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ON THE BOOLEAN MINIMAL REALIZATION PROBLEM IN THE MAX-PLUS ALGEBRA*

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Abstract

The max-plus algebra is one of the frameworks that can be used to model discrete event systems. One of the open problems in the max-plus-algebraic system theory for discrete event systems is the minimal realization problem. In this paper we present some results for a simplified version of the general minimal realization problem: the boolean minimal realization problem, i.e., we consider models in which the entries of the system matrices are either equal to the max-plus-algebraic zero element or to the max-plus-algebraic identity element.

1 Introduction

1.1 Overview

In general, models that describe the behavior of a discrete event system (DES) are nonlinear, but there exists a class of DESs — the max-linear DESs — for which the model becomes "linear" when it is formulated in the max-plus algebra [1, 2, 3].

One of the open problems in the max-plus-algebraic system theory for DESs is the minimal realization problem, which can be stated as follows: given the impulse response of a max-linear DES, determine a model of smallest possible size the impulse response of which coincides with the given impulse response. The minimal realization problem in the max-plus algebra is the central topic of this paper. First we introduce the so-called canonical representation of an impulse response. Then we consider sequences of consecutive max-plus-algebraic matrix power. Finally, we present some results in connection with the boolean minimal realization problem and we derive a lower bound for the minimal system order of a boolean max-linear DES as a function of the length of the transient part of its impulse response.

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1.2 Notation

Let A be an m by n matrix. The jth column of A is denoted by $A_{.,j}$. Let $\alpha \subseteq \{1, 2, \ldots, m\}$ and $\beta \subseteq \{1, 2, \ldots, n\}$. The submatrix of A obtained by removing all rows (columns) of A except for those indexed by α (β) is denoted by $A_{\alpha,.}$ ($A_{.,\beta}$).

1.3 The max-plus algebra

One of the frameworks that can be used to model DESs is the max-plus algebra [1, 3]. The basic operations of the max-plus algebra are the maximum (represented by $\oplus$) and the addition (represented by $\otimes$):

$$x \oplus y = \max(x, y)$$
$$x \otimes y = x + y$$

with $x, y \in \mathbb{R} \cup \{-\infty\}$. The reason for choosing these symbols is that many properties from conventional linear algebra can be translated to the max-plus algebra simply by replacing $+$ by $\oplus$ and $\times$ by $\otimes$. Therefore, we call $\oplus$ the max-plus-algebraic sum and $\otimes$ the max-plus-algebraic product. Define $\varepsilon = -\infty$ and $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$. The structure $\mathbb{R}_{\max} = (\mathbb{R}_\varepsilon, \oplus, \otimes)$ is called the *max-plus algebra*.
The operations $\oplus$ and $\otimes$ are extended to matrices in the usual way. So if $A, B \in \mathbb{R}_\varepsilon^{m \times n}$ then we have

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij}$$

for all i, j. If $A \in \mathbb{R}_\varepsilon^{m \times p}$ and $B \in \mathbb{R}_\varepsilon^{p \times n}$ then

$$(A \otimes B)_{ij} = \bigoplus_{k=1}^{p} a_{ik} \otimes b_{kj}$$

for all i, j. The matrix E_n is the n by n max-plus-algebraic identity matrix: $(E_n)_{ii} = 0$ for all i and $(E_n)_{ij} = \varepsilon$ for all i, j with $i \neq j$. The m by n zero matrix in the max-plus algebra is denoted by $\mathcal{E}_{m \times n}$: $(\mathcal{E}_{m \times n})_{ij} = \varepsilon$ for all i, j. The kth max-plus-algebraic matrix power of a matrix $A \in \mathbb{R}_\varepsilon^{n\times n}$ with $k \in \mathbb{N}$ is defined as follows:

$$A^{\otimes^0} = E_n \text{ and } A^{\otimes^k} = \underbrace{A \otimes A \otimes \ldots \otimes A}_{k \text{ times}} \text{ if } k > 0\,.$$

Define $\mathbb{B} = \{0, \varepsilon\}$. A matrix with entries in $\mathbb{B}$ is called a max-plus-algebraic *boolean* matrix.

