Model reduction and energy analysis as a tool to detect spurious modes

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Abstract

Linear system identification is an important and useful approach in experimental modal analysis. It allows for the extraction of resonance frequencies, damping ratios and mode shapes of a vibrating structure. In order to reduce the bias on the estimates, the model order is usually chosen quite high. This often results in the appearance of non-physical, or so-called spurious modes. In this paper we will show that many spurious modes can be removed from the obtained model using ideas from model reduction and looking at the energy contained in these modes.

1 Introduction

System identification is an important tool for the analysis of forcefully or ambient excited vibrating structures [1]. In its standard form, a linear model for the structure is built starting from available observations. Based on this model, modal characteristics like resonance frequencies and mode shapes can be estimated. Usually, the order of the models involved is chosen quite high in order to reduce the bias on the estimates and allow for the model to capture all relevant characteristics of the structure, even in the presence of large amounts of measurement noise. Unfortunately, as the order of the model is increased, so will be the number of identified modes. This will in many cases inevitably result in the appearance of so-called spurious modes. In this paper we will describe a technique to detect these spurious modes and remove them from the model. The technique is heavily based on energy considerations and bears resemblance with the well known theory of balanced model reduction [6, 7].

In Section 1, we will introduce a particular notion of energy and describe its relation to the theory of balanced model reduction. In Section 2, the ideas from balanced model reduction will be applied to models in the modal form for use in modal analysis. In Section 3, a concrete implementation of the proposed technique will be described, and in Section 4 some real life examples will demonstrate that the proposed technique leads to elimination of spurious modes in many cases. Finally, some conclusions will be drawn in Section 5. Some common notations that will be used throughout this text are the following. $\mathcal{E}[\cdot]$ will be used to denote the expected value of an expression. $A(i{:}j, k{:}l)$ denotes a submatrix of A, bounded by the i^{th} and j^{th} row and k^{th} and l^{th} column. If a colon (:) is used on its own (e.g. $A(:, k{:}l)$) all available rows and/or columns are included in the submatrix. The same principle applies to vectors. $\big($eg. $x(3{:}5) = \begin{bmatrix} x(3) & x(4) & x(5) \end{bmatrix}\big)$. Further extensions are trivial.

2 A notion of energy and balanced model reduction

In order to introduce the basic concepts of balanced model reduction we will consider deterministic linear time invariant models. In discrete time, such a model may be represented as follows:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \,, \\ y_k &= Cx_k + Du_k \,, \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times n}$ and $D \in \mathbb{R}^{l\times m}$. $y_k \in \mathbb{R}^l$ represents the output and $u_k \in \mathbb{R}^m$ the input at discrete time k. We will assume that an infinite amount of data is available ($k \in [0, \infty)$).

The output energy of a model of the form (1) in a time interval $[0, N\text{-}1]$ is given as $\sum_{k=0}^{N-1} \|y_k\|^2$. Likewise the input energy is given as $\sum_{k=0}^{N-1} \|u_k\|^2$. It is straightforward to see that

$$\begin{aligned} J_y &= \sum_{k=0}^{N-1} \|y_k - Du_k\|^2 \\ &= x_0^T \left(\sum_{k=0}^{N-1} (A^k)^T C^T C A^k \right) x_0 \\ &= x_0^T \Gamma_N^T \Gamma_N x_0 \,, \end{aligned} \tag{2}$$

where

$$\Gamma_N = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{N-1} \end{bmatrix} \tag{3}$$

is the so-called observability matrix of order N of (1). Introducing the observability Grammian of order N:

$$Q_N = \Gamma_N^T \Gamma_N, \tag{4}$$

equation (2) can further be reduced to

$$J_y = x_0^T Q_N x_0 \,. \tag{5}$$

It can now be seen that the observability Grammian Q_N in (5) can be interpreted in terms of an "oriented" energy. Large entries in Q_N will signalise a large contribution of certain parts of the state to the output energy whereas small elements in Q_N can be associated to small contributions. This point can be made more precise by taking the singular value decomposition of Γ_N:

$$\Gamma_N = U_{\Gamma_N} \Sigma_{\Gamma_N} V_{\Gamma_N}^T \,. \tag{6}$$

With (6), equation (5) can be rewritten as

$$J_y = (x_0^T V_{\Gamma_N})(\Sigma_{\Gamma_N})(V_{\Gamma_N}^T x_0) \tag{7}$$

The conclusion is straightforward. If an important part of the initial state lies along right singular vectors corresponding to small singular values, it will contribute little to J_y. If however, the initial state is lying along singular vectors corresponding to large singular values, it will contribute a lot.

