Katholieke Universiteit Leuven

Departement Elektrotechniek ESAT-SISTA/TR 98-62

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Remco de Vries Bart De Schutter Bart De Moor [1]

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[1]K.U.Leuven, Dept. of Electrical Engineering (ESAT), Research group SISTA, Kardinaal Mercierlaan 94, 3001 Leuven, Belgium, Tel. 32/16/32 17 95 Fax 32/16/32 19 70, WWW: `http://www.esat.kuleuven.ac.be/sista`. E-mail: `remco.devries,bart.deschutter,bart.demoor@esat.kuleuven.ac.be`. This research was sponsored by the Concerted Action Project of the Flemish Community, entitled "Model-based Information Processing Systems" (GOA-MIPS), by the Belgian program on interuniversity attraction poles (IUAP P4-02 and IUAP P4-24), by the ALAPEDES project of the European Community Training and Mobility of Researchers Program, and by the European Commission Human Capital and Mobility Network SIMONET. The scientific responsibility rests with its authors. Bart De Schutter is a senior research assistant with the FWO (Fund for Scientific Research-Flanders) and Bart De Moor is a senior research associate with the FWO.

MINIMAL REALIZATIONS AND STATE SPACE TRANSFORMATIONS IN THE SYMMETRIZED MAX-ALGEBRA

Remco de Vries * **Bart De Schutter** *,[1] **Bart De Moor** *,[2]

* *Katholieke Universiteit Leuven, Dept. ESAT-SISTA,*
Kardinaal Mercierlaan 94, 3001 Heverlee, Belgium,
Email: {*remco.devries,bart.deschutter,bart.demoor*}*@esat.kuleuven.ac.be*

Abstract: State space transformations in the max-algebraic system theory for Discrete Event Systems (DES) are discussed. Some transformations between different realizations of a given impulse response of a DES are suggested and their limitations are illustrated. It is explained why a general result seems hard to prove.
Résumé: Nous discutons certaines transformations d'état dans la théorie des systèmes pour une classe de systèmes à évènements discrets (SED). On propose des transformations entre différentes réalisations de la réponse impulsionelle d'un SED max-algébrique donné. Nous illustrons les limitations de ces transformations et nous expliquons pourquoi un resultat général sera difficile à prouver.

Keywords: discrete event systems, transformation matrices, realization theory

1. INTRODUCTION

A class of Discrete Event Systems (DES), e.g. systems which involve synchronization, can be described by linear models provided that the usual addition is replaced by maximization and multiplication by addition. The resulting algebraic structure is called the max-algebra and a max-algebraic system theory has been developed for this class of DES, see (Baccelli *et al.*, 1992).

One of the problems in the system theory for DES is the minimal realization problem. Given an impulse response of a system, find a state space description of minimal dimension of which the behavior is equal to the given impulse response. The minimal realization problem for DES was introduced in (Olsder, 1986), see also (Olsder and de Vries, 1988). Other results are given in e.g. (Cuninghame-Green, 1991), (Wang *et al.*, 1995) and (De Schutter and De Moor, 1995). Up until now however, no general solution of the minimal realization problem exists.

In conventional system theory a state space transformation always exists between two minimal realizations of a given impulse response. It is investigated whether a similar statement holds true in the max-algebraic system theory of DES. Since in the max-algebra the inverse of a matrix only exists for a small class of matrices, the search for state space transformations will be extended to the symmetrized max-algebra, which is the linear closure of the max-algebra. The symmetrized max-algebra was first introduced in (Max Plus, 1990), see also (Baccelli *et al.*, 1992) and (Gaubert, 1992).

This paper is organized as follows. In section 2 the max-algebra and the symmetrized max-algebra are discussed. Furthermore the linear, in max-algebra sense, models for a class of DES are introduced. In section 3 the minimal realization

[1] Senior Research Assistant with the FWO (Fund for Scientific Research-Flanders)
[2] Senior Research Associate with the FWO

problem is discussed and it is shown why the similarity transformation problem is of interest. This problem will be discussed in section 4. First some results from conventional system theory are recalled. Then possible similarity transformations for max-algebraic systems are introduced. In section 5 some concluding remarks are made. An extended version of this paper appears as (De Vries *et al.*, 1997).

