Subsystem Identification
for Nonlinear Model Updating

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Abstract— We consider model updating by adding correction terms to the model equations in the state space form. Two classes of errors, namely model errors in the dynamics equation and model errors in the output equation, are considered. The model errors are assumed to arise from an unknown nonlinear subsystem. First, the states of the true system are estimated using unbiased minimum-variance filters. Next, the state estimates are used to obtain least squares estimates of the unmodeled terms. Finally, these least squares estimates are used to identify the correction subsystem. We discuss model updating for the case in which the unknown subsystem is either a static nonlinear function or a dynamic nonlinear system. A few illustrative examples are also provided.

I. INTRODUCTION

Both first principle (that is, analytical) models and empirical (that is, identified) models are approximate. The required accuracy of a model is application dependent. In this paper, we assume that an *initial model* in state space form is available and that the fidelity of the initial model is insufficient. We update the initial model by adding correction terms to the model equations. This technique is of particular interest when the initial model is a large-scale analytical model or a computer simulation, in which case it is convenient to add correction terms rather than replace the initial model.

In [9] model updating is based on adding a small correction model (delta model) in parallel, cascade or feedback with the initial model. However, such methods have several limitations. Often the delta model is the same dynamic order as the sum of orders of the initial model and the true system. Such a high order correction can be expensive and inefficient, especially when the true system and initial model are high

order systems while the error is relatively low order. Another disadvantage of this method is that the delta model is not physically meaningful.

Alternatively, the approaches of [6] corrects a structural dynamics model of a structure with truncated modes by appending an analytically derived correction model in parallel. In [2] to preserve the structure of the structural model, the parameters of the model are updated directly by using connectivity constraints. Furthermore, in [5] a method is outlined for modifying an existing controller based on knowledge of deviations in the plant. However, the aim of [5] is not to correct the model itself, but rather to correct the controller such that it handles deviations in the plant.

In [1] model updating for state space models, which have an additive error in the dynamics equation, is considered. Specifically, systems of the form

$$x_{k+1} = A_k x_k + H_k d_k + B u_k + w_k, \qquad (I.1)$$
$$y_k = C_k x_k + D_k u_k + v_k. \qquad (I.2)$$

are considered, where A_k, B_k, C_k, D_k, and H_k are known, while d_k is an unknown signal arising from an unknown linear subsystem. This framework can be represented as shown in Figure 1. Since the states x_k, which are inputs to the subsystem, are not measured, a filter is first designed to estimate the states. For an arbitrary unknown signal d_k, the traditional Kalman filter state estimates are biased. Hence, the filter developed in [3], which delivers unbiased estimates of the states in spite of arbitrary unknown inputs, is used to estimate the states of (I.1)-(I.2). Based on these state estimates, the unknown signal d_k is estimated and thus the linear subsystem is identified. Both recursive and batch model updating are considered.
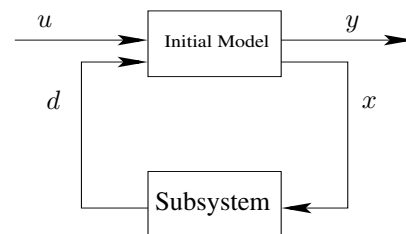

Fig. 1. Block diagram representation of model error in dynamics equation

In the present paper, we extend the approach of [1], by

letting the unknown subsystem be either a static nonlinear function, or a nonlinear dynamical subsystem. Furthermore, we consider additive model errors in the output equation. This class of model errors can be represented in the state space form as

$$
\begin{aligned}
x_{k+1} &= A_k x_k + B u_k + w_k, & \text{(I.3)} \\
y_k &= C_k x_k + H_k d_k + D_k u_k + v_k, & \text{(I.4)}
\end{aligned}
$$

where A_k, B_k, C_k, D_k, and G_k are known, while d_k is an unknown signal arising from an unknown nonlinear subsystem. Figure 2 shows a block diagram representation of the model updating problem for model errors in the output equation (I.3)-(I.4). Again, we let the subsystem be a static nonlinear function or a nonlinear dynamical system.

