JOINT STATE AND BOUNDARY CONDITION ESTIMATION IN LINEAR DATA ASSIMILATION USING BASIS FUNCTION EXPANSION

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ABSTRACT
This paper addresses the problem of joint state and boundary condition estimation in linear data assimilation. By approximating the equations of an optimal estimator for linear discrete-time state space systems with unknown inputs, an efficient recursive filtering technique is developed. Unlike existing boundary condition estimation techniques, the filter makes no assumption about the initial value or the time evolution of the boundary conditions. However, the derivation is based on the assumption that measurements at the boundary are available. Furthermore, it is assumed that the spatial form of the boundary condition can be expanded as a linear combination of a limited number of predefined basis vectors. A simulation example on a linear heat conduction model shows the effectiveness of the method.

KEY WORDS
Boundary condition estimation, Kalman filtering, unknown input estimation, data assimilation.

1 Introduction

The term "data assimilation" refers to methodologies that estimate the state of a large-scale physical system from incomplete and inaccurate measurements [1, 2, 3, 4]. The Kalman filter, well known from linear control theory, is the optimal algorithm for assimilating measurements into a linear model. This technique recursively updates the state estimate when new measurements become available. However, for large-scale systems, the task of state estimation is very challenging. The required spatial resolution leads to large-scale models, obtained by discretizing partial differential equations (PDEs), with a huge number of state variables, from 10^4 to 10^7 [1, 2]. As a consequence, the number of computations and the required storage for the Kalman filter become prohibitive. Therefore, several suboptimal filtering schemes for use in realistic data assimilation applications have been developed [1, 3]. Extensions of these techniques to joint state and parameter estimation have been proposed in [2, 4]. However, the applicability of these methods is limited by the assumption that a model for the time evolution of the unknown parameters is available.

The estimation of boundary conditions has been intensively studied in inverse heat conduction problems. In [5, 6] it is assumed that the initial state and the functional form in space and time of the boundary condition are known. The unknown parameters in the functional form are then estimated using least-squares estimation. An extension to simultaneous boundary condition and initial state estimation, can be found in [7]. Approaches using the Kalman filter are developed in [8, 9]. Finally, in [10] an efficient algorithm for estimating the boundary condition in large-scale heat conduction problems is developed. The algorithm is based on the Kalman filter and uses model reduction techniques to reduce the computational burden of the Kalman filter. However, the applicability of the previous methods is limited by the assumption that a model for the time evolution of the unknown boundary conditions is available.

This paper extends existing techniques by estimating unknown arbitrary boundary conditions without making an assumption about their time evolution. More precisely, we consider the problem of jointly estimating the system state and unknown arbitrary boundary conditions in large-scale linear models. Instead of reducing the dimension of the model, we use suboptimal filtering techniques to reduce the computational burden. Our data assimilation technique is based on an optimal filter for linear discrete-time state space systems with unknown inputs. In contrast to existing techniques, it makes no assumption about the initial value or the time evolution of the boundary condition. The boundary condition may be strongly time-varying. However, it is assumed that the spatial form of the boundary condition can be expanded as a linear combination of a limited number of basis vectors. Furthermore, it is assumed that measurements at the boundary are available.

This paper is outlined as follows. In section 2, we formulate the problem in more detail. Next, in section 3, we establish a connection between boundary condition estimation and unknown input estimation and we summarize the optimal unknown input filter developed in [11, 12]. In section 4, we extend this filter to large-scale systems by approximating the optimal filter equations. Finally, in section 5, we consider an inverse heat conduction problem.

2 Problem formulation

Consider a set of linear PDEs with partially unknown boundary conditions. By spatial discretization over n points, the PDE is transformed into a state space model of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + G(t)d(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ represents the state vector, $u(t) \in \mathbb{R}^{m_u}$ represents the known boundary conditions and inputs and $d(t) \in \mathbb{R}^{m_d}$ represents the unknown boundary conditions and inputs.