2. MAX-ALGEBRA AND EXTENTIONS

In this section a brief overview of the max-algebra and of the symmetrized max-algebra is given. For an extensive discussion see (Baccelli *et al.*, 1992).

Let $\varepsilon = -\infty$ and $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{\varepsilon\}$. For $a, b \in \mathbb{R}_\varepsilon$ the operations $\oplus$ and $\otimes$ are defined by

$$\begin{aligned} a \oplus b &= \max(a, b), \\ a \otimes b &= a + b. \end{aligned}$$

The set $\mathbb{R}_\varepsilon$ together with the operations $\oplus$ and $\otimes$ will be denoted by $\mathbb{R}_{\max}$ and is called the max-algebra. In $\mathbb{R}_{\max}$, ε is the neutral element for $\oplus$ while the neutral element for $\otimes$ is 0.

The max-algebra operations are extended to matrices as follows. If $A, B \in \mathbb{R}_\varepsilon^{m \times n}$ then $(A \oplus B)_{ij} = a_{ij} \oplus b_{ij}$ for all i, j. For $A \in \mathbb{R}_\varepsilon^{m \times p}$ and $B \in \mathbb{R}_\varepsilon^{p \times n}$, $(A \otimes B)_{ij} = \bigoplus_{k=1}^{p} a_{ik} \otimes b_{kj}$ for all i, j. Let E_n or just E denote the $n \times n$ max-algebraic unit matrix. Its entries are: $E_{ij} = \varepsilon$ for $i \neq j$ and $E_{ii} = 0$ $(i, j = 1, \ldots, n)$.

A problem with $\mathbb{R}_{\max}$ is that there exist no inverse elements w.r.t. $\oplus$. Therefore, in (Max Plus, 1990) $\mathbb{S}_{\max}$, the linear closure of $\mathbb{R}_{\max}$, is introduced. This structure is called the symmetrized max-plus algebra. Here the basic notions of $\mathbb{S}_{\max}$ are given.

The set $\mathbb{S}$ consists of the following three subsets: $\mathbb{S}^\oplus = \mathbb{R}_\varepsilon$, the max-positive numbers; $\mathbb{S}^\ominus = \{\ominus a | a \in \mathbb{R}_\varepsilon\}$, the max-negative numbers; and $\mathbb{S}^\bullet = \{a^\bullet = a \oplus (\ominus a) | a \in \mathbb{R}_\varepsilon\}$, the balanced numbers. The elements in the set $\mathbb{S}^\vee = \mathbb{S}^\oplus \cup \mathbb{S}^\ominus$ will be called signed. For $x, y \in \mathbb{R}_\varepsilon$ let

$$\begin{aligned} x \oplus (\ominus y) &= x \quad \text{if } x > y, \\ x \oplus (\ominus y) &= \ominus y \quad \text{if } x < y, \\ x \oplus (\ominus y) &= x^\bullet \quad \text{if } x = y. \end{aligned}$$

Furthermore, for any $x, y \in \mathbb{S}$ holds

$$\ominus(x \oplus y) = (\ominus x) \oplus (\ominus y), \; x \otimes (\ominus y) = \ominus(x \otimes y),$$
$$(\ominus x) \otimes (\ominus y) = x \otimes y, \; \ominus(\ominus x) = x.$$

These properties allow us to write $a \oplus (\ominus b) = a \ominus b$.

Let $a \in \mathbb{S}$. Define its max-positive part $a^\oplus$ and its max-negative part $a^\ominus$ as follows. If $a \in \mathbb{S}^\oplus$ then $a^\oplus = a$ and $a^\ominus = \varepsilon$. If $a \in \mathbb{S}^\ominus$ then $a^\oplus = \varepsilon$ and $a^\ominus = \ominus a$. If $a \in \mathbb{S}^\bullet$, $\exists b \in \mathbb{R}$ such that $a^\oplus = a^\ominus = b$. Any element $a \in \mathbb{S}$ can then be written as $a = a^\oplus \ominus a^\ominus$.