To update the initial model, we first estimate the states of the true system without knowledge of the signal d_k. To obtain unbiased estimates of the states, we use the *unbiased minimum-variance filter* presented in [3] for (I.1)-(I.2), and the *unbiased minimum-variance filter for output correction (output correction filter)* derived in this paper for (I.3)-(I.4). Once we obtain unbiased estimates of the states of the true system, we then use these state estimates to obtain a least squares estimate of the unknown signal d_k. Finally, the correction subsystem is identified using a basis function expansion, or subspace identification methods [4, 7, 8, 10].
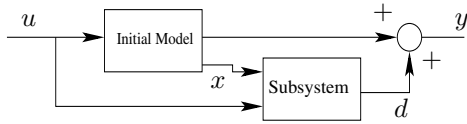

Fig. 2. Block diagram representation of model error in output equation

In the current approach the identified subsystem is driven by the estimates of the states of the true system, and hence is physically meaningful. Additionally, this approach has the advantage over the method in [9] that the order of the correction subsystem is the same as the dynamic order of the model error.

This paper is organized as follows. Sections 2, 3, 4 and 5 deal with model errors in the dynamics equation. In section 2, the problem is presented, while in section 3 the details of the unbiased minimum-variance filter is described briefly. Sections 4 and 5 discuss model updating methods using the unbiased minimum-variance filter. Sections 6, 7 and 8 deal with model errors in the output equation. Section 6 describes the problem, section 7 presents the derivation of the output correction filter, and section 8 discusses model updating using the output correction filter. In section 9 a nonlinear identification method using the model updating technique is described. And finally in sections 10 and 11 illustrative examples are presented, while sections 12 and 13 are conclusions and appendix respectively.

II. DYNAMICS EQUATION SUBSYSTEM IDENTIFICATION

Consider the system

$$
\begin{aligned}
x_{k+1} &= A_k x_k + H_k d_k + B_k u_k + w_k, & \text{(II.1)} \\
y_k &= C_k x_k + D_k u_k + v_k, & \text{(II.2)}
\end{aligned}
$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, $y_k \in \mathbb{R}^l$ and $d_k \in \mathbb{R}^p$. Throughout this paper we assume that measurements of the inputs u_k and outputs y_k are available, while $w_k \in \mathbb{R}^n$ and $v_k \in \mathbb{R}^l$ are unknown white noise sequences with known covariances Q_k and R_k, respectively. We consider model correction for (II.1)-(II.2) when A_k, B_k, C_k and D_k are known to within a similarity transformation, and signal d_k is unknown. For now, H_k is assumed to be known, but the case in which H_k is unknown is discussed later. Also, without loss of generality, we assume $\text{rank}(H_k) = p$. The signal d_k is assumed to be an output of an unknown subsystem that is driven by the system states x_k and the model inputs u_k. We focus on the case in which the unknown subsystem is either a static nonlinear function or a nonlinear time-invariant system.

The model updating technique consists of three key steps. In the first step an unbiased minimum-variance filter is designed based on the known initial model and assuming no knowledge of d_k, to obtain unbiased estimates $\hat{x}_k$ of the states x_k. In the second step, using the estimates $\hat{x}_k$, we obtain a least-squares estimate $\hat{d}_k$ of the unknown signal d_k. In the final step, the correction subsystem, which has inputs $\hat{x}_k$ and u_k and outputs $\hat{d}_k$, is identified. When the unknown subsystem is a static nonlinear function, a basis function expansion of the estimated states $\hat{x}_k$ is used to identify the correction subsystem. Subspace identification is used to identify the correction subsystem when the unknown subsystem is a nonlinear dynamic system. We first present a brief description of the unbiased minimum-variance filter [3] used in the first step.

III. UNBIASED MINIMUM-VARIANCE FILTER

Consider the system (II.1)-(II.2) where A_k, B_k, C_k, D_k, and H_k are known, measurements of u_k and y_k are available, while $d_k \in \mathbb{R}^p$ is unknown. The filter derived in [3] is of the form

$$
\begin{aligned}
\hat{x}_{k+1|k+1} &= \hat{x}_{k+1|k} \\
&\quad + L_{k+1}(y_{k+1} - C_{k+1}\hat{x}_{k+1|k} - D_{k+1}u_{k+1}), & \text{(III.1)} \\
\hat{x}_{k+1|k} &= A_k \hat{x}_{k|k} + B_k u_k, & \text{(III.2)}
\end{aligned}
$$

where $L_{k+1} \in \mathbb{R}^{n \times p}$. The error covariance matrix is defined as

$$
P_{k+1|k+1} \triangleq \mathcal{E}[e_{k+1} e_{k+1}^{\mathrm{T}}], \qquad \text{(III.3)}
$$

where $\mathcal{E}$ is the expected value, and $e_{k+1} \triangleq x_{k+1} - \hat{x}_{k+1|k+1}$ is the estimation error.