For simulation on a computer, the continuous-time model (1) is usually discretized in time, resulting in

$$x_{k+1} = A_k x_k + B_k u_k + G_k d_k + H_k w_k, \tag{2}$$

where $x_k \simeq x(kT_s), u_k \simeq u(kT_s), d_k \simeq d(kT_s)$ with T_s the sampling time and where the process noise $w_k \in \mathbb{R}^{m_w}$ has been introduced to represent stochastic uncertainties in the state equation, e.g. due to the discretization. The process noise is assumed to be a zero-mean white random signal with covariance matrix $Q_k = \mathbb{E}[w_k w_k^{\mathsf{T}}]$.

Let p linear combination of the state vector be measured, then the model (2) can be extended to

$$x_{k+1} = A_k x_k + B_k u_k + G_k d_k + H_k w_k, \tag{3}$$

$$y_k = C_k x_k + v_k, \tag{4}$$

where $y_k \in \mathbb{R}^p$ represents the vector of measurements. The measurement noise v_k has been introduced to represent stochastic errors in the measurement process. The measurement noise is assumed to be a zero-mean white random signal with covariance matrix $R_k = \mathbb{E}[v_k v_k^{\mathsf{T}}]$, uncorrelated with w_k.

We assume that all external inputs are known, such that d_k represents only unknown boundary conditions. Under this assumption, the first objective of this paper is to derive a recursive filter which jointly estimates the system state x_k and the vector of unknown boundary conditions d_k when new measurements become available. In contrast to existing methods, we assume that no prior knowledge about the unknown boundary condition is available. It can be any type of signal and may for example be strongly time-varying.

In data assimilation applications, the PDEs are usually discretized over a huge spatial grid, resulting in a state vector of very large dimension n. Consequently, the standard filtering techniques can not be applied and approximations have to be made. Therefore, the second objective is to extend the joint state and boundary condition estimator to large-scale data assimilation problems where $n \gg m, p$ by approximation the optimal filter equations.

The first objective is addressed in Section 3, the second objective in Section 4.

3 Relation to unknown input filtering

Note that d_k enters the system (3)-(4) like an unknown input. The problem of joint state and boundary condition estimation is thus equivalent to joint input and state estimation. An optimal filter for systems with unknown inputs which assumes that no prior knowledge about the unknown input is available, was first developed in [11]. The derivation in [11] is however limited to optimal state estimation. An extension to joint optimal input and state estimation can be found in [12]. In this section, we summarize the filter developed in [11, 12].

The filter equations can be written in three steps: 1) the time update of the state estimate, 2) the estimation of the unknown boundary condition and 3) the measurement update of the state estimate.

3.1 Time update

Let the optimal unbiased estimate of x_{k-1} given measurements up to time $k-1$ be given by $\hat{x}_{k-1|k-1}$, and let $P_{k-1|k-1}$ denote its covariance matrix, then the time update is given by

$$\hat{x}_{k|k-1} = A_{k-1}\hat{x}_{k-1|k-1} + B_{k-1}u_{k-1}, \tag{5}$$

$$P_{k|k-1} = A_{k-1}P_{k-1|k-1}A_{k-1}^{\mathsf{T}} + H_{k-1}Q_{k-1}H_{k-1}^{\mathsf{T}}. \tag{6}$$

Note that the unknown boundary condition d_{k-1} can not be estimated using measurements up to time $k-1$. Therefore, the state estimate $\hat{x}_{k|k-1}$ is biased. Furthermore, note that $P_{k|k-1}$ is not the covariance matrix of $\hat{x}_{k|k-1}$.

