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The extended linear complementarity problem and its applications in the analysis and control of discrete event systems and hybrid systems

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Abstract

In this paper we first present a mathematical programming problem, the Extended Linear Complementarity Problem (ELCP). Next we discuss how the ELCP can be used to solve some basic problems in the system theory for a class of discrete event systems that can be modeled using the max-plus algebra, which has maximization and addition as basic operations. Finally we show that the ELCP also appears in the analysis of certain classes of hybrid systems.

1. The Extended Linear Complementarity Problem

The Extended Linear Complementarity Problem (ELCP) is an extension of the Linear Complementarity Problem, which is one of the fundamental problems in mathematical programming [2]. The ELCP is defined as follows:

> Given $A \in \mathbb{R}^{p \times n}$, $B \in \mathbb{R}^{q \times n}$, $c \in \mathbb{R}^p$, $d \in \mathbb{R}^q$ and m subsets $\phi_1, \phi_2, \ldots, \phi_m$ of $\{1, 2, \ldots, p\}$, find $x \in \mathbb{R}^n$ such that
>
> $$\sum_{j=1}^{m} \prod_{i \in \phi_j} (Ax - c)_i = 0$$
>
> subject to $Ax \geqslant c$ and $Bx = d$, or show that no such x exists.

The ELCP can be considered as a system of linear equations and inequalities ($Ax \geqslant c$, $Bx = d$), where we can distinguish m groups of linear inequalities (one group for each index set ϕ_j) such that in each group at least one inequality should hold with equality (i.e., its residue $(Ax - c)_i$ should be equal to 0). In [4, 5] we have developed an algorithm to compute the complete solution set of an ELCP. This algorithm yields a description of the solution set of an ELCP by vertices, extreme rays and a basis of the linear subspace corresponding to the largest affine subspace of the solution set. In that way it provides a geometrical insight in the solution set of the ELCP and related problems. In [4, 5] we have also shown that the general ELCP is NP-hard.

2. The Extended Linear Complementarity Problem and discrete event systems

2.1. Discrete event systems

The formulation of the ELCP arose from our work in the study of discrete event systems (DESs). Typical examples of DESs are flexible manufacturing systems, subway traffic networks, parallel processing systems, telecommunication networks and logistic systems. The class of the DESs essentially contains man-made systems that consist of a finite number of resources (e.g., machines, communications channels, or processors) that are shared by several users (e.g., product types, information packets, or jobs) all of which contribute to the achievement of some common goal (e.g., the assembly of products, the end-to-end transmission of a set of information packets, or a parallel computation).

One of the most characteristic features of a DES is that its dynamics are event-driven as opposed to time-driven: the behavior of a DES is governed by events rather than by ticks of a clock. An event corresponds to the start or the end of an activity. If we consider a production system then possible events are: the completion of a part on a machine, a machine breakdown, or a buffer becoming empty.

In general the description of the behavior of a DES leads to a model that is nonlinear in conventional algebra. However, there exists a class of DESs for which the model is "linear" when we express it in the max-plus algebra [1, 3], which has maximization and addition as basic operations. DESs that can be described by such a "linear" model are called *max-linear DESs*.

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Loosely speaking we could say that the class of max-linear DESs corresponds to the class of deterministic time-invariant DESs in which only synchronization and no concurrency occurs.

2.2. The max-plus algebra and max-linear DESs

The basic operations of the max-plus algebra are the maximum (represented by $\oplus$) and the addition (represented by $\otimes$):

$$
\begin{aligned}
x \oplus y &= \max(x, y) \\
x \otimes y &= x + y
\end{aligned}
$$

with $x, y \in \mathbb{R}$. The operations $\oplus$ and $\otimes$ are extended to matrices in the usual way. So if $A, B \in \mathbb{R}^{m \times n}$ then we have

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij}$$

for all i, j. If $A \in \mathbb{R}^{m \times p}$ and $B \in \mathbb{R}^{p \times n}$ then

$$(A \otimes B)_{ij} = \bigoplus_{k=1}^{p} a_{ik} \otimes b_{kj}$$

for all i, j.

Let $x, r \in \mathbb{R}$. The rth max-plus-algebraic power of x is denoted by $x^{\otimes^r}$ and corresponds to rx in conventional algebra.

If we use the notation introduced above we get a model of the following form for max-linear DESs:

$$
\begin{aligned}
x(k+1) &= A \otimes x(k) \oplus B \otimes u(k) && (1) \\
y(k) &= C \otimes x(k) \ , && (2)
\end{aligned}
$$

where the vector x represents the state, u the input and y the output of the system. For a manufacturing system $u(k)$ would typically represent the time instants at which raw material is fed to the system for the $(k+1)$st time; $x(k)$ the time instants at which the machines start processing the kth batch of intermediate products; and $y(k)$ the time instants at which the kth batch of finished products leaves the system.