Proposition III.1. ([3]) Suppose L_{k+1} satisfies

$$
(I - L_{k+1}C_{k+1})H_k = 0. \qquad \text{(III.4)}
$$

Then the state estimates given by (III.1) are unbiased estimates of the states of (II.1), and the error covariance matrix satisfies

$$
\begin{aligned}
P_{k+1|k+1} = & L_{k+1}R_{k+1}L_{k+1}^{\mathrm{T}} \\
& +(I - L_{k+1}C_{k+1})P_{k+1|k}(I - L_{k+1}C_{k+1})^{\mathrm{T}},
\end{aligned}
\tag{III.5}
$$

where

$$
P_{k+1|k} \triangleq A_k P_{k|k} A_k^{\mathrm{T}} + Q_k.
\tag{III.6}
$$

Furthermore, assuming that R_{k+1} is positive definite, the minimum-variance gain L_{k+1} satisfying (III.4) is

$$
L_{k+1} = H_k \Pi_k + F_{k+1}\tilde{R}_{k+1}^{-1}(I - V_{k+1}\Pi_k),
\tag{III.7}
$$

where $\tilde{R}_{k+1}$, V_{k+1}, F_{k+1} and Π_k are defined as

$$
\begin{aligned}
\tilde{R}_{k+1} &\triangleq C_{k+1}P_{k+1|k}C_{k+1}^{\mathrm{T}} + R_{k+1}, & \text{(III.8)} \\
V_{k+1} &\triangleq C_{k+1}H_k, & \text{(III.9)} \\
F_{k+1} &\triangleq P_{k+1|k}C_{k+1}^{\mathrm{T}}, & \text{(III.10)} \\
\Pi_k &\triangleq (V_{k+1}^{\mathrm{T}}\tilde{R}_{k+1}^{-1}V_{k+1})^{-1}V_{k+1}^{\mathrm{T}}\tilde{R}_{k+1}^{-1}. & \\
& & \text{(III.11)}
\end{aligned}
$$

Since H_k has rank p, (III.4) implies

$$
\mathrm{rank}(C_{k+1}H_k) = p
\tag{III.12}
$$

and thus $l \geq p$. When $l = p$, L_{k+1} is uniquely determined by the constraint (III.4). Furthermore, using (III.7)-(III.11), the covariance update equation (III.5) becomes

$$
\begin{aligned}
P_{k+1|k+1} = & P_{k+1|k} - F_{k+1}\tilde{R}_{k+1}^{-1}F_{k+1}^{\mathrm{T}} + \\
& (H_k - F_{k+1}\tilde{R}_{k+1}^{-1}V_{k+1})(V_{k+1}^{\mathrm{T}}\tilde{R}_{k+1}^{-1}V_{k+1})^{-1} \\
& \times (H_k - F_{k+1}\tilde{R}_{k+1}^{-1}V_{k+1})^{\mathrm{T}}.
\end{aligned}
\tag{III.13}
$$

IV. Static Nonlinear Correction

In this section, we consider the case in which the unknown subsystem is a static nonlinear function, that is, $d_k = h(x_k)$ in (II.1), where $h\colon \mathbb{R}^n \to \mathbb{R}^p$. By treating the unknown nonlinear function $h(x_k)$ as an unknown external input, we use the unbiased minimum-variance filter (III.1)-(III.2) with the gain (III.7) to obtain unbiased estimates of the states of the true system.

Next, we use the estimate $\hat{x}_{k+1|k}$ given by (III.2) to obtain a least squares estimate $\hat{d}_k$ of $d_k = h(x_k)$.