3.2 Estimation of unknown boundary condition

Once the measurement y_k is available, the unknown boundary condition d_{k-1} can be estimated. Defining the innovation $\tilde{y}_k = y_k - C_k\hat{x}_{k|k-1}$, it follows from (3-4) that

$$\tilde{y}_k = C_k G_{k-1} d_{k-1} + e_k, \tag{7}$$

where e_k is given by

$$e_k = C_k A_{k-1}(x_{k-1} - \hat{x}_{k-1|k-1}) + C_k w_{k-1} + v_k. \tag{8}$$

Let $\hat{x}_{k-1|k-1}$ be unbiased, then it follows from (8) that $\mathbb{E}[e_k] = 0$. Consequently, it follows that the minimum-variance unbiased estimate of d_{k-1} based on $\tilde{y}_k$ is obtained from (7) by weighted least-squares estimation with weighting matrix equal to the inverse of

$$\tilde{R}_k = \mathbb{E}[e_k e_k^{\mathsf{T}}], \tag{9}$$

$$= C_k P_{k|k-1} C_k^{\mathsf{T}} + R_k. \tag{10}$$

The optimal estimate of d_{k-1} is thus given by

$$\hat{d}_{k-1} = (F_k^{\mathsf{T}}\tilde{R}_k^{-1}F_k)^{-1}F_k^{\mathsf{T}}\tilde{R}_k^{-1}\tilde{y}_k, \tag{11}$$

where $F_k = C_k G_{k-1}$. The variance of $\hat{d}_{k-1}$ is given by

$$D_{k-1} = (F_k^{\mathsf{T}}\tilde{R}_k^{-1}F_k)^{-1}. \tag{12}$$

Note that the inverses in (11) and (12) exist under the assumption that

$$\text{rank } C_k G_{k-1} = \text{rank } G_{k-1} = m_d. \tag{13}$$

Note that (13) implies $n \geq m_d$ and $p \geq m_d$. For (13) to hold, (linear combinations of) measurements of all boundary states must be available.

3.3 Measurement update

As shown in [12], the update of the state estimate $\hat{x}_{k|k-1}$ with the measurement y_k resulting in the minimum-variance unbiased state estimate $\hat{x}_{k|k}$, can be written as

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \tilde{y}_k + (I - K_k C_k)\bar{G}_{k-1}\hat{d}_{k-1}, \tag{14}$$

where the expression for the gain matrix K_k equals the expression for the Kalman gain,

$$K_k = P_{k|k-1} C_k^{\mathsf{T}} \tilde{R}_k^{-1}. \tag{15}$$

The covariance matrix of $\hat{x}_{k|k}$ can be written as

$$P_{k|k} = \bar{P}_{k|k} + (I - K_k C_k) G_{k-1} D_{k-1} G_{k-1}^{\mathsf{T}} (I - K_k C_k)^{\mathsf{T}}, \tag{16}$$

where the expression for $\bar{P}_{k|k}$ equals the measurement update of the Kalman filter,

$$\bar{P}_{k|k} = (I - K_k C_k) P_{k|k-1}. \tag{17}$$

3.4 Computational burden

Consider the case where $n, p \gg m_d, m_w$. In this case, a direct implementation of the filter takes $\mathcal{O}(n^3 + p^3 + n^2 p)$ flops. The storage requirements are $\mathcal{O}(n^2 + p^2)$ memory elements.

4 Suboptimal filtering

In data assimilation applications, the discrete-time model (3)-(4) is usually obtained by discretizing PDEs over a huge spatial grid. This results in a state vector of very large dimension, from $n = 10^4$ in tidal flow forecasting [1] to $n = 10^7$ in weather forecasting. The number of measurements ranges from $p = 10^2$ to $p = 10^5$. Consequently, the filter summarized in section 3 can not be used in these applications. Therefore, in section 4.1, we reduce the number of computations and the storage requirements by approximating the filter equations.

A second disadvantage of the filter in Section 3 is that, especially in 2D and 3D problem, the existence condition (13) may not be satisfied. In Section 4.2, we relax this existence condition by expanding the unknown boundary condition as a linear combination of basis functions.