The reason for choosing the symbols $\oplus$ and $\otimes$ to represent respectively maximization and addition is that many properties from conventional linear algebra can be translated to the max-plus algebra simply by replacing $+$ by $\oplus$ and $\times$ by $\otimes$. Note that the model (1) – (2) closely resembles the state space model for linear time-invariant discrete-time systems. This analogy between $\oplus$ and $+$ and between $\otimes$ and $\times$ allows us to translate many concepts, properties and techniques from conventional linear algebra and linear system theory to the max-plus algebra and the system theory for max-linear DESs. However, there are also some major differences that prevent a straightforward translation of properties, concepts and algorithms from conventional linear algebra and linear system theory to max-plus algebra and max-plus-algebraic system theory for DESs.

2.3. The max-plus algebra and the ELCP

Consider the following problem:

> Given $p_1 + p_2$ positive integers $m_1, m_2, \ldots, m_{p_1+p_2}$ and real numbers a_{ki}, b_k and c_{kij} for $k = 1, 2, \ldots, p_1 + p_2$, $i = 1, 2, \ldots, m_k$ and $j = 1, 2, \ldots, n$, find $x \in \mathbb{R}^n$ such that
>
> $$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j^{\otimes^{c_{kij}}} = b_k \qquad (3)$$
>
> for $k = 1, 2, \ldots, p_1$, and
>
> $$\bigoplus_{i=1}^{m_k} a_{ki} \otimes \bigotimes_{j=1}^{n} x_j^{\otimes^{c_{kij}}} \leqslant b_k \qquad (4)$$
>
> for $k = p_1 + 1, p_1 + 2, \ldots, p_1 + p_2$.

We call (3) – (4) a *system of multivariate max-plus-algebraic polynomial equalities and inequalities.* Note that the exponents may be negative or real.

In [4, 8] we have shown that the problem of solving a system of multivariate max-plus-algebraic polynomial equalities and inequalities can be recast as an ELCP. This allows us to solve many problems in the max-plus algebra and in the system theory for max-linear DESs such as computing max-plus-algebraic matrix factorizations, computing max-plus-algebraic singular value decompositions and max-plus-algebraic QR decompositions, constructing matrices with a given max-plus-algebraic characteristic polynomial, performing state space transformations for max-linear DESs, computing minimal state space realizations of max-linear DESs, and so on [4, 6, 7]. Although the analogues of these problems in conventional linear algebra and linear system theory are easy to solve, the max-plus-algebraic problems are not that easy to solve and for almost all of them the ELCP approach is at present the only way to solve the problem.

Although the general ELCP is NP-hard we have recently developed some fast heuristic procedures (with an average execution time that is polynomial in the size of the problem) to solve some of the problems mentioned above (see [9]).

3. The Extended Linear Complementarity Problem and hybrid systems

3.1. Hybrid systems

Hybrid systems arise from the interaction between DESs and continuous-variable systems (these are systems that can be modeled using difference or differential equations). In general we could say that a hybrid system can be in one of several "regimes" whereby in each regime the behavior of the system can be described by a system of difference or differential equations — this corresponds to the continuous-variable

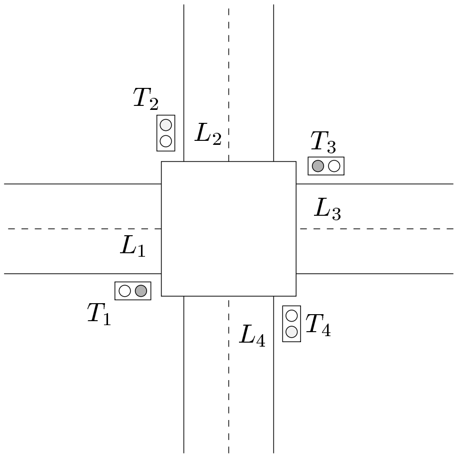

Figure 1: A traffic-light-controlled intersection of two two-way streets.

Period	T_1	T_2	T_3	T_4
$t_0 - t_1$	red	green	red	green
$t_1 - t_2$	green	red	green	red
$t_2 - t_3$	red	green	red	green
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 1: The traffic light switching scheme.

aspect of the hybrid system, — and that the system switches from one regime to another due to the occurrence of events — this corresponds to the DES aspect of the hybrid system.