1.4 Max-plus-algebraic system theory

In [1, 2, 3] it has been shown that there exists a class of DESs that can be modeled by a max-plus-algebraic model of the following form:

$$x(k+1) = A \otimes x(k) \;\oplus\; B \otimes u(k) \tag{1}$$

$$y(k) = C \otimes x(k)\;\;. \tag{2}$$

The vector x represents the state, u is the input vector and y is the output vector of the system. For a manufacturing system, $u(k)$ would typically represent the time instants at which raw material is fed to the system for the $(k+1)$st time, $x(k)$ the time instants at which the machines start processing the kth batch of intermediate products, and $y(k)$ the time instants at which the kth batch of finished products leaves the system. Note that the model (1) – (2) closely resembles the state space model for linear time-invariant discrete-time systems. Therefore, a DES that can be modeled by (1) – (2) will be called a *max-linear time-invariant DES*.

The number of components of the state vector x will be called the order of the state space model. For an nth order state space model of a max-linear time-invariant DES with m inputs and l outputs we have $A \in \mathbb{R}_\varepsilon^{n\times n}$, $B \in \mathbb{R}_\varepsilon^{n\times m}$ and $C \in \mathbb{R}_\varepsilon^{l\times n}$. The matrices A, B and C are called the system matrices of the model. We shall characterize a model of the form (1) – (2) by the triple (A, B, C) of system matrices. A system with one input and one output is called a single-input single-output (SISO) system. A system with more than one input and more than one output is called a multi-input multi-output (MIMO) system.

Consider a DES that can be described by an nth order state space model of the form (1) – (2) with $A \in \mathbb{R}_\varepsilon^{n\times n}$, $B \in \mathbb{R}_\varepsilon^{n\times m}$ and $C \in \mathbb{R}_\varepsilon^{l\times n}$. If we apply a max-plus-algebraic unit impulse: $e(k) = 0$ if $k = 0$, and $e(k) = \varepsilon$ if $k \neq 0$, to the ith input of the system and if $x(0) = \mathcal{E}_{n\times 1}$, we get $y(k) = C \otimes A^{\otimes^{k-1}} \otimes B_{.,i}$ for $k = 1, 2, \ldots$ as the output of the DES. Note that $y(k)$ corresponds to the ith column of the matrix $G_{k-1} = C \otimes A^{\otimes^{k-1}} \otimes B$ for $k = 1, 2, \ldots$ Therefore, the sequence $\{G_k\}_{k=0}^{\infty}$ is called the *impulse response* of the DES. The G_k's are called the impulse response matrices.

The impulse response of a max-linear time-invariant DES can be characterized as follows:

Theorem 1 *If $\{G_k\}_{k=0}^{\infty}$ is the impulse response of a max-linear time-invariant DES with m inputs and l outputs then*

$$\forall i \in \{1, 2, \ldots, l\}\,, \forall j \in \{1, 2, \ldots, m\}\,, \exists c \in \mathbb{N}_0\,,$$

$$\exists \lambda_1, \lambda_2, \ldots, \lambda_c \in \mathbb{R}_\varepsilon, \exists k_0 \in \mathbb{N} \textit{ such that } \forall k \in \mathbb{N}:$$

$$(G_{k_0+kc+c+s-1})_{ij} = \lambda_s{}^{\otimes^c} \otimes (G_{k_0+kc+s-1})_{ij} \tag{3}$$

for $s = 1, 2, \ldots, c$.

Proof: This is a consequence of, e.g., Corollary 1.1.9 of [4, p. 166] or of Proposition 1.2.2 of [5]. ■

If a sequence $G = \{G_k\}_{k=0}^{\infty}$ exhibits a behavior of the form (3) then we say that the sequence G is *ultimately periodic*. If $G = \{G_k\}_{k=0}^{\infty}$ is an ultimately periodic sequence then the smallest possible c for which (3) holds is called the *period* of G. If $G = \{G_k\}_{k=0}^{\infty}$ is the impulse response of a max-linear time-invariant DES and if the triple (A, B, C) is a state space realization of the DES, then it can be shown that the period of G is a divisor of the cyclicity $c(A)$ of the system matrix A.

