$$

$$x_{2k+2} = \max(x_{2k+1} + b_2 \delta_{2k+1}, 0) \qquad \text{for } k \in \beta(N) \ , \tag{13}$$

with

$$\alpha(N) = \left\{0, 1, \ldots, \left\lfloor \frac{N-1}{2} \right\rfloor \right\} \quad \text{and} \quad \beta(N) = \left\{0, 1, \ldots, \left\lfloor \frac{N}{2} \right\rfloor - 1 \right\} \ .$$

Using the procedure given above the system (9) – (13) can be rewritten as a system of the form

$$Ax^* + B\delta^* + c \ge 0 \tag{14}$$

$$x^* \ge 0 \tag{15}$$

$$Ex^* + D\delta^* + f \ge 0 \tag{16}$$

$$(Ax^* + B\delta^* + c)^T x^* = 0 \ , \tag{17}$$

which is again a special case of an ELCP. In order to determine the optimal traffic light switching scheme we then have to minimize the objective function J over the solution set of this ELCP. The algorithm of [4, 5] to compute the solution set of a general ELCP requires exponential execution times. However, in [8] we have developed efficient methods to determine suboptimal traffic light switching schemes for the model (14) – (17).

Remark: The model we have derived is different from the models used by most other researchers due to the fact that we consider red-green cycle lengths that may vary from cycle to cycle. Furthermore, we consider non-saturated intersections, i.e., we allow queue lengths to become equal to 0 during the green cycle. For more information on other models that describe the evolution of the queue lengths at a traffic-light-controlled intersection and on optimal traffic light control the interested reader is referred to [10, 11, 12, 14] and the references given therein. ◇

3 Complementary slackness systems

In [13, 15, 16] Schumacher and van der Schaft consider a class of hybrid systems — the "complementary-slackness systems" — typical examples of which are electrical networks with diodes, or mechanical systems subject to geometric inequality constraints. They develop a method to determine the uniqueness of smooth continuations and to solve the associated mode selection problem for these complementary-slackness systems. When the underlying system is a linear system, then this leads to a Linear Dynamic Complementarity Problem which can also be considered as a special case of the ELCP (see [9]).

4 Conclusions and further research

We have introduced the Extended Linear Complementarity Problem (ELCP) and indicated how it can be used in the modeling and analysis of certain classes of hybrid systems. Topics for further research include: development of efficient algorithms for the special cases of the

ELCP that appear in the analysis of specific classes of hybrid systems, investigation of the use of the ELCP to model and to analyze other classes of hybrid systems, and extension of our model for a traffic-light-controlled intersection to networks of intersections.

Acknowledgments

This research was sponsored by the Concerted Action Project of the Flemish Community, entitled "Model-based Information Processing Systems" (GOA-MIPS), by the Belgian program on interuniversity attraction poles (IUAP P4-02 and IUAP P4-24), and by the ALAPEDES project of the European Community Training and Mobility of Researchers Program.

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