Where the observability Grammian gives a measure of state-output energy, the so-called controllability Grammian is its system theoretic dual: The controllability Grammian can be shown to characterize input-state energy relations. Assuming that the initial state is $x_0 = 0$, it is straightforward to see that at time N:

$$x_N = \Delta_N \begin{bmatrix} u_{N-1} \\ \vdots \\ u_0 \end{bmatrix} = \Delta_N U, \tag{8}$$

with $\Delta_N = \begin{bmatrix} B & AB & A^2B & \dots & A^{N-1}B \end{bmatrix}$, the so-called controllability matrix of order N of the system. One could now search for the input of minimum energy that brings the system from an initial state $x_0 = 0$ to an end state x_N at time N by solving the following least squares problem:

$$\min_U \|x_N - \Delta_N U\|^2, \tag{9}$$

which has as a solution

$$U = \Delta_N^\dagger x_N. \tag{10}$$

where $\Delta_N^\dagger$ is the Moore-Penrose inverse of Δ_N. The input energy corresponding to this solution is given by

$$\begin{aligned} J_U &= \sum_{k=0}^{N-1} \|u_k\|^2 \\ &= U^T U \\ &= x_N^T (\Delta_N \Delta_N^T)^\dagger x_N \\ &= x_N^T P_N^\dagger x_N \,. \end{aligned} \tag{11}$$

In (11) the controllability Grammian of order N, $P_N = \Delta_N \Delta_N^T$ was introduced. Again we could take a singular value decomposition of Δ_N yielding

$$\Delta_N = U_{\Delta_N} \Sigma_{\Delta_N} V_{\Delta_N}^T. \tag{12}$$

The system (1) will be called controllable if Δ_N is of full rank for $N \geq n$. Assuming the system is controllable and $N \geq n$, equation (9) will have exactly one solution and $P_N^\dagger = P_N^{-1}$. With these assumptions (12) can be substituted in (11) to obtain:

$$\sum_{k=0}^{N-1} \|u_k\|^2 \tag{13}$$

$$= x_N^T U_{\Delta_N} \Sigma_{\Delta_N}^{-1} V_{\Delta_N}^T V_{\Delta_N} \Sigma_{\Delta_N}^{-1} U_{\Delta_N}^T x_N \tag{14}$$

$$= x_N^T U_{\Delta_N} \begin{bmatrix} \frac{1}{\sigma_1^2} & & \\ & \ddots & \\ & & \frac{1}{\sigma_n^2} \end{bmatrix} U_{\Delta_N}^T x_N \tag{15}$$

$$= x_N^T P_N^{-1} x_N \,. \tag{16}$$

Obviously the elements of P_N serve as a measure of degree of controllability of the state. If for instance

x_N equals the first left singular vector, then the minimum energy needed to control the state from $x_0 = 0$ to x_N equals $\frac{1}{\sigma_1^2}$. If however x_N equals the singular vector corresponding to the smallest singular value, the energy needed equals $\frac{1}{\sigma_n^2}$. Hence state vectors that are lying in the direction of singular vectors corresponding to large singular values are relatively easy to control compared to states that are lying in singular subspaces corresponding to small singular values.

From the definitions $P_N = \Delta_N \Delta_N^T$ and $Q_N = \Gamma_N^T \Gamma_N$ it is easy to demonstrate that they satisfy the following Lyapunov equations:

$$AP_N^T A^T - P_N = A^N B B^T {A^N}^T - BB^T, \quad (17)$$

$$A^T Q_N A - Q_N = {A^N}^T C^T C A^N - C^T C. \quad (18)$$

When the system is stable, taking the limit for $N \rightarrow \infty$ results in the following Lyapunov equations for the infinite Grammians:

$$AP_\infty A^T - P_\infty = -BB^T, \quad (19)$$

$$A^T Q_\infty A - Q_\infty = -C^T C. \quad (20)$$

Efficient algorithms exist to solve both Lyapunov equations for P_∞ and Q_∞ directly from the system matrices A, B, C, and D. [2, 3, 4, 5]