Proposition 1 *A sequence $G = \{G_k\}_{k=0}^{\infty}$ with $G_k \in \mathbb{R}_\varepsilon^{l\times m}$ for all k is the impulse response of a max-linear time-invariant DES if and only if it is an ultimately periodic sequence.*

Proof: A proof of this proposition for SISO systems can be found in [1, 4, 5].
For MIMO systems the "only if" part corresponds to Theorem 1. To prove the "if" part for MIMO systems we consider each sequence $\{(G_k)_{ij}\}_{k=0}^{\infty}$ separately. Since such a sequence corresponds to a SISO system, we can apply the first part of this proof and afterwards merge all SISO systems into one large MIMO system (see also [6]). ■

Based on Theorem 1 we now introduce a new concept, the so-called *canonical representation* of the impulse response of a max-linear time-invariant DES or — which is equivalent — of an ultimately periodic sequence. We shall only do this for impulse responses of SISO systems. The extension to MIMO systems is straightforward. The goal of introducing this canonical representation is to get a concise, unique representation of an ultimately periodic sequence. Consider an ultimately periodic sequence of real numbers $g = \{g_k\}_{k=0}^{\infty}$. First we determine the smallest possible $c \in \mathbb{N}_0$ for which (3) holds. The λ_s's are then defined uniquely[1] (up to a circular permutation of the indices). Next, we determine the smallest possible $k_0 \in \mathbb{N}$ such that (3) holds for all $k \geqslant 0$. Now we can

[1] Provided that for a subsequence of the form ε, ε, ε, ..., we take λ_s equal to ε.

uniquely represent the sequence g by the (k_0+2c+1)-tuple $(c, \lambda_1, \lambda_2, \ldots, \lambda_c, g_0, g_1, \ldots, g_{k_0+c-1})$. The subsequence $g_0, g_1, \ldots, g_{k_0-1}$ will be called the *transient part* of g.

Example 2 Consider the sequence

$$g = 0, 0, 0, 0, 0, 1, 0, 3, 0, 5, 0, 7, 0, 10, \ldots$$

This is an ultimately periodic sequence with period $c = 2$, $\lambda_1 = 0$, $\lambda_2 = 2$ and $k_0 = 4$. The transient part of g is the subsequence $g_0, g_1, g_2, g_3 = 0, 0, 0, 0$. □

1.5 Graph theory

In order to define some max-plus-algebraic concepts, we also need some results from graph theory, which will be presented in this section[2].

A directed graph is called strongly connected if for any two different vertices v_i, v_j there exists a path from v_i to v_j. A maximal strongly connected subgraph (m.s.c.s.) $\mathcal{G}_{\text{sub}}$ of a directed graph $\mathcal{G}$ is a strongly connected subgraph that is maximal, i.e., if we add any extra node (and the corresponding arcs) of $\mathcal{G}$ to $\mathcal{G}_{\text{sub}}$ then $\mathcal{G}_{\text{sub}}$ is no longer strongly connected.

The cyclicity of an m.s.c.s. is the greatest common divisor of the lengths of all the circuits of the given m.s.c.s. If an m.s.c.s. or a graph contains no circuits then its cyclicity is equal to 0 by definition. The *cyclicity* $c(\mathcal{G})$ of a graph $\mathcal{G}$ is the least common multiple of the nonzero cyclicities of its m.s.c.s.'s.