4.1 Reduced rank filtering

Several suboptimal filtering schemes based on the Kalman filter have been proposed. Usually a square-root formulation is adopted. Potter and Stern [13] introduced the idea of factoring the error covariance matrix P_k into Cholesky factors, $P_k = S_k S_k^{\mathsf{T}}$, and expressing the Kalman filter equations in terms of the Cholesky factor S_k, rather than P_k. Suboptimal square-root filters gain speed, but loose accuracy by propagating a non-square $S_k \in \mathbb{R}^{n \times q}$ with very few columns, $q \ll n$. The value of q in data assimilation applications is typically in the order of 10^2. This leads to a huge decrease in computation times and storage requirements, while the computed error covariance matrix remains positive definite at all times. One of these suboptimal filters which is successfully used in practice, is the reduced rank square root filter (RRSQRT) [1]. This algorithm is based on an optimal lower rank approximation of the error covariance matrix and has the interesting property that is algebraically equivalent to the Kalman filter for $q = n$.

In this section, we consider the case where $n \gg p, m_d$ and we extend the filter of section 3 to large-scale system based on the ideas of [1]. The resulting algorithm is algebraically equivalent to the filter of section 3 for $q = n$ and consists of four steps: 1) the time update of the state estimate, 2) the estimation of the unknown boundary condition, 3) the measurement update of the state estimate and 4) a step where the rank of the covariance matrix is reduced.

4.1.1 Time update

We assume that the matrix $H_k Q_k H_k^{\mathsf{T}}$ is of low rank r, with $r \leq m_w \ll n$, such that a square-root factor $H_k Q_k^{1/2} \in \mathbb{R}^{n \times r}$ can easily be found. Let $S_{k-1|k-1}^{\star}$ be a Cholesky factor of $P_{k-1|k-1}$, then time update (5)-(6) is written as

$$\hat{x}_{k|k-1} = A_{k-1}\hat{x}_{k-1|k-1} + B_{k-1}u_{k-1}, \tag{18}$$

$$S_{k|k-1}^{\star} = \begin{bmatrix} A_{k-1}S_{k-1|k-1}^{\star} & H_{k-1}Q_{k-1}^{1/2} \end{bmatrix}. \tag{19}$$

Like [1], we approximate (19), but strongly reduce the computational load by replacing $S_{k-1|k-1}^{\star}$ by a Cholesky factor $S_{k-1|k-1} \in \mathbb{R}^{n \times q}$ of an optimal rank-q approximation of the error covariance matrix $P_{k-1|k-1}$ with $q \ll n$. Finally, note that the number of columns in the covariance square-root, and hence the rank of the error covariance matrix grows from q to $q + r$.

4.1.2 Estimation of unknown boundary condition

If p is large, the most time consuming step in the estimation of the unknown boundary condition is the inversion of $\tilde{R}_k$. We now show how this inverse can be efficiently computed. Defining $V_k = C_k S_{k|k-1}$, it follows by applying the matrix inversion lemma to (10) that

$$\tilde{R}_k^{-1} = R_k^{-1} - R_k^{-1} V_k (I_{q+r} + V_k^{\mathsf{T}} R_k^{-1} V_k)^{-1} V_k^{\mathsf{T}} R_k^{-1}. \tag{20}$$

Under the assumption that R_k^{-1} is available or easy to compute, (20) requires only the inversion of a $(q+r)\times(q+r)$ matrix.

4.1.3 Measurement update

It follows from (16) that a square-root formulation of the measurement update can be written as

$$S^{\star}_{k|k} = [\bar{S}^{\star}_{k|k} \;\; (G_{k-1} - K_k F_k) D^{1/2}_{k-1}], \quad (21)$$

where $\bar{S}^{\star}_{k|k}\bar{S}^{\star\mathsf{T}}_{k|k} = \bar{P}_{k|k}$ and $D^{1/2}_{k-1}D^{\mathsf{T}/2}_{k-1} = D_{k-1}$. Like in the time update, we approximate (21) by replacing $\bar{S}^{\star}_{k|k}$ by a Cholesky factor $\bar{S}_{k|k} \in \mathbb{R}^{n\times(q+r)}$ of an optimal rank-$(q+r)$ approximation of the error covariance matrix $\bar{P}_{k|k}$, $q \ll n$.