The infinite observability and controllability Grammians serve as a basis for many model reduction schemes. An important concept, introduced by Moore [6, 7], is that of balanced model reduction. In short, the transfer matrix $D + C(zI - A)^{-1}B$ corresponding to any state-space model of the form (1) is invariant under a state transformation $x_k \rightarrow x_k^T = Tx_k$, where T is an arbitrary but non-singular matrix, provided the system matrices are adapted accordingly $(A, B, C, D) \rightarrow (TAT^{-1}, TB, CT^{-1}, D)$. It has been shown by Moore in [6] that there always exist a representation (A^b, B^b, C^b, D) transforming the infinite controllability and observability Grammian to be diagonal and equal. The corresponding transformation T can be obtained from the eigenvalue decomposition of the product $Q_\infty P_\infty$ as $T^{-T} Q_\infty P_\infty T^T = \Sigma^2$ [6]. In this representation,

$$\begin{aligned} x_{k+1}^b &= A^b x_k^b + B^b u_k, \\ y_k &= C^b x_k^b + D u_k, \end{aligned} \quad (21)$$

which is known as the balanced representation, we then have:

$$P_\infty^b = TP_\infty T^T = \Sigma, \quad (22)$$

$$Q_\infty^b = T^{-T} Q_\infty T^{-1} = \Sigma, \quad (23)$$

$$\Sigma = \text{diag}\begin{bmatrix}\sigma_1 & \sigma_2 & \dots & \sigma_n\end{bmatrix}. \quad (24)$$

By formally substituting a diagonal Grammian in (5) and (11), it can be seen that the result of diagonalizing the Grammians and making them equal is that the importance of each state coefficient in a model of the form (21) can now be assessed. Indeed, from (5) it can be seen that the most important coefficients in the state, in terms of degree of observability, or more loosely, contribution to the total output energy, are those coefficients related to the largest singular values in Σ. Likewise, it can be seen from (11) that these will precisely be the state coefficients with the highest degree of controllability. The balanced model reduction scheme is now the following:

1. Solve the Lyapunov equations (19-20) for P_∞ and Q_∞.
2. Calculate the balancing transformation T from the eigenvalue decomposition $T^{-T} Q_\infty P_\infty T^T = \Sigma^2$.
3. Balance the model $(A, B, C, D) \rightarrow (TAT^{-1}, TB, CT^{-1}, D)$.
4. Partition Σ in dominant and small singular values:

$$\Sigma = \left[\begin{array}{ccc|ccc} \sigma_1 & & & & & \\ & \ddots & & & 0 & \\ & & \sigma_r & & & \\ \hline & & & \sigma_{r+1} & & \\ & 0 & & & \ddots & \\ & & & & & \sigma_n \end{array}\right], \quad (25)$$

and partition (A, B, C, D) accordingly:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right]. \quad (26)$$

The balanced model reduced model is:

$$(A_r, B_r, C_r, D) = (A_{11}, B_1, C_1, D). \quad (27)$$

Noting the full order transfer matrix $D + C^b(zI - A^b)^{-1}B^b$ of (21) as $H^b(z)$, and the lower order transfer matrix resulting from the truncation as $H_r^b(z)$ it has been reported in the literature that [6]:

$$\|H^b - H_r^b\|_\infty < 2\left(\sum_{i=r+1}^{n} \sigma_i\right), \quad (28)$$

where for a transfer matrix H, $\|H\|_\infty = \max_\omega \bar{\sigma}(H(e^{j\omega}))$, with $\bar{\sigma}(X)$ the maximal singular

value of a matrix X, is the so-called infinity norm of H. Hence, it follows from (28) that balanced model reduction allows to reduce the order of the model, with minimal changes to the transfer matrix in the L_∞ sense.

It has been shown in [8], however, that balanced model reduction is far from optimal in L^2 sense. The L^2 norm of a transfer matrix H is calculated as

$$\begin{aligned} \|H\|_2^2 &= \sum_{k=1}^{\infty} \text{tr}\left[(CA^{k-1}B)(CA^{k-1}B)^T\right] && (29)\\ &+ \text{tr}\left[DD^T\right], && (30)\end{aligned}$$

which, using the Grammians P_∞ and Q_∞, reduces to:

$$\|H\|_2^2 - \text{tr}\left[DD^T\right] = \text{tr}\left[CP_\infty C^T\right] = \text{tr}\left[B^T Q_\infty B\right], \quad (31)$$

and in the balanced form

$$\|H\|_2^2 - \text{tr}\left[DD^T\right] = \sum_{i=1}^{n} \sigma_i C^b(:,i)^T C^b(:,i). \quad (32)$$

Based on these derivations, it was suggested in [8] that, in case one is interested in L^2 model reduction rather than L_∞ model reduction it is recommended to partition the state space model according to the magnitudes of $\sigma_i C^b(:,i)^T C^b(:,i)$, the so-called balanced gains, in stead of σ_i alone.