3.2. Traffic-light-controlled intersections

Consider a single intersection of two two-way streets with controllable traffic lights on each corner (see Figure 1). There are four lanes L_1, L_2, L_3 and L_4, and on each corner of the intersection there are traffic lights (T_1, T_2, T_3 and T_4). For sake of simplicity we assume that the traffic lights can be either green or red. The average arrival rate of cars in lane L_i is λ_i. When the traffic light is green, the average departure rate in lane L_i is μ_i. Let t_0, t_1, t_2, t_3, ... be the time instants at which the traffic lights switch from green to red or vice versa. The traffic light switching scheme is shown in Table 1. Define $\delta_k = t_{k+1} - t_k$. Let $l_i(t)$ be the queue length (i.e., the number of cars waiting) in lane L_i at time instant t.

Let us now write down a model that describes the evolution of the queue lengths (as continuous variables) as a function of time. This will then yield the equations that give the relation between the switching time instants and the queue lengths at the switching time instants.

Consider lane L_1. When the traffic light T_1 is red, there are arrivals at lane L_1 and no departures. As a consequence, we have

$$\frac{dl_1(t)}{dt} = \lambda_1 \tag{5}$$

for $t \in (t_{2k}, t_{2k+1})$ with $k \in \mathbb{N}$, and

$$l_1(t_{2k+1}) = l_1(t_{2k}) + \lambda_1 \delta_{2k}$$

for $k = 0, 1, 2, \ldots$ When the traffic light T_1 is green, there are arrivals and departures at lane L_1. Since the net arrival rate is $\lambda_1 - \mu_1$ and since the queue length $l_1(t)$ cannot be negative, we have:

$$\frac{dl_1(t)}{dt} = \begin{cases} \lambda_1 - \mu_1 & \text{if } l_1(t) > 0 \\ 0 & \text{if } l_1(t) = 0 \end{cases} \tag{6}$$

for $t \in (t_{2k+1}, t_{2k+2})$ with $k \in \mathbb{N}$. So

$$l_1(t_{2k+2}) = \max\bigl(l_1(t_{2k+1}) + (\lambda_1 - \mu_1)\delta_{2k+1}, 0\bigr)$$

for $k = 0, 1, 2, \ldots$ Note that we also have

$$l_1(t_{2k+1}) = \max\bigl(l_1(t_{2k}) + \lambda_1 \delta_{2k}, 0\bigr)$$

for $k = 0, 1, 2, \ldots$ since $l_1(t) \geqslant 0$ for all t.
We can write down similar equations for $l_2(t_k)$, $l_3(t_k)$ and $l_4(t_k)$. So if we define

$$\begin{aligned} x_k &= \begin{bmatrix} l_1(t_k) & l_2(t_k) & l_3(t_k) & l_4(t_k) \end{bmatrix}^T \\ b_1 &= \begin{bmatrix} \lambda_1 & \lambda_2 - \mu_2 & \lambda_3 & \lambda_4 - \mu_4 \end{bmatrix}^T \\ b_2 &= \begin{bmatrix} \lambda_1 - \mu_1 & \lambda_2 & \lambda_3 - \mu_3 & \lambda_4 \end{bmatrix}^T , \end{aligned}$$

then we have

$$x_{2k+1} = \max(x_{2k} + b_1 \delta_{2k}, 0) \tag{7}$$
$$x_{2k+2} = \max(x_{2k+1} + b_2 \delta_{2k+1}, 0) \tag{8}$$

for $k = 0, 1, 2, \ldots$

The traffic-light-controlled intersection can be considered as a hybrid system with time and the queue lengths as state variables. The system can operate in two regimes characterized by differential equations of the form (5) or (6) depending on the value of a discrete control variable that can have the value "red" or "green". The events in this hybrid system are the switchings from "red" to "green" or vice versa.

Now we show that the system (7)–(8) can be reformulated as an ELCP. First consider (7) for an arbitrary index k. This equation can be rewritten as follows:

$$\begin{gathered} x_{2k+1} \geqslant x_{2k} + b_1 \delta_{2k} \\ x_{2k+1} \geqslant 0 \\ (x_{2k+1})_i = (x_{2k} + b_1 \delta_{2k})_i \quad \text{or} \quad (x_{2k+1})_i = 0 \\ \text{for } i = 1, 2, 3, 4\,, \end{gathered}$$

or equivalently

$$\begin{gathered} x_{2k+1} - x_{2k} - b_1 \delta_{2k} \geqslant 0 \\ x_{2k+1} \geqslant 0 \\ (x_{2k+1} - x_{2k} - b_1 \delta_{2k})_i \, (x_{2k+1})_i = 0 \qquad \text{for all } i\,. \end{gathered}$$

Since a sum of nonnegative numbers is equal to 0 if and only if all the numbers are equal to 0, this system of equations is equivalent to:

$$
\begin{aligned}
x_{2k+1} - x_{2k} - b_1 \delta_{2k} &\geqslant 0 \\
x_{2k+1} &\geqslant 0 \\
\sum_{i=1}^{4} \left(x_{2k+1} - x_{2k} - b_1 \delta_{2k} \right)_i \left(x_{2k+1} \right)_i &= 0 \ .
\end{aligned}
$$

We can repeat this reasoning for (8) and for each index k.