Proposition IV.1. Let $\hat{x}_{k|k}$ be an unbiased estimate of the states x_k of (II.1). Then

$$
\hat{d}_k = H_k^{\dagger}L_{k+1}(y_{k+1} - C_{k+1}\hat{x}_{k+1|k} - D_{k+1}u_{k+1}), \tag{IV.1}
$$

is an unbiased estimate of d_k.

Proof. See appendix.

Next, to approximate $\hat{d}_k$ by a static nonlinear subsystem,

we use a basis function expansion

$$
\sum_{i=1}^{r}\lambda_{ij}f_{ij}(\hat{x}_{k|k}) = \hat{d}_k^j, \quad j = 1,\dots,p,
\tag{IV.2}
$$

where $\hat{d}_k^j$ is the jth component of $\hat{d}_k$, r is the number of basis functions, $f_{ij}\colon \mathbb{R}^n \to \mathbb{R}$ are the basis functions, and λ_{ij} are the coefficients of the basis function expansion obtained by means of standard least squares.

Finally the initial model is updated using the static correction subsystem identified in (IV.2), so that the final corrected model is

$$
x_{k+1} = A_k x_k + H_k \begin{bmatrix} \sum_{i=1}^{r}\lambda_{i1}f_{i1}(x_k) \\ \vdots \\ \sum_{i=1}^{r}\lambda_{ip}f_{ip}(x_k) \end{bmatrix} + B_k u_k,
\tag{IV.3}
$$

$$
y_k = C_k x_k + D_k u_k,
\tag{IV.4}
$$

where the λ_{ij}'s are calculated in (IV.2).

In the derivation of the filter, we have assumed that H_k is known. When H_k is unknown, one choice of H_k that satisfies (III.12) is $H_k = [C_{k+1}^{\mathrm{R}}]_{(:,1:p)}$, where $[C_{k+1}^{\mathrm{R}}]_{(:,1:p)}$ denotes the first p columns of a right inverse of C_{k+1}.

V. Dynamic Nonlinear Correction

When d_k is an output of an unknown nonlinear subsystem, we use subspace identification algorithms to identify the correction subsystem. Consider again the system (II.1)-(II.2), where d_k is generated from the time-invariant nonlinear dynamical subsystem

$$
\begin{aligned}
z_{k+1} &= A_d z_k + B_d f_1(x_k, u_k) + w_{d_k}, & \text{(V.1)} \\
d_k &= C_d z_k + D_d f_2(x_k, u_k) + v_{d_k}. & \text{(V.2)}
\end{aligned}
$$

To apply the model updating technique, the first two steps of the model updating procedure are again repeated to obtain the estimates $\hat{x}_{k|k}$ of the states x_k and the estimates $\hat{d}_k$ of the unknown signal d_k. In the third step, instead of using a basis function expansion, we use a subspace-based Hammerstein identification algorithm [4, 8] to identify the correction subsystem. In the subsystem (V.1) and (V.2) the inputs are the system states x_k and the initial-model inputs u_k, and the outputs are d_k. Thus to identify the correction subsystem using subspace algorithm, the inputs are chosen to be the estimates of the states $\hat{x}_{k|k}$ and the known inputs u_k, while the outputs are chosen to be the estimates $\hat{d}_k$ obtained from (IV.1).

Let $\hat{A}_d$, $\hat{B}_d$, $\hat{C}_d$, $\hat{D}_d$, $\hat{f}_1$ and $\hat{f}_2$ represent the estimates of A_d, B_d, C_d, D_d, f_1 and f_2 respectively, obtained from subspace identification. Then the corrected model is

$$
\begin{aligned}
x_{k+1} &= A_k x_k + H_k \hat{d}_k + B_k u_k & \text{(V.3)} \\
\hat{z}_{k+1} &= \hat{A}_d \hat{z}_k + \hat{B}_d \hat{f}_1(x_k, u_k), & \text{(V.4)} \\
\hat{d}_k &= \hat{C}_d \hat{z}_k + \hat{D}_d \hat{f}_2(x_k, u_k), & \text{(V.5)} \\
y_k &= C_k x_k + D_k u_k. & \text{(V.6)}
\end{aligned}
$$