3 Model reduction in the modal form

The model reduction techniques given in section 2 work on models in their balanced form, meaning that both the observability and controllability Grammians are diagonal and equal. In modal analysis, however, one usually works with the modal form, given as.

$$\begin{aligned} x_{k+1}^m &= \Lambda x_k^m + B^m u_k, \\ y_k &= C^m x_k^m + D u_k, \end{aligned} \quad (33)$$

where Λ is diagonal and mainly consists of pairs of complex conjugated eigenvalues $\lambda, \overline{\lambda}$, being the poles of the system. The modal characteristics of the structure under study can easily be obtained as follows:

$$\begin{aligned} f_i &= \arg\left(\lambda_i \frac{T_s}{2\pi}\right), && (34)\\ d_i &= \frac{\ln(|\lambda_i|)}{\sqrt{\ln(|\lambda_i|)^2 + \arg(\lambda_i)^2}}, && (35)\\ v_i &= C'(:,i), && (36)\end{aligned}$$

with f_i, d_i and v_i the resonance frequency, damping value and mode shapes corresponding to the i^{th} pole $\Lambda(i,i) = \lambda_i$. T_s is the sampling rate.

It has been shown in [9] that for nearly undamped structures, the Grammians P_∞^m and Q_∞^m in the modal form are almost diagonal, meaning that the modal and balanced form are "close" to each other in some sense for nearly undamped systems. In this case, the model reduction procedures outlined in Section 2 could be applied to (33), effectively removing those modes from the model that are least relevant in describing the structures dynamics. As it seems only reasonable to assume that spurious modes take little part in the accurate description of the structures dynamics, model reduction might offer a technique to remove spurious modes in the nearly undamped case.

In many practical cases, however, the structure under study is not nearly undamped, and the modal form may deviate quite considerably from the balanced form, even for relatively lightly damped structures. In such cases the Grammians P_∞^m and Q_∞^m will not be diagonal and the total output energy will consist of a sum of contributions from the different states, including cross-terms:

$$\begin{aligned} J_y &= x_0^T Q_\infty^m x_0, && (37)\\ &= \sum_{i=1}^{n} x_0^2(i) Q_\infty^m(i,i), && (38)\\ &+ \sum_{i\neq j} x_0(i) x_0(j) Q_\infty^m(i,j). && (39)\end{aligned}$$

Due to the crossterms in (39) the individual contribution of every state component to the total output energy is hard to assess, which is why modal reduction techniques can in general not be used in the modal form. Formally we can still calculate the norms of the difference of the transfer matrices of the full order model $H^m(z)$ and a truncated model $H_r^m(z)$:

$$\|H^m - H_r^m\|_\infty, \quad \|H^m - H_r^m\|_{L^2}. \quad (40)$$

It will be shown by means of some examples that in many cases, calculating (40) gives enough information to successfully remove most spurious modes in a model.

4 Concrete implementation

Let's assume we have a discrete linear model given in modal form:

$$\begin{aligned} x_{k+1}^m &= \Lambda x_k^m + B^m u_k, \\ y_k &= C^m x_k^m + D u_k, \end{aligned} \quad (41)$$

with the dimensions involved the same as in prior sections. Select mode i from (41). The modal subsystem corresponding to this mode is given as

$$\begin{bmatrix} x_{k+1}(i) \\ \bar{x}_{k+1}(i) \end{bmatrix} = \begin{bmatrix} \lambda_i & 0 \\ 0 & \bar{\lambda}_i \end{bmatrix} \begin{bmatrix} x_k(i) \\ \bar{x}_k(i) \end{bmatrix} + \begin{bmatrix} b_i \\ \bar{b}_i \end{bmatrix} u_k, \quad (42)$$

$$y_k = \begin{bmatrix} c & \bar{c}_i \end{bmatrix} \begin{bmatrix} x_k(i) \\ \bar{x}_k(i) \end{bmatrix}, \quad (43)$$

with $b_i = B^m(i,:)$ and $c_i = C^m(:,i)$. We introduce the following notations:

$$A_i = \begin{bmatrix} \lambda_i & 0 \\ 0 & \bar{\lambda}_i \end{bmatrix}, \quad (44)$$