Consider $A \in \mathbb{R}_\varepsilon^{n \times n}$. The precedence graph of A, denoted by $\mathcal{G}(A)$, is a weighted directed graph with set of vertices $\{1, 2, \ldots, n\}$ and an arc (j, i) with weight a_{ij} for each $a_{ij} \neq \varepsilon$. The average weight of a path is defined as the sum of the weights of the arcs that compose the path divided by the length of the path. An elementary circuit of $\mathcal{G}(A)$ is called critical if it has maximum average weight among all circuits. The critical graph $\mathcal{G}^{\text{c}}(A)$ consists of those nodes and arcs of $\mathcal{G}(A)$ that belong to a critical circuit of $\mathcal{G}(A)$.

The cyclicity of a matrix $A \in \mathbb{R}_\varepsilon^{n \times n}$ is denoted by $c(A)$ and is equal to the cyclicity of the critical graph of the precedence graph of A. So $c(A) = c(\mathcal{G}^{\text{c}}(A))$. Note that if $A \in \mathbb{B}^{n \times n}$ then every circuit in $\mathcal{G}(A)$ is critical, which implies that $c(A) = c(\mathcal{G}^{\text{c}}(A)) = c(\mathcal{G}(A))$.

Definition 1 (Irreducibility) *We say that a matrix $A \in \mathbb{R}_\varepsilon^{n \times n}$ with $n \geqslant 2$ is irreducible if its precedence graph is strongly connected, i.e., if*

$$(A \oplus A^{\otimes^2} \oplus \cdots \oplus A^{\otimes^{n-1}})_{ij} \neq \varepsilon$$

for all i, j with $i \neq j$.
By definition a 1 by 1 matrix is always irreducible.

Definition 2 (Max-plus-algebraic eigenvalue) *Let $A \in \mathbb{R}_\varepsilon^{n \times n}$. If there exist a number $\lambda \in \mathbb{R}_\varepsilon$ and a vector $v \in \mathbb{R}_\varepsilon^n$ with $v \neq \mathcal{E}_{n \times 1}$ such that $A \otimes v = \lambda \otimes v$ then we say that λ is a max-plus-algebraic eigenvalue of A and that v is a corresponding max-plus-algebraic eigenvector of A.*

It can be shown [1, 2] that every square matrix with entries in $\mathbb{R}_\varepsilon$ has at least one max-plus-algebraic eigenvalue and that irreducible matrices have only one max-plus-algebraic eigenvalue. The max-plus-algebraic eigenvalue has the following graph-theoretic interpretation. Consider $A \in \mathbb{R}_\varepsilon^{n \times n}$. If $\lambda_{\max}$ is the maximal average weight over all elementary circuits of $\mathcal{G}(A)$, then $\lambda_{\max}$ is a max-plus-algebraic eigenvalue of A. For formulas and algorithms to determine max-plus-algebraic eigenvalues and eigenvectors the interested reader is referred to [1] and the references cited therein.

2 Sequences of consecutive max-plus-algebraic matrix powers

Theorem 3 *If $A \in \mathbb{R}_\varepsilon^{n \times n}$ is irreducible, then*

$$\exists k_0 \in \mathbb{N} \text{ such that}$$
$$\forall k \geqslant k_0 \,:\, A^{\otimes^{k+c}} = \lambda^{\otimes^c} \otimes A^{\otimes^k} \quad (4)$$

where λ is the (unique) max-plus-algebraic eigenvalue of A and c is the cyclicity of A.

Proof: See, e.g., [1, 2, 5]. ■

Now we give some extra propositions in connection with the cyclicity of a general matrix and with the integer k_0 that appears in Theorem 3 for a boolean matrix.

For the cyclicity of a general matrix we have the following upper bounds [9, 10][3].

Lemma 1 *If $A \in \mathbb{R}_\varepsilon^{n \times n}$ then $c(A) \leqslant \exp\left(\dfrac{n}{e}\right)$.*

Lemma 2 *If $A \in \mathbb{R}_\varepsilon^{n \times n}$ with $n \geqslant 4$ then*

$$c(A) \leqslant \exp\left(\sqrt{n \log n}\left(1 + \frac{\log\log n - 0.975}{2\log n}\right)\right) . \quad (5)$$

[2] The definitions of basic graph-theoretic concepts like graph, directed graph, path, circuit, ..., can be found in, e.g., [7, 8].