Since $\ominus$ is not cancellative, $a \ominus a \neq \varepsilon$ for $a \neq \varepsilon$, balances (∇) are used in the symmetrized max-algebra instead of equalities. For $a, b \in \mathbb{S}$ the balance relation is defined as

$$a \nabla b \Leftrightarrow a^\oplus \oplus b^\ominus = a^\ominus \oplus b^\oplus.$$

From this definition it follows that for $a, b, c \in \mathbb{S}$ $a \nabla b \oplus c \Leftrightarrow a \ominus b \nabla c$. This implies that $a \nabla b \Leftrightarrow a \ominus b \nabla \varepsilon$. For $a, b \in \mathbb{S}^\vee$ it follows that $a \nabla b \Leftrightarrow a = b$. These results imply $a^\bullet = a \ominus a \nabla \varepsilon$. A problem with the balance relation is that it is not transitive, e.g. $1 \nabla 1^\bullet$, $1^\bullet \nabla \ominus 1$ but $1 \not\nabla \ominus 1$.

The extension of $\mathbb{S}_{\max}$ to matrices is similar to the extension of $\mathbb{R}_{\max}$ to matrices. In $\mathbb{S}_{\max}$ the determinant of an $n \times n$ matrix A is defined (as usual) as, see (Baccelli *et al.*, 1992),

$$\det(A) = \bigoplus_{\sigma} \operatorname{sgn}(\sigma) \otimes \bigotimes_{i=1}^{n} A_{i\sigma(i)}.$$

where $\operatorname{sgn}(\sigma)$ is the signature of the permutation σ. If σ is even, $\operatorname{sgn}(\sigma) = 0$. If σ is odd, $\operatorname{sgn}(\sigma) = \ominus 0$. Next, define the transpose of the matrix of cofactors $A^\natural$ by $A^\natural_{ij} = \operatorname{cof}_{ji}(A)$, where $\operatorname{cof}_{ji}(A)$ is equal to the determinant of the matrix obtained from A by deleting its j-th row and i-th column. This matrix satisfies $A \otimes A^\natural \nabla \det(A) \otimes E_n$ (Baccelli *et al.*, 1992, Thm. 3.76). The 'inverse' of a matrix A, denoted by $A^\#$, is then defined as $A^\# \otimes \det(A) = A^\natural$, provided that $\det(A) \not\nabla \varepsilon$.

Within the max-algebra structure, a class of Discrete Event Systems can be described by linear equations, see e.g. (Cohen *et al.*, 1985) and (Baccelli *et al.*, 1992) as follows

$$x(k+1) = A \otimes x(k) \oplus B \otimes u(k) \tag{1}$$
$$y(k) = C \otimes x(k). \tag{2}$$

For a production network $x_i(k)$ typically denotes the time instant machine i becomes active for the k-th time, $u(k)$ denotes the time instants outside resources become available and $y(k)$ the time instants at which the k-th production cycle is finished. The entries of A, B and C represent transportation and/or production times. A model of the form (1)-(2) will be characterized by the triple (A, B, C) of system matrices. In this paper only single input single output (SISO) systems are considered.

If a max-algebraic unit impulse, defined by $u(k) = \varepsilon$ for $k \neq 0$ and $u(0) = 0$, is applied to the system (1)-(2) and if $x_0 = \varepsilon$, the output of the system becomes $y(k) = C \otimes A^{k-1} \otimes B$ $(k \geq 1)$. Define

$$g_k = C \otimes A^{k-1} \otimes B \qquad k = 1, 2, \ldots \quad (3)$$

These values are called the Markov parameters and the sequence $\{g_k\}_{k=1}^{\infty}$ is the impulse response of the system.

3. THE MINIMAL REALIZATION PROBLEM

The minimal realization problem is the following. Given a sequence of Markov parameters $\{g_k\}_{k=1}^{\infty}$, find matrices A, B, C such that $C \otimes A^{k-1} \otimes B = g_k$ for $k = 1, 2, \ldots$ and such that the dimension of A is as small as possible.