$$B_i = \begin{bmatrix} b_i \\ \bar{b}_i \end{bmatrix}, \quad (45)$$

$$C_i = \begin{bmatrix} c_i & \bar{c}_i \end{bmatrix}, \quad (46)$$

$$H_i(z) = C_i(zI_2 - A_i)^{-1}B_i. \quad (47)$$

With these notations, we have

$$\|H_i\|_{L^2}^2 = \text{tr}\left(C_i P_{i\infty} C_i^T\right), \quad (48)$$

$$\|H_i\|_\infty = \max_\omega \bar{\sigma}\left(H_i(e^{j\omega})\right), \quad (49)$$

where $P_{i\infty}$ can be obtained by solving the discrete Lyapunov equation: [6]

$$A_i P_{i\infty} {A_i}^T - P_{i\infty} = -B_i {B_i}^T, \quad (50)$$

and numerical procedures are widely available for the calculation of the infinity norm [10]. Clearly equations (48-49) can be seen as giving a measure of the model reduction error when removing the mode corresponding to λ_i. It should be noted that for practical applications, algorithms to solve the lyapunov equation (50) might break down due to the complex nature of the matrices A_i, B_i, C_i. In these cases one might prefer to work with a state space representation where, in stead of being diagonal, A^m is composed out of 2 by 2 block matrices of the form:

$$\tilde{A}_i = \begin{bmatrix} \text{Real}(\lambda_i) & \text{Imag}(\lambda_i) \\ -\text{Imag}(\lambda_i) & \text{Real}(\lambda_i) \end{bmatrix} \quad (51)$$

and B_i and C_i are adapted accordingly.

Note that certain components of H_i in (47)will become zero if a zero is present in the full order model H^m at position λ_i, a so called pole-zero cancellation. Suppose for instance that $H^m(k,l)(\lambda_i)$ is zero, meaning that the transfer function from input l to output k contains zeros at positions λ_i and $\bar{\lambda}_i$, then we derive from:

$$H^m(k,l)(\lambda_i) \quad (52)$$

$$= D(k,l) + C^m(k,:)(\lambda_i I_n - \Lambda)^{-1} B^m(:,l) \quad (53)$$

$$= D(k,l) + \sum_{p=1}^{n} \frac{C^m(k,p)B^m(p,l)}{\lambda_i - \lambda_p} \quad (54)$$

$$= \frac{\prod_{q=1}^{n}(\lambda_i - \lambda_q)D(k,l)}{\prod_{q=1}^{n}(\lambda_i - \lambda_q)} \quad (55)$$

$$+ \frac{\sum_{p=1}^{n}\prod_{q\neq p}(\lambda_i - \lambda_q)C^m(k,p)B^m(p,l)}{\prod_{q=1}^{n}(\lambda_i - \lambda_q)} = 0, \quad (56)$$

that $C^m(k,i)B^m(i,l) = c_i(k)b_i(l) = \bar{c}_i(k)\bar{b}_i(l) = 0$. Hence:

$$H_i(k,l)(z) = \begin{bmatrix} c_i(k) & \bar{c}_i(k) \end{bmatrix} \quad (57)$$

$$\times \left(zI_2 - \begin{bmatrix} \lambda_i & 0 \\ 0 & \bar{\lambda}_i \end{bmatrix} \right)^{-1} \quad (58)$$

$$\times \begin{bmatrix} b_i(l) \\ \bar{b}_i(l) \end{bmatrix} = 0. \quad (59)$$

It has been proposed in the literature to look at pole/zero cancellations occurring in every component of H in order to detect spurious poles [11]. The above calculations show that a cancellation for λ_i in every component of H leads to $\|H_i\|_{L^2} = \|H_i\|_\infty = 0$. Hence this type of pole/zero cancellations is easily detected using model reduction in the modal form.

Extension of the results to the stochastic case is trivial. For stochastic systems, a state space model in so-called forward innovation form, is given as:

$$\begin{aligned} x_{k+1}^F &= Ax_k^F + Ke_k, \\ y_k &= Cx_k^F + e_k, \\ \mathcal{E} e_k e_l^T &= \Sigma_e \delta_{kl}, \end{aligned} \quad (60)$$

with e_k being a white noise sequence with zero mean and covariance matrix Σ_e, $A \in \mathbb{R}^{n\times n}$, $K \in \mathbb{R}^{n\times l}$, and $C \in \mathbb{R}^{l\times n}$. For this type of models we have

$$H^F(z) = I + C^F(zI_n - A^F)^{-1}K^F, \quad (61)$$

and (40) can again be used to get an idea of the importance of a certain mode. For completeness it is worth mentioning that a notion of stochastic balancing