[3] Note that Theorem 2 of [10] erroneously states that (5) holds if $n \geqslant 3$, but in fact this condition only holds if $n > 3$ since $c(A) = 3$ for $A = \begin{bmatrix} \varepsilon & \varepsilon & 0 \\ 0 & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon \end{bmatrix}$ whereas evaluating the right-hand side of (5) yields 2.967.

When $n \geqslant 26$ the bound of Lemma 2 is tighter than the bound of Lemma 1.

For general (possibly not irreducible) *boolean* matrices we can improve the result of Theorem 3 by giving an upper bound for the integer k_0:

Theorem 4 *Let $A \in \mathbb{B}^{n\times n}$ and let c be the cyclicity of A. We have $A^{\otimes^{k+c}} = A^{\otimes^k}$ for all $k \geqslant 2n^2-3n+2$. If A is irreducible then we have $A^{\otimes^{k+c}} = A^{\otimes^k}$ for all $k \geqslant n^2-2n+2$.*

It is easy to verify that the max-plus-algebraic eigenvalue of a max-plus-algebraic boolean matrix is either 0 or ε. That is why λ does not appear in Theorem 4. In particular, if the max-plus-algebraic eigenvalue of $A \in \mathbb{B}^{n\times n}$ is equal to ε then we have $A^{\otimes^k} = \mathcal{E}_{n\times n}$ for all $k \geqslant n$.
The extension of Theorem 4 to general matrices with entries in $\mathbb{R}_\varepsilon$ is a topic of current research. The following example shows that — in contrast to boolean matrices, where the upper bound for the integer k_0 of Theorem 3 only depends on the size of the matrix — for a general matrix A with entries in $\mathbb{R}_\varepsilon$ an upper bound for k_0 also depends on the range and resolution (i.e., on the size of the representation) of the non-ε entries of A.

Example 5 Let $N \in \mathbb{N}$ and consider

$$A(N) = \begin{bmatrix} -1 & -N \\ 0 & 0 \end{bmatrix} .$$

The matrix $A(N)$ is irreducible and has cyclicity 1 and max-plus-algebraic eigenvalue 0. We have

$$\big(A(N)\big)^{\otimes^k} = \begin{bmatrix} \max(-k,-N) & -N \\ 0 & 0 \end{bmatrix}$$

for each $k \in \mathbb{N}_0$. This implies that the smallest integer k_0 for which (4) holds, is given by $k_0 = N$, i.e., k_0 depends on the range of the non-ε entries of $A(N)$.
A similar example can be found in [1, p. 152]. This example shows that in general k_0 depends on the resolution of the non-ε entries of the matrix A. $\square$

3 The minimal state space realization problem

If $G = \{G_k\}_{k=0}^{\infty}$ is an ultimately periodic sequence with $G_k \in \mathbb{R}_\varepsilon^{l\times m}$ for all k, then it follows from Proposition 1 that G is the impulse response of a max-linear time-invariant DES with m inputs and l outputs. Now consider the following problem:

> Given an ultimately periodic sequence $G = \{G_k\}_{k=0}^{\infty}$ with $G_k \in \mathbb{R}_\varepsilon^{l\times m}$ for all k and an integer r, find, if possible, matrices $A \in \mathbb{R}_\varepsilon^{r\times r}$, $B \in \mathbb{R}_\varepsilon^{r\times m}$ and $C \in \mathbb{R}_\varepsilon^{l\times r}$ such that (A,B,C) is a realization of G, i.e., $G_k = C \otimes A^{\otimes^k} \otimes B$ for all $k \in \mathbb{N}$.

This problem is called the *state space realization problem.* If we make r as small as possible, then the problem is called the *minimal* state space realization problem and the resulting value of r is called the minimal system order.