The term $\bar{S}_{k|k}$ can then be computed using any existing suboptimal measurement update for the Kalman filter. We use the update of the ensemble transform Kalman filter [3], which is based on the Potter formulation of the measurement update. Let $\bar{\bar{P}}_{k|k}$ denote the optimal rank $q+r$ approximation of $\bar{P}_{k|k}$, then the Potter formulation of the measurement update can be written as

$$\bar{\bar{P}}_{k|k} = S_{k|k-1}\left(I - V_k^{\mathsf{T}} E_k^{-1} V_k\right) S^{\mathsf{T}}_{k|k-1}, \quad (22)$$

where $E_k = V_k V_k^{\mathsf{T}} + R_k$. For convenience of notation, we define the matrix $T_k \in \mathbb{R}^{(q+r)\times(q+r)}$ by

$$T_k = (I - V_k^{\mathsf{T}} E_k^{-1} V_k). \quad (23)$$

Let the square-root factorization of T_k be given by

$$T_k = N_k N_k^{\mathsf{T}}, \quad (24)$$

then it follows from (22) that $\bar{S}_{k|k}$ is given by

$$\bar{S}_{k|k} = S_{k|k-1} N_k. \quad (25)$$

If $p \gg q$, the computation time can be reduced by avoiding the inversion of E_k. First, compute the matrix $W_k = V_k^{\mathsf{T}} R_k^{-1} V_k$. Let the eigenvalue decomposition of W_k be given by

$$W_k = U_k \Lambda_k U_k^{\mathsf{T}}, \quad (26)$$

then using (23) and (26), it is straightforward to show that

$$T_k = U_k\left(I_{q+r} + \Lambda_k\right)^{-1} U_k^{\mathsf{T}}. \quad (27)$$

Consequently, it follows from (24) that N_k is given by

$$N_k = U_k (I_{q+r} + \Lambda_k)^{-1/2}. \quad (28)$$

Under the assumption that $m_d \ll n$, the second term in (21), $(G_{k-1} - K_k F_k) D^{1/2}_{k-1}$, can be efficiently computed by substituting

$$K_k = S_{k|k-1} V_k^{\mathsf{T}} \tilde{R}_k^{-1} \quad (29)$$

and (20) in (21) and computing the matrix products from the left to the right.

Note that the rank of the error covariance matrix grows from $q+r$ to $q+r+m_d$ during the measurement update.

4.1.4 Reduction step

The augmentation of the rank during the time update and the measurement update could quickly blow up computation times. Like [1], the number of columns in $S_{k|k}$ is reduced from $q+r+m_d$ back to q by truncating the error covariance matrix $P_{k|k} = S_{k|k}S^{\mathsf{T}}_{k|k}$ after the q largest eigenvalues and corresponding eigenvectors. The eigenvalue decomposition of $P_{k|k}$ can efficiently be computed from the one of the much smaller matrix $S^{\mathsf{T}}_{k|k}S_{k|k} \in \mathbb{R}^{(q+r+m_d)\times(q+r+m_d)}$. Let the eigenvalue-decomposition of $S^{\mathsf{T}}_{k|k}S_{k|k}$ be given by

$$S^{\mathsf{T}}_{k|k}S_{k|k} = X_k \Omega_k X_k^{\mathsf{T}}, \quad (30)$$

then it is straightforward to show that

$$(S_{k|k} X_k \Omega_k^{-1/2}) \Omega_k (S_{k|k} X_k \Omega_k^{-1/2})^{\mathsf{T}} \quad (31)$$

is the eigenvalue-decomposition of $P_{k|k}$. Consequently,

$$\widetilde{S}_{k|k} = \left[S_{k|k} X_k\right]_{:,1:q} \quad (32)$$

is a square-root of the optimal rank-q approximation of $P_{k|k}$. Since $q, r, m_d \ll n$ this procedure is much faster than an eigenvalue decomposition directly on $P_{k|k}$.