So if we consider N switching time instants and if we define

$$
\begin{aligned}
x^* &= \left[\begin{array}{cccc} x_1^T & x_2^T & \cdots & x_N^T \end{array} \right]^T \\
\delta^* &= \left[\begin{array}{cccc} \delta_0 & \delta_1 & \cdots & \delta_{N-1} \end{array} \right]^T \ ,
\end{aligned}
$$

we finally get a description of the form

$$
\begin{aligned}
Ax^* + B\delta^* + c &\geqslant 0 && (9) \\
x^* &\geqslant 0 && (10) \\
(Ax^* + B\delta^* + c)^T x^* &= 0 \ . && (11)
\end{aligned}
$$

It is easy to verify that the system (9) – (11) is a special case of an ELCP.

Now we can compute traffic light switching schemes that minimize objective functions such as average queue length, worst case queue length, average waiting time, and so on. Furthermore, we can impose extra conditions such as minimum and maximum durations for the green and the red time[1], maximum queue lengths[2], and so on. Define

$$
\begin{aligned}
\alpha(N) &= \left\{ 0, 1, \ldots, \left\lfloor \frac{N-1}{2} \right\rfloor \right\} \\
\beta(N) &= \left\{ 0, 1, \ldots, \left\lfloor \frac{N}{2} \right\rfloor - 1 \right\} \ .
\end{aligned}
$$

Using the procedure given above the extra conditions

$$
\begin{aligned}
\delta_{\mathrm{min,r}} \leqslant \delta_{2k} \leqslant \delta_{\mathrm{max,r}} & \qquad \text{for } k \in \alpha(N) \\
\delta_{\mathrm{min,g}} \leqslant \delta_{2k+1} \leqslant \delta_{\mathrm{max,g}} & \qquad \text{for } k \in \beta(N) \\
x_k \leqslant x_{\max} & \qquad \text{for } k = 1, 2, \ldots, N,
\end{aligned}
$$

can be rewritten as a system of inequalities of the form

$$
Ex^* + D\delta^* + f \geqslant 0 \ .
$$

This finally leads to the following problem:

$$
\text{minimize } J
$$

subject to

$$
\begin{aligned}
Ax^* + B\delta^* + c &\geqslant 0 && (12) \\
x^* &\geqslant 0 && (13) \\
Ex^* + D\delta^* + f &\geqslant 0 && (14) \\
(Ax^* + B\delta^* + c)^T x^* &= 0 \ . && (15)
\end{aligned}
$$

Note that the system (12) – (15) is a special case of an ELCP. In order to determine the optimal traffic light switching scheme we have to minimize the objective function J over the solution set of this ELCP. The algorithm of [4, 5] to compute the solution set of a general ELCP requires exponential execution times. However, in [10] we have developed efficient methods to determine suboptimal traffic light switching schemes for the model (12) – (15).

Remark: The model we have derived is different from the models used by most other researchers due to the fact that we consider red-green cycle lengths that may vary from cycle to cycle. Furthermore, we also consider non-saturated intersections, i.e., we allow queue lengths to become equal to 0 during the green cycle. For more information on other models that describe the evolution of the queue lengths at a traffic-light-controlled intersection and on optimal traffic light control the interested reader is referred to [12, 13, 14, 16] and the references given therein. ◇

3.3. Complementary-slackness problems

In [15, 17, 18] Schumacher and van der Schaft consider a class of hybrid systems — the "complementary-slackness systems" — typical examples of which are electrical networks with diodes, or mechanical systems subject to geometric inequality constraints. They develop a method to determine the uniqueness of smooth continuations and to solve the associated mode selection problem for these complementary-slackness systems. When the underlying system is a linear system, then this leads to a Linear Dynamic Complementarity Problem which can also be considered as a special case of the ELCP (see [11]).

4. Conclusions and further research

We have introduced the Extended Linear Complementarity Problem (ELCP) and indicated how it can be used in the modeling and analysis of certain classes of discrete event systems and hybrid systems. Topics for further research include: development of efficient algorithms for the special cases of the ELCP that appear in the analysis of discrete event systems and hybrid systems, investigation of the use of the ELCP to model and to analyze other classes of hybrid systems, and extension of our model for a traffic-light-controlled intersection to networks of intersections.

[1] A green time that is too short is wasteful. If the red time is too long, drivers tend to believe that the signals have broken down.

[2] This could correspond to an upper bound on the available storage space due to the distance to the preceding junction or to the layout of the intersection.

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