VI. Output Equation Subsystem Identification

In this section we consider model errors in the output equation, that is the unknown signal d_k appears in the output equation. Consider the system

$$x_{k+1} = A_k x_k + B_k u_k + w_k, \qquad (VI.1)$$

$$y_k = C_k x_k + G_k d_k + D_k u_k + v_k, \qquad (VI.2)$$

where A_k, B_k, C_k, D_k and G_k are known to within a similarity transformation, while $d_k \in \mathbb{R}^q$ is an unknown signal arising from an unknown nonlinear subsystem. Again, without loss of generality we assume rank$(G_k) = q$. For this situation, since the unbiased minimum-variance filter cannot be used, we derive an *unbiased minimum-variance filter for output correction (output correction filter)* that provides unbiased minimum-variance estimates of the system states when there are arbitrary unknown terms in the output equation.

VII. Unbiased Minimum-variance Filter for Output Correction

Consider again the system (VI.1)-(VI.2), where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^l$ and $d \in \mathbb{R}^q$. We consider a filter of the form

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k}$$
$$+ L_{k+1}(y_{k+1} - C_{k+1}\hat{x}_{k+1|k} - D_{k+1}u_{k+1}),$$
$$(VII.1)$$

$$\hat{x}_{k+1|k} = A_k \hat{x}_{k|k} + B_k u_k. \qquad (VII.2)$$

The state estimation error is

$$e_k = x_{k+1} - \hat{x}_{k+1|k+1}, \qquad (VII.3)$$

and the error covariance matrix is defined as

$$P_{k+1|k+1} = \mathcal{E}\left[e_{k+1}e_{k+1}^{\mathrm{T}}\right]. \qquad (VII.4)$$

The filter is unbiased if and only if

$$\mathcal{E}[x_{k+1} - \hat{x}_{k+1|k+1}] = 0, \qquad (VII.5)$$

or

$$\mathcal{E}[A_k e_k + w_k - L_{k+1}(C_{k+1}A_k e_k + C_{k+1}w_k$$
$$+ v_{k+1} + G_{k+1}d_{k+1})] = 0. \qquad (VII.6)$$

Condition (VII.6) hold for all signal d_k only if

$$L_{k+1}G_{k+1} = 0, \qquad (VII.7)$$

which implies $l > q$.

Lemma VII.1. *If (VII.7) is satisfied, the error covariance $P_{k+1|k+1}$ is given by*

$$P_{k+1|k+1} = L_{k+1}\tilde{R}_{k+1}L_{k+1}^{\mathrm{T}} - F_{k+1}L_{k+1}^{\mathrm{T}}$$
$$- L_{k+1}C_{k+1}P_{k+1|k} + P_{k+1|k}, \qquad (VII.8)$$

where $P_{k+1|k}$, $\tilde{R}_{k+1}$ and F_{k+1} are defined as

$$P_{k+1|k} \triangleq A_k P_{k|k} A_k^{\mathrm{T}} + Q_k, \qquad (VII.9)$$

$$\tilde{R}_{k+1} \triangleq C_{k+1} P_{k+1|k} C_{k+1}^{\mathrm{T}} + R_{k+1}, \qquad (VII.10)$$

$$F_{k+1} \triangleq P_{k+1|k} C_{k+1}^{\mathrm{T}}. \qquad (VII.11)$$

Proof. See appendix.

Next, we define the cost function J as the trace of the error covariance matrix

$$J(L_{k+1}) = \mathrm{tr}\mathcal{E}[e_{k+1}e_{k+1}^{\mathrm{T}}]$$
$$= \mathrm{tr}P_{k+1|k+1}. \qquad (VII.12)$$

Proposition VII.1. *The gain L_{k+1} in the filter (VII.1), which minimizes the cost function (VII.12) and satisfies the constraint (VII.7), is given by*

$$L_{k+1} = \Big[F_{k+1} - F_{k+1}\tilde{R}_{k+1}^{-1}G_{k+1}$$
$$\times (G_{k+1}^{\mathrm{T}}\tilde{R}_{k+1}^{-1}G_{k+1})^{-1}G_{k+1}^{\mathrm{T}}\Big]\tilde{R}_{k+1}^{-1}. \qquad (VII.13)$$