4.2 Basis function expansion

In this section, we relax the existence condition (13) by making an assumption about the unknown boundary condition. We assume that the unknown boundary condition at time instant k can be written as a linear combination of N, with $N \ll m_d$, prescribed basis vectors $\phi_{i,k} \in \mathbb{R}^{m_d}, i = 1\ldots N$,

$$d_k = \sum_{i=1}^{N} a_{i,k}\phi_{i,k}. \quad (33)$$

Defining the vector of coefficients $a_k \in \mathbb{R}^N$ by $a_k = [a_{1,k} \;\; a_{2,k} \ldots a_{N,k}]^{\mathsf{T}}$, and defining the matrix $\Phi_k = [\phi_{1,k} \; \phi_{2,k} \ldots \phi_{N,k}]$, (33) is rewritten as

$$d_k = \Phi_k a_k. \quad (34)$$

Substituting (34) in (3), yields

$$x_{k+1} = A_k x_k + B_k u_k + \bar{G}_k a_k + H_k w_k, \quad (35)$$

where $\bar{G}_k = G_k \Phi_k$. The problem of estimating the unknown boundary condition d_k has thus been transformed to estimating the vector of coefficients a_k. This vector can be estimated using the method developed in [12] if and only if

$$\text{rank } C_k \bar{G}_{k-1} = \text{rank } \bar{G}_{k-1} = N, \text{ for all k.} \quad (36)$$

If $N \ll m_d$, the rank condition (36) is in practice less strong than the condition (13). Loosely speaking, it states that (linear combinations of) measurements of N boundary states must be available. Furthermore, for $N \ll m_d$, the number of computations in the second step of the algorithm is strongly reduced.

5 Simulation example

Consider heat conduction in a two-dimensional plate, governed by the PDE

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + u(x,y,t), \qquad (37)$$

where $T(x,y,t)$ denotes the temperature at position (x,y) and time instant t, $u(x,y,t)$ represents the external heat sources and α is the heat conduction coefficient of the plate. The dimension of the plate is $L_x = 1$m by $L_y = 2$m, the heat conduction coefficient is $\alpha = 10^{-4}$ W/Km^2 and the external heat input is given by

$$u(x,y,t) = \frac{1}{2} e^{-\left(\frac{(x-L_x/2)^2}{2\sigma^2} + \frac{(y-L_y/2)^2}{2\sigma^2} \right)} \qquad (38)$$

with $\sigma = 10^{-1}$, which represents the influence of a flame centered under the middle of the plate. The boundary condition at $x = 0$ is unknown. The other boundary conditions are given by

$$T(L_x,y,t) = T(x,0,t) = T(x,L_y,t) = 300. \qquad (39)$$

The initial condition is given by $T(x,y,0) = 300$.

The PDE (37) is discretized in space and time using finite differences with $\Delta x = \Delta y = 0.025$m and $\Delta t = 2$s, resulting a linear discrete-time state space model of order $n = 3200$. Process noise with variance 10^{-4} is introduced. The matrix G is chosen such that $d_k \in \mathbb{R}^{80}$ represents the unknown boundary condition at $x = 0$. It is assumed that $p = 36$ measurements are available. The measurement locations are indicated by the stars in Fig. 2. The variance of the measurement noise is $R = 10^{-2} I_p$. Note that 12 measurements at the unknown boundary are available. Consequently, the rank condition (13) is not satisfied. Therefore, we expand the unknown boundary condition as a linear combination of basis functions. Note that at most $N = 12$ basis functions can be used in order to satisfy the rank condition (36). We choose as basis functions the orthogonal Chebyshev polynomials.

In a first experiment, we consider the case of constant boundary conditions and set up a simple problem in order to test the efficiency and performance of the filter developed in section 4. We use the method of twin-experiments. First, we simulate the discretized model and add process noise and measurement noise to the state and output. The boundary condition at $x = 0$ is a linear combination of the first 4 Chebyshev polynomials. The coefficients used in the simulation are given in Table 5. Next, we apply the filter where we assume that the initial state and the boundary condition at $x = 0$ are unknown and thus have to be estimated by the filter. By expanding the boundary condition as a linear combination of the first 4 Chebyshev polynomials, the problem boils down to the joint estimation of the state and the coefficients in the expansion. The true and estimated values of the coefficients are shown in Table 5. The estimated values are obtained by averaging over 10 consecutive estimates. The rank of the error covariance matrix was chosen $q = 25$. For larger values of q, results are only slightly more accurate. However, for smaller values of q, performance quickly degrades.