The minimal state space realization problem for max-linear time-invariant DESs has been studied by many authors and for some specific cases the problem can be solved (see, e.g., [11, 12, 13, 14] and the reference given therein). However, at present it is still an open problem whether there exist tractable methods to solve the general minimal state space realization problem.

3.1 The minimal system order

If $G = \{G_k\}_{k=0}^{\infty}$ is a sequence with $G_k \in \mathbb{R}_\varepsilon^{l\times r}$ for all k, then we define the (semi-infinite) block Hankel matrix

$$H(G) \stackrel{\text{def}}{=} \begin{bmatrix} G_0 & G_1 & G_2 & \dots \\ G_1 & G_2 & G_3 & \dots \\ G_2 & G_3 & G_4 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} .$$

The following proposition is a generalization to the MIMO case of Proposition 2.3.1 of [4, p. 175]. It is also an adaptation to max-linear systems of a similar theorem for nonnegative linear systems [15, Theorem 5.4.10].

Proposition 2 *Let $G = \{G\}_{k=0}^{\infty}$ be the impulse response of a max-linear time-invariant DES with m inputs and l outputs. Let n be the smallest integer for which there exist matrices $A \in \mathbb{R}_\varepsilon^{n\times n}$, $U \in \mathbb{R}_\varepsilon^{\infty\times n}$ and $V \in \mathbb{R}_\varepsilon^{n\times\infty}$ such that*

$$\begin{aligned} H(G) &= U \otimes V && (6)\\ U \otimes A &= \overline{U} && (7) \end{aligned}$$

where $\overline{U}$ is the matrix obtained by removing the first l rows of U. Then n is equal to the minimal system order.

Proof: See [16]. ■

It can be shown that if we have a minimal decomposition of the form (6) – (7) of $H(G)$ then the triple $(A, U_{\{1,2,\dots,l\},\cdot}, V_{\cdot,\{1,2,\dots,m\}})$ is a minimal state space realization of the given impulse response.

Remark 1 *Proposition 2 also holds if we replace $H(G)$ in (6) by the matrix that contains the first m columns of $H(G)$ and if V is an n by m matrix.*

Definition 3 (Schein rank [4])
Consider $A \in \mathbb{R}_\varepsilon^{m\times n}$ with $A \neq \mathcal{E}_{m\times n}$. The smallest

integer r for which there exist matrices $U \in \mathbb{R}_\varepsilon^{m\times r}$ and $V \in \mathbb{R}_\varepsilon^{r\times n}$ such that $A = U \otimes V$ is called the max-plus-algebraic Schein rank of A and it is denoted by $\operatorname{rank}_{\oplus,\text{Schein}}(A)$. *By definition we have* $\operatorname{rank}_{\oplus,\text{Schein}}(\mathcal{E}) = 0$.

Proposition 2 implies that the max-plus-algebraic Schein rank of $H(G)$ is a lower bound for the minimal system order. However, the following theorem shows that, unless P = NP, this lower bound cannot be computed in a number of operations that increases polynomially with the size of $H(G)$ — even if $H(G)$ is a boolean matrix. For basic definitions and more information on NP-completeness the reader is referred to [17].

Theorem 6 *Determining the max-plus-algebraic Schein rank of a max-plus-algebraic boolean matrix is an NP-hard problem.*

Proof: See [16]. ■

Other lower and upper bounds for the minimal system order are given in [4, 5]. However, at present, there do not exist efficient (i.e., polynomial time) algorithms to compute a non-trivial lower bound for the minimal system order for a given ultimately periodic sequence.

Since the general minimal realization problem is still an open problem, we consider a simplified version of this problem in the next section.

4 The boolean minimal realization problem

A max-linear time-invariant DES for which all the entries of all the impulse response matrices belong to $\mathbb{B} = \{0, \varepsilon\}$ is called a *boolean max-linear time-invariant DES*. It is easy to verify that if we have an rth order state space realization (A, B, C) of a boolean max-linear time-invariant DES where the entries of A, B, C belong to $\mathbb{R}_\varepsilon$, then there also exists an rth order state space realization $(\tilde{A}, \tilde{B}, \tilde{C})$ such that the entries of $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ belong to $\mathbb{B}$.