Proof. The cost function J can be written as

$$J(L_{k+1}) = \mathrm{tr}P_{k+1|k+1}$$
$$= \mathrm{tr}[L_{k+1}\tilde{R}_{k+1}L_{k+1}^{\mathrm{T}} - F_{k+1}L_{k+1}^{\mathrm{T}}$$
$$- L_{k+1}C_{k+1}P_{k+1|k} + P_{k+1|k}]. \qquad (VII.14)$$

Thus the optimization problem is to minimize the cost function (VII.14) subject to the constraint (VII.7). If $\Lambda_k \in \mathbb{R}^{n \times q}$ is the matrix of lagrange multipliers, the Lagrangian is then

$$\mathcal{L}(L_{k+1}) \triangleq J(L_{k+1}) + 2\mathrm{tr}[L_{k+1}G_{k+1}\Lambda_{k+1}^{\mathrm{T}}]. \qquad (VII.15)$$

Differentiating with respect to L_{k+1} and setting it to zero, we get

$$2\tilde{R}_{k+1}L_{k+1}^{\mathrm{T}} - 2C_{k+1}P_{k+1|k} + 2G_{k+1}\Lambda_{k+1}^{\mathrm{T}} = 0, \qquad (VII.16)$$

while differentiating with respect to Λ_{k+1} and setting it to zero yields the constraint (VII.7). Combining (VII.16) and (VII.7) in matrix form we get

$$\begin{bmatrix} \tilde{R}_{k+1} & G_{k+1} \\ G_{k+1}^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} L_{k+1}^{\mathrm{T}} \\ \Lambda_{k+1}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} C_{k+1}P_{k+1|k} \\ 0 \end{bmatrix}. \qquad (VII.17)$$

For a unique solution to exist we need left hand side matrix to be full rank. Further, assuming R_k to be positive definite, we can write (VII.16) as

$$L_{k+1}^{\mathrm{T}} = \tilde{R}_{k+1}^{-1}(C_{k+1}P_{k+1|k} - G_{k+1}\Lambda_{k+1}^{\mathrm{T}}). \qquad (VII.18)$$

Using (VII.18) in (VII.7), we get the following expression for the matrix of Lagrange multipliers

$$\Lambda_{k+1} = F_{k+1}\tilde{R}_{k+1}^{-1}G_{k+1}\left(G_{k+1}^{\mathrm{T}}\tilde{R}_{k_1}^{-1}G_{k+1}\right)^{-1}. \qquad (VII.19)$$

Substituting the above expression for Λ_{k+1} back in (VII.18),

the optimal solution for L_{k+1} is

$$L_{k+1} = \left[F_{k+1} - F_{k+1}\tilde{R}_{k+1}^{-1}G_{k+1} \right.$$
$$\left. \times (G_{k+1}^{T}\tilde{R}_{k+1}^{-1}G_{k+1})^{-1}G_{k+1}^{T} \right] \tilde{R}_{k+1}^{-1}. \quad \text{(VII.20)}$$

$\square$

Furthermore, using the expression (VII.13), the covariance update equation (VII.8) becomes

$$P_{k+1|k+1} = P_{k+1|k} - F_{k+1}\tilde{R}_{k+1}^{-1}[I - G_{k+1}$$
$$(G_{k+1}^{T}\tilde{R}_{k+1}^{-1}G_{k+1})^{-1}G_{k+1}^{T}\tilde{R}_{k+1}^{-1}]F_{k+1}^{T}$$
$$\text{(VII.21)}$$

VIII. Model Updating Using Output Correction Filter

Now, the output correction filter derived in the previous section is directly applicable for model updating for the system (VI.1) and (VI.2). In the first step, again the states of the system (VI.1) and (VI.2) are estimated using the output correction filter (VII.1)-(VII.2) and (VII.13). Once the estimates $\hat{x}_{k|k}$ of the states x_k are obtained, these state estimates are used to estimate the unknown signal d_k as

$$\hat{d}_k = G_k^{\dagger}(y_k - C_k\hat{x}_{k|k} - D_k u_k). \quad \text{(VIII.1)}$$