A starting point is the construction of the semi-infinite Hankel matrix H corresponding with the Markov parameters $\{g_i\}_{i=1}^{\infty}$ and given by

$$H = \begin{pmatrix} g_1 & g_2 & g_3 & \cdots \\ g_2 & g_3 & g_4 & \cdots \\ g_3 & g_4 & g_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Let $H_{.,j}$ denote the jth column of H.

The following theorem is an adaptation of a similar theorem from conventional linear system theory (see e.g. (Sontag, 1990)).

Theorem 1. Given an impulse response $\{g_i\}_{i=1}^{\infty}$ such that for the corresponding Hankel matrix

$$H_{.,i} \oplus a_1 \otimes H_{.,i-1} \oplus \cdots \oplus a_n \otimes H_{.,i-n} \nabla \varepsilon, \quad (4)$$

for $i > n$, $a_i \in \mathbb{S}$ and where n is the smallest integer for which this or another dependency of this form is possible. Then the triple

$$A = \begin{pmatrix} \varepsilon & 0 & \varepsilon & \ldots & \varepsilon \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \varepsilon \\ \varepsilon & \ldots & \ldots & \varepsilon & 0 \\ \ominus a_n & \cdots & & \cdots & \ominus a_1 \end{pmatrix}, \quad B = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix},$$
$$C = \begin{pmatrix} 0 & \varepsilon & \ldots & \varepsilon \end{pmatrix} \quad (5)$$

is a minimal realization of $\{g_i\}_{i=1}^{\infty}$ which satisfies

$$C \otimes A^{i-1} \otimes B \nabla g_i, \qquad i = 1, 2, \ldots \quad (6)$$

Proof: Direct calculation shows that the impulse response of the given system balances the given impulse response. Suppose a lower dimensional realization exists. Then a smaller number of successive columns of the Hankel matrix are linear independent since the resulting A-matrix satisfies its own characteristic equation (see (Olsder and Roos, 1988) and (De Schutter, 1996)) which contradicts the statement of the theorem. □

Remark: Since in general a relation of the form (6) instead of (3) holds, the realization given by (5) will be called a minimal balancing realization. The realization given by (5) will be referred to as the realization in companion form.

The use of Theorem 1 seems to be rather limited since the entries of the matrix A in (5) are not necessarily in $\mathbb{R}_\varepsilon$. The problem now becomes whether from (5) a realization of the given impulse response can be derived such that the entries of the resulting matrices are all in $\mathbb{R}_\varepsilon$. In conventional system theory a state space or similarity transformation is used to transform the realization in companion form to a desired form. Therefore, similarity transformations in the max-algebraic system theory will be discussed.

The following proposition provides another similarity between conventional system theory and the max-algebraic system theory for DES. The result is used in the next section.

Proposition 2. Let the triple (A, B, C) be a minimal balancing realization of order n of a given sequence of Markov parameters $\{g_i\}_{i=1}^{\infty}$. Define matrices O and R as follows

$$O = \begin{pmatrix} C \\ C \otimes A \\ \vdots \\ C \otimes A^{n-1} \end{pmatrix}, \quad R = \begin{pmatrix} B & A \otimes B & \ldots & A^{n-1} \otimes B \end{pmatrix}.$$

Then $\det(O) \not\nabla \varepsilon$ and $\det(R) \not\nabla \varepsilon$.

Proof: see (De Vries *et al.*, 1997). □

In conventional system theory a minimal realization is both reachable and observable. For max-algebraic systems there is no such interpretation.

Proposition 2 is valid for matrices in $\mathbb{S}_{\max}$. For matrices with entries in $\mathbb{R}_\varepsilon$ this proposition only holds if the minimal balancing realization and a minimal realization for which all entries of the system matrices are in $\mathbb{R}_\varepsilon$, are of the same order.

The opposite of Proposition 2 is not true. If a realization is not minimal, this does not necessarily imply that either $\det(O) \nabla \varepsilon$ or $\det(R) \nabla \varepsilon$, see (De Vries *et al.*, 1997).