In a second experiment, we consider time-varying boundary conditions. The true boundary condition at $x = 0$ varies sinusoidally in time and in space. We expand the unknown boundary condition as a linear combination of the first 8 Chebyshev polynomials (which gives the best results in this experiment) and let the filter estimate the time-varying coefficients. Figure 1 compares the convergence speed of the RRSQRT to the joint state and boundary condition estimator for $q = 25$. In the RRSQRT, the boundary condition is assumed to be known. We conclude from Fig. 1 that the joint state and boundary condition estimator converges as fast as in the case where the boundary condition is known. The error in the state estimates are shown in Fig. 2. The stars indicate the locations where measurements were taken. The figure on the left hand side show the error after 50 steps. The figure on the right after 250 steps, i.e. when the filter has converged. The estimation is largest in the neighborhood of the unknown boundary.

Table 1. Comparison between true and estimated value of the coefficients in the basis function expansion. The estimated values shown are obtained by averaging over 10 consecutive estimates.

	true value	estimated value
a_1	300	299,967
a_2	15	15,003
a_3	-50	-49,999
a_4	-25	-25,015

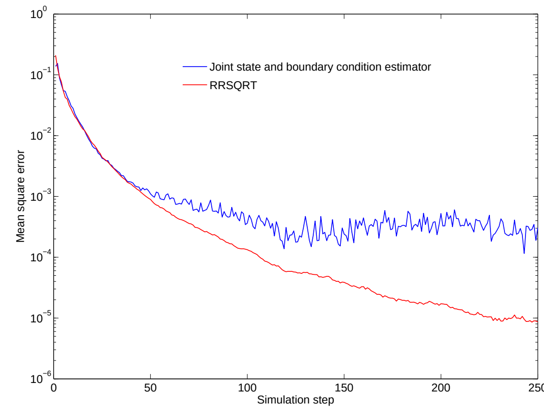

Figure 1. Comparison between the convergence speed of the reduced rank square root filter (RRSQRT), where the boundary condition at $x = 0$ is assumed to be known, and the joint state and boundary condition estimator developed in section 4. Results are shown for $q = 25$.

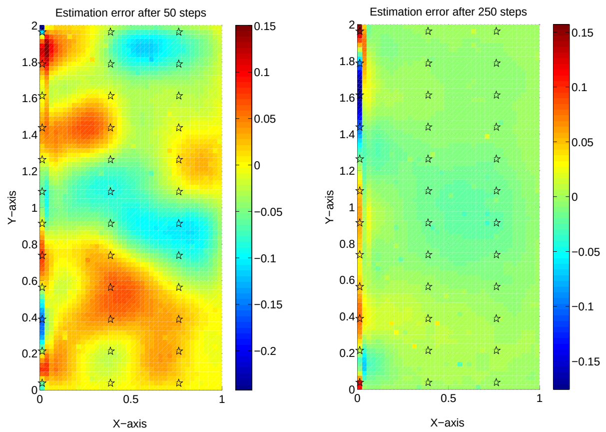

Figure 2. Estimation error at simulation step 50 (left) and simulation step 250 (right). The stars indicate the locations where measurements were taken.

6 Conclusion and remarks

This paper has studied the problem of joint state and boundary condition estimation in linear data assimilation. A suboptimal filter was developed which is based on the assumption that no prior information about the time evolution of the boundary condition is available. However, it is assumed that the spatial form of the boundary condition can be expanded as a linear combination of a limited number of basis vectors. Furthermore, it is assumed that measurements at the boundary are available. A simulation example using a linear heat conduction model indicates that the filter converges almost as fast as in the case where the boundary condition is known. Furthermore, the filter is able to accurately estimate time-varying boundary conditions. However, it remains to be seen how the method performs on real data and in more complex (nonlinear) data assimilation applications.

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