It is straightforward to check that the above estimate $\hat{d}_k$ is an unbiased estimate of d_k by taking the expected value of the both sides. And finally, the correction subsystem is identified using a basis function expansion or subspace identification. The final corrected model for the case in which the correction subsystem is a nonlinear dynamical system is

$$x_{k+1} = A_k x_k + B_k u_k, \quad \text{(VIII.2)}$$
$$z_{k+1} = \hat{A}_z z_k + \hat{B}_d f_1(x_k, u_k), \quad \text{(VIII.3)}$$
$$\hat{d}_k = \hat{C}_z z_k + \hat{D}_d f_2(x_k, u_k), \quad \text{(VIII.4)}$$
$$y_k = C_k x_k + D_k u_k + G_k \hat{d}_k. \quad \text{(VIII.5)}$$

IX. Nonlinear Identification

When no initial model is available, we perform linear subspace identification to obtain an initial model. Further based on this initial model, we design a unbiased minimum-variance filter or a output correction filter by assuming a suitable H_k or G_k respectively. And the rest of the procedure to estimate d_k and then identify a correction subsystem remains the same. Although this procedure is not supported by theoretically rigorous results, numerical examples suggest that this technique can be effective. The class of systems that can be potentially identified by this method include systems with nonlinearities in states, and thus can be useful.

X. Example: Model Updating for Van der Pol Oscillator

We consider a discrete-time model of the Van der Pol oscillator with an external driver

$$\begin{bmatrix} x_{1,k+1} \\ x_{2,k+1} \end{bmatrix} = \begin{bmatrix} x_{1,k} + T_s x_{2,k} \\ x_{2,k} + T_s[(1 - x_{1,k}^2)x_{2,k} - x_{1,k} + u_k] \end{bmatrix},$$
$$\text{(X.1)}$$

where T_s is the sampling interval. We assume that the linear part of the dynamics is known perfectly, that is, the initial model is the linear part of the equations while $d_k = T_s x_{1,k}^2 x_{2,k}$. Measurements of the state x_2 are available, thus the output matrix is $C_k = \begin{bmatrix} 0 & 1 \end{bmatrix}$. Since the nonlinear term appears only in the equation of the second state we take $H_k = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The rank condition (III.12) is satisfied, hence we use the unbiased minimum-variance filter to estimate the states of the system. Figure 3 shows a plot of the actual states, the states of the initial model, and the estimates of the states from the unbiased minimum-variance filter. It is seen from the plot that the state estimates from the unbiased minimum-variance filter based on the initial model approximates the actual states closely. Once the estimates of the states are obtained we then obtain a least squares estimate $\hat{d}_k$ of the unknown signal d_k by using (IV.1). Then we use a basis function expansion of the state estimates to approximate $\hat{d}_k$ by a nonlinear function of the states as shown in Figure 4. As seen from Figure 5 the corrected model output closely approximates the actual output. And finally, Figure 6 shows the corrected-model output for an independent set of inputs. The corrected-model output shows the correct limit-cycle type behavior.

XI. Conclusions

In this paper, we discussed model updating by adding correction terms to the model equations. Two classes of errors namely model errors in the dynamics equation and model errors in the output equation were discussed. First, the states of the true system were estimated using unbiased minimum-variance filters. Next, the state estimates were used to obtain least squares estimates of the unmodeled terms. Finally, these least squares estimates were used to identify the correction subsystem. The case in which unknown subsystem was either a static nonlinear function or a dynamic nonlinear system was discussed. Finally a nonlinear identification method based on the model updating technique was discussed. A few illustrative examples were also provided.

XII. Appendix

Proof of Proposition IV.1 : Since $\hat{x}_{k|k}$ is an unbiased estimate of x_k, the state estimation error e_k satisfies

$$\mathcal{E}[e_k] = 0. \quad \text{(XII.1)}$$

Since $l \geq p$, we can define $\hat{d}_k$ as

$$\hat{d}_k = H_k^{\dagger}L_{k+1}(y_{k+1} - C_{k+1}\hat{x}_{k+1|k} - D_{k+1}u_{k+1}), \quad \text{(XII.2)}$$

where $\dagger$ denotes the Moore-Penrose generalized inverse. Next, we use (III.1) and (XII.2) to get

$$\hat{d}_k = H_k^{\dagger}(\hat{x}_{k+1|k+1} - \hat{x}_{k+1|k})$$
$$= H_k^{\dagger}(x_{k+1} + e_{k+1} - A_k\hat{x}_{k|k} - B_k u_k)$$
$$= H_k^{\dagger}(x_{k+1} - A_k x_k - B_k u_k + e_{k+1} - A_k e_k)$$
$$= H_k^{\dagger}(H_k d_k + w_k + e_{k+1} - A_k e_k). \quad \text{(XII.3)}$$