4. SIMILARITY TRANSFORMATIONS

4.1 *Conventional system theory*

In conventional system theory, see e.g (Sontag, 1990), it is known that when a similarity transformation is applied to the system, represented by the triple (A_1, B_1, C_1), the resulting system (A_2, B_2, C_2) will have the same behavior as the original system. Furthermore, it is known that between any two minimal realizations (A_1, B_1, C_1) and (A_2, B_2, C_2) of a given impulse response a similarity transformation exists. If the similarity transformation is represented by an invertible matrix T then the relation between the two systems is in both cases given by $A_1 = TA_2T^{-1}$, $B_1 = TB_2$, $C_1 = C_2T^{-1}$.

In the max-algebra the inverse of a matrix only exists for matrices which can be written as the product of a diagonal matrix and a permutation matrix. Only for such matrices state space transformations can be defined in a similar way as in the conventional system theory. Therefore, a more general formulation will be given in which no inverse matrices are needed.

4.2 *A transformation in the max-algebra*

In (De Schutter, 1996) two transformations are proposed which make it possible to derive system equivalence for a broader class of triples of system matrices. But it is also shown that such transformations may not exist between two different realizations of the same impulse response. Here a more general transformation is introduced.

Proposition 3. Let the triples (A_1, B_1, C_1) and (A_2, B_2, C_2) be such that: $T \otimes A_2 = A_1 \otimes T$, $T \otimes B_2 = B_1$ and $C_2 = C_1 \otimes T$, for some matrix T. Then both triples are equivalent (i.e. they exhibit the same input/output behavior).

Proof: It follows that $C_2 \otimes A_2^k \otimes B_2 = C_1 \otimes A_1^k \otimes B_1$ by substituting the given relations. □

It can also be shown that two triples (A_1, B_1, C_1) and (A_2, B_2, C_2) are equivalent when a matrix S exists such that $S \otimes A_1 = A_2 \otimes S$, $S \otimes B_1 = B_2$ and $C_1 = C_2 \otimes S$. The matrices S and T do not have to be square. Unfortunately, these transformations may not always exist between two different realizations of the same impulse response.

Example 4. Consider the following triples

$$A_1 = \begin{pmatrix} 6 & 9 \\ 0 & 5 \end{pmatrix}, B_1 = \begin{pmatrix} 0 \\ -4 \end{pmatrix}, C_1^T = \begin{pmatrix} 9 \\ 15 \end{pmatrix}, \quad (7)$$

$$A_2 = \begin{pmatrix} 6 & 10 \\ -1 & 5 \end{pmatrix}, B_2 = \begin{pmatrix} 0 \\ -4 \end{pmatrix}, C_2^T = \begin{pmatrix} 9 \\ 15 \end{pmatrix}. \quad (8)$$

Both triples are minimal realizations of

$$\{g_i\}_{i=1}^{\infty} = 11, 16, 21, 27, 33, 39, 45, 51, \ldots \quad (9)$$

Between (A_1, B_1, C_1) and (A_2, B_2, C_2) a state space transformation T exists such that $T \otimes A_2 = A_1 \otimes T$, $T \otimes B_2 = B_1$ and $C_2 = C_1 \otimes T$. The matrix

$$T = \begin{pmatrix} 0 & 4 \\ -6 & 0 \end{pmatrix} \quad (10)$$

satisfies these relations. It can be shown that no matrix $S \in \mathbb{R}_\varepsilon^{2\times 2}$ exists such that $S \otimes A_1 = A_2 \otimes S$, $S \otimes B_1 = B_2$ and $C_1 = C_2 \otimes S$.

Another minimal realization of (9) is,

$$A_5 = \begin{pmatrix} 6 & 10 \\ 0 & 5 \end{pmatrix}, B_5 = \begin{pmatrix} 0 \\ -4 \end{pmatrix}, C_5^T = \begin{pmatrix} 8 \\ 15 \end{pmatrix}. \quad (11)$$

It turns out that the equations $T \otimes A_5 = A_1 \otimes T$, $T \otimes B_5 = B_1$ and $C_5 = C_1 \otimes T$ with the entries of the matrix T as the unknowns, do not have a solution. Hence, there is no T-transformation between (A_1, B_1, C_1) and (A_5, B_5, C_5). It can also be shown that no S-transformation exists.