4.1 Comparing boolean impulse responses

The following corollaries are direct consequences of Theorem 4.

Corollary 1 *Consider a boolean max-linear time-invariant DES with minimal system order n and impulse response $G = \{G_k\}_{k=0}^{\infty}$. Let c be the period of G. Then we have $G_{k+c} = G_k$ for all $k \geqslant 2n^2 - 3n + 2$.*

Corollary 2 *Let $G = \{G_k\}_{k=0}^{\infty}$ and $F = \{F_k\}_{k=0}^{\infty}$ be impulse responses of boolean max-linear time-invariant DESs with minimal system order less than or equal to n. Let c be the maximum of the period of G and the period of F. If $G_k = F_k$ for $k = 0, 1, \ldots, 2n^2 - 3n + 1 + c$ then $G_k = F_k$ for all $k \in \mathbb{N}$.*

The last corollary gives an explicit upper bound on the number of terms that two impulse responses of boolean max-linear time-invariant DESs should have in common in order to coincide completely.

4.2 A lower bound for the minimal system order

Let $G = \{G_k\}_{k=0}^{\infty}$ be the impulse response of a boolean max-linear time-invariant DES. The max-plus-algebraic Schein of the matrix $H(G)$ is a lower bound for the minimal system order by Proposition 2 From Theorem 6 it follows that, unless P=NP, this lower bound cannot be computed efficiently. However, for a boolean impulse response the following lemma provides an easily computable lower bound for the minimal system order:

Lemma 3 *Consider a boolean max-linear time-invariant DES with minimal system order n and impulse response $G = \{G_k\}_{k=0}^{\infty}$. Let c be the period of G. Let L be the length of the transient part of the impulse response, i.e., L is equal to the smallest integer K for which we have $G_{k+c} = G_k$ for all $k \geqslant K$. If $L \geqslant 2$ then $n \geqslant \dfrac{3 + \sqrt{8L - 7}}{4}$.*

Proof: From Corollary 1 it follows that

$$L \leqslant 2n^2 - 3n + 2 \ . \tag{8}$$

If is easy to verify that this condition holds for every $n \in \mathbb{N}$ if $L = 0$ or if $L = 1$. So from now on we assume that $L \geqslant 2$. The zeros of the function f defined by $f(n) = 2n^2 - 3n + 2 - L$ are $n_1 = \dfrac{3 + \sqrt{8L - 7}}{4}$ and $n_2 = \dfrac{3 - \sqrt{8L - 7}}{4}$. Since $n_2 \leqslant 0$ if $L \geqslant 2$ and since n is always positive, the function f will be nonnegative if $n \geqslant n_1$. Hence, condition (8) will only be satisfied if $n \geqslant n_1$. ■

The results presented in this section and in Section 2 can be used to show that the decision problem that corresponds to the boolean realization problem (i.e., deciding whether or not a boolean realization of a given order exists) is decidable, and that the boolean minimal realization problem can be solved in a number of elementary operations that is bounded from above by an exponential of the square of (any upper bound of) the minimal system order [16].

However, at present it is still an open problem whether or not the boolean minimal realization problem can be solved in polynomial time.

5 Conclusions

In the paper we have considered the minimal state space realization problem for max-linear time-invariant discrete event systems (DESs). We have introduced a canonical representation of the impulse response of a max-linear time-invariant DES and we have presented some propositions in connection with sequences of consecutive max-plus-algebraic matrix powers. Next we have directed our attention to the boolean minimal realization problem. We have derived an upper bound on the number of terms that two impulse response of boolean max-linear time-invariant DESs should have in common in order to coincide completely, and we have given an easily computable lower bound for the minimal system order of a boolean max-linear time-invariant DES.

In our future research we hope to extend some of the results of this paper to general max-linear time-invariant DESs.

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