Further, taking expected value on both sides of (XII.3), yields

$$\mathcal{E}[\hat{d}_k] = \mathcal{E}[H_k^\dagger(H_k d_k + w_k + e_{k+1} - A_k e_k)],$$
(XII.4)

Finally, using (XII.1) and the fact that w_k is zero-mean, we get

$$\mathcal{E}[\hat{d}_k] = H_k^\dagger H_k \mathcal{E}[d_k] = \mathcal{E}[d_k].$$
(XII.5)

□

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Fig. 4. Model Updating for Van der Pol Oscillator Example. Plot showing the actual unmodeled term d_k and it's approximation by a basis function expansion.

Fig. 5. Model Updating for Van der Pol Oscillator Example. Plot showing the output of the true system, the output of the initial model and the corrected-model output.
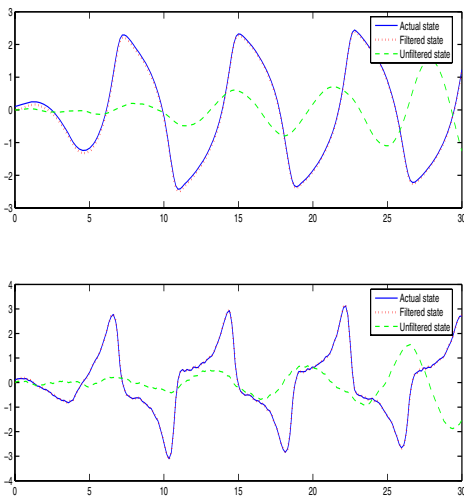

Fig. 3. Model Updating for Van der Pol Oscillator Example. Plot showing the states of the true system, the states of the initial model and the estimates of the states from the unbiased minimum-variance filter.
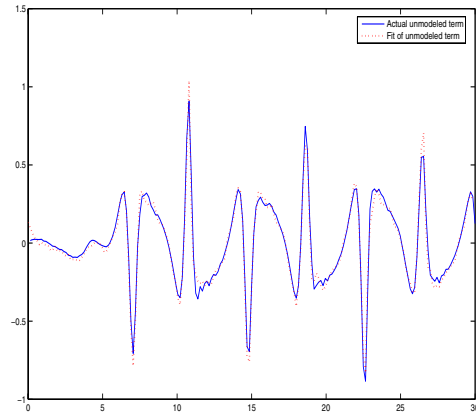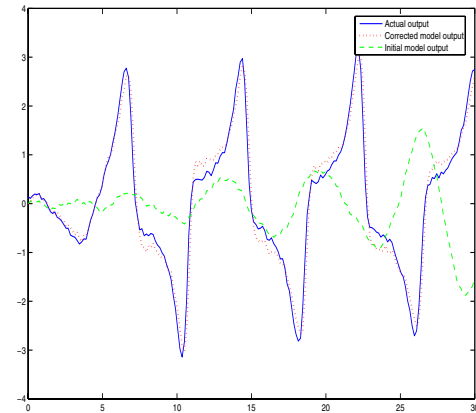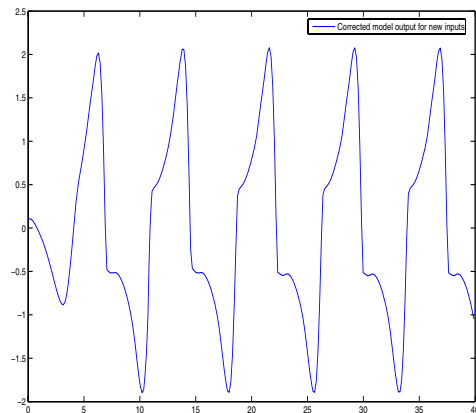

Fig. 6. Model Updating for Van der Pol Oscillator Example. Corrected-model output for an independent set of inputs. The output shows the correct limit cycle type response.