Since a state space transformation cannot always be found in the max-algebra, we will extend our search to the symmetrized max-algebra.

4.3 *Balancing similarity transformations*

From conventional system theory it is known (see (Sontag, 1990)) that between two minimal realizations (A_1, B_1, C_1) and (A_2, B_2, C_2) of an impulse response a unique state space transformation exists. The transformation matrix T is given by (for SISO systems) $T = (O_1)^{-1}O_2 = R_1(R_2)^{-1}$ in which O_i and R_i $(i = 1, 2)$ are the observability respectively the controllability matrices of the given systems. In the following similar results are derived for systems in the max-algebra.

Proposition 5. Let the triple (A', B', C') be an n-dimensional realization of a sequence of Markov parameters $\{g_i\}_{i=1}^{\infty}$ such that all entries of the matrices are in $\mathbb{R}_\varepsilon$. Let (A, B, C) be the n-dimensional minimal balancing realization of the same sequence in companion form. Assume that A' satisfies the characteristic polynomial of A. Then a transformation matrix T such that $T \otimes A' \nabla A \otimes T$, $T \otimes B' \nabla B$, $C' \nabla C \otimes T$ is given by

$$T = \begin{pmatrix} C' \\ C' \otimes A' \\ \vdots \\ C' \otimes (A')^{n-1} \end{pmatrix}. \quad (12)$$

Proof: Some computations show that $T \otimes A' \nabla A \otimes T$ where it is used that A' satisfies the characteristic polynomial of A. Furthermore, it follows immediately that $T \otimes B' = B$ and $C \otimes T = C'$. Note that equality holds in these relations. □

In the following conjecture a possible similarity transformation between any two minimal realizations of a given impulse response is given. Only for a part of this conjecture a proof exists yet.

Conjecture 6. Let (A_1, B_1, C_1) and (A_2, B_2, C_2) be two minimal realizations and let (A, B, C) be the minimal balancing realization of a sequence of Markov parameters. If A_1 and A_2 are of the same order as A and satisfy the characteristic polynomial of A, then a state space transformation matrix T exists such that $T \otimes B_2 \,\nabla\, B_1$ and $C_2 \,\nabla\, C_1 \otimes T$. Under certain conditions T also satisfies $T \otimes A_2 \,\nabla\, A_1 \otimes T$. Transformation matrices are given by $T = T_o = O_1^{\#} \otimes O_2$ with $O_i = \begin{pmatrix} C_i \\ C_i \otimes A_i \\ \vdots \\ C_i \otimes A_i^{n-1} \end{pmatrix}$ and $T = T_r = R_1 \otimes (R_2)^{\#}$ with $R_i = (\, B_i \;\; A_i \otimes B_i \;\; \ldots \;\; A_i^{n-1} \otimes B_i \,)$ $(i = 1, 2)$.

Proof: Since the triples are minimal realizations, $(O_1)^{\#}$ and $(R_2)^{\#}$ exist according to Proposition 2.

Since both triples are realizations of the same sequence of Markov parameters it follows that $O_1 \otimes B_1 = O_2 \otimes B_2$ and hence $(O_1)^{\#} \otimes O_1 \otimes B_1 = (O_1)^{\#} \otimes O_2 \otimes B_2$. Since $(O_1)^{\#} \otimes O_1 \,\nabla\, E$, this implies $B_1 \,\nabla\, T_o \otimes B_2$. Similarly it is shown that $C_2 \,\nabla\, C_1 \otimes T_r$.

Let (A, B, C) be the realization according to Theorem 1. From Proposition 5 it follows that $C_1 = C \otimes O_1$ and $C_2 = C \otimes O_2$. Multiplication of the former equality with $(O_1)^{\#} \otimes O_2$ yields $C_1 \otimes (O_1)^{\#} \otimes O_2 = C \otimes O_1 \otimes (O_1)^{\#} \otimes O_2 \,\nabla\, C \otimes O_2 = C_2$. Analogously it is shown that $B_1 \,\nabla\, T_r \otimes B_2$.

The relations $T \otimes A_2 \,\nabla\, A_1 \otimes T$ with $T = T_o$ respectively $T = T_r$ remain to be shown. A major problem in this case is the fact that the balance relation is not necessarily transitive. In (De Vries *et al.*, 1997) some rather technical conditions are derived which should be satisfied. □

Remark: For the triple (A, B, C) given by (5) in Theorem 1 $O = E_n$. So the transformation in Proposition 5 is a special case of the transformation in Conjecture 6.

Example 7. Consider the triples (7) and (11). In Example 4 it was shown that no S- or T-transformation exist between these triples. But

$$T_o = \begin{pmatrix} 0 & 5 \ominus 5 \\ \ominus(-6) & 0 \end{pmatrix}, \tag{13}$$

satisfies $T_o \otimes A_5 \nabla A_1 \otimes T_o$, $B_1 \nabla T_o \otimes B_5$, $C_5 \nabla C_1 \otimes T_o$. It turns out that indeed $T_o = (O_1)^{\#} \otimes O_5$.

Another transformation matrix between (A_1, B_1, C_1) and (A_5, B_5, C_5) is given by the matrix $T_r = R_1 \otimes R_5^{\#} = \begin{pmatrix} 0 & 4^{\bullet} \\ -5^{\bullet} & 0 \end{pmatrix}$. Note that $T_r \,\nabla\, T_o$.

Transformation matrices T_o and T_r between (A_1, B_1, C_1) and (A_2, B_2, C_2) given by (8) are

$$T_o = \begin{pmatrix} 0 & 5^{\bullet} \\ -6^{\bullet} & 0 \end{pmatrix} \text{ and } T_r = \begin{pmatrix} 0 & 4^{\bullet} \\ -5^{\bullet} & 0 \end{pmatrix}.$$

Again $T_o \,\nabla\, T_r$. Note that the matrix T, given by (10), satisfies both $T \,\nabla\, T_o$ and $T \,\nabla\, T_r$. It is conjectured that such a result holds in general.

Example 8. Consider the following sequence of Markov parameters

$$\{g_i\}_{i=1}^{\infty} = 3, 5, 8, 9, 14, 15, 20, 21, \ldots. \tag{14}$$

A relation between those parameters is given by $g_{i+3} \ominus 2 \otimes g_{i+2} \ominus 6 \otimes g_{i+1} \oplus 8 \otimes g_i \,\nabla\, \varepsilon$, $i = 1, 2, \ldots$. There is no relation of the form (4) between any three consecutive Markov parameters. According to Theorem 1 a minimal balancing realization is given by

$$A = \begin{pmatrix} \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & 0 \\ \ominus 8 & 6 & 2 \end{pmatrix},\; B = \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix},\; C^T = \begin{pmatrix} 0 \\ \varepsilon \\ \varepsilon \end{pmatrix}.$$

A realization of $\{g_i\}_{i=1}^{\infty}$ which does have all its entries in $\mathbb{R}_\varepsilon$ is given by the triple

$$A_1 = \begin{pmatrix} 2 & \varepsilon & \varepsilon \\ \varepsilon & 1 & 3 \\ 0 & 3 & \varepsilon \end{pmatrix},\; B_1 = \begin{pmatrix} 1 \\ \varepsilon \\ 0 \end{pmatrix},\; C_1^T = \begin{pmatrix} 2 \\ \varepsilon \\ 2 \end{pmatrix}.$$

According to Proposition 5 a similarity transformation between (A, B, C) and (A_1, B_1, C_1) is

$$T = \begin{pmatrix} C_1 \\ C_1 \otimes A_1 \\ C_1 \otimes A_1^2 \end{pmatrix} = \begin{pmatrix} 2 & \varepsilon & 2 \\ 4 & 5 & \varepsilon \\ 6 & 6 & 8 \end{pmatrix}.$$

With this matrix T it follows that

$$T \otimes A_1 = \begin{pmatrix} 4 & 5 & \varepsilon \\ 6 & 6 & 8 \\ 8 & 11 & 9 \end{pmatrix},\; A \otimes T = \begin{pmatrix} 4 & 5 & \varepsilon \\ 6 & 6 & 8 \\ 10^{\bullet} & 11 & 10^{\bullet} \end{pmatrix}$$

and hence $T \otimes A_1 \,\nabla\, A \otimes T$. Since $A \otimes T$ contains balanced entries $T \otimes A_1 \neq A \otimes T$. Furthermore, $T \otimes B_1 = B$ and $C \otimes T = C_1$.

Another triple which realizes (14) is

$$A_2 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 3 \\ 0 & 3 & 1 \end{pmatrix},\; B_2 = \begin{pmatrix} 0 \\ -3 \\ -1 \end{pmatrix},\; C_2^T = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}.$$

A transformation matrix T_o between (A_1, B_1, C_1) and (A_2, B_2, C_2) is given by $T_o = O_1^{\#} \otimes O_2 = \begin{pmatrix} 1 & -1^{\bullet} & 1^{\bullet} \\ 0^{\bullet} & 1 & 0^{\bullet} \\ -1^{\bullet} & -1^{\bullet} & 1 \end{pmatrix}$.

The matrix $T_r = R_1 \otimes R_2^{\#}$ is another transformation matrix. It also satisfies $T_r \,\nabla\, T_o$.

It is not completely clear under which conditions Conjecture 6 is valid. In the following example it does not hold.

Example 9. Consider the systems

$$A_1 = \begin{pmatrix} 5 & -1 & 0 \\ 3 & -3 & 5 \\ -3 & -3 & -4 \end{pmatrix}, \; B_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \; C_1^T = \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}$$

and

$$A_2 = \begin{pmatrix} \varepsilon & \varepsilon & 5 \\ -2 & \varepsilon & 0 \\ 0 & \varepsilon & 5 \end{pmatrix}, \; B_2 = \begin{pmatrix} 0 \\ -5 \\ -5 \end{pmatrix}, \; C_2^T = \begin{pmatrix} 2 \\ 7 \\ 2 \end{pmatrix}.$$

Both systems are minimal realizations of

$$\{g_k\}_{k=1}^{\infty} = 2, 5, 7, 12, 17, 22, 27, 32, 37, 42, \ldots.$$

With matrix T_o from Conjecture 6 it follows

$$T_o \otimes A_2 = \begin{pmatrix} 4 & \varepsilon & 9 \\ 5 & \varepsilon & 9^{\bullet} \\ 2^{\bullet} & \varepsilon & 7^{\bullet} \end{pmatrix}, \; A_1 \otimes T_o = \begin{pmatrix} 4 & 6^{\bullet} & 9 \\ 5 & \ominus 6 & 7^{\bullet} \\ 1^{\bullet} & 4 & 1^{\bullet} \end{pmatrix}.$$

So, in this case $T_o \otimes A_2 \not\nabla A_1 \otimes T_o$. A reason could be that between any four consecutive Markov parameters several relations are possible. Therefore, it is possible that (A_1, B_1, C_1) and (A_2, B_2, C_2) of which A_1 and A_2 have different characteristic polynomials, as in this example, both satisfy $C_i \otimes A_i^{k-1} \otimes B_i = g_k$ for $i = 1, 2$ and $k = 1, 2, \ldots$. Note that $T \otimes B_2 \nabla B_1$ and $C_2 \nabla C_1 \otimes T$.

5. CONCLUDING REMARKS

In this paper similarity transformations between different realizations of a given impulse response were discussed. In certain cases the existence of a similarity transformation could be proved. The transformations which were found resemble the transformations which exist in the conventional system theory. There is no general result yet. The intransitivity of the balance relation is the major obstacle. It is not obvious how to solve this problem, since the intransitivity of the balance relation follows immediately from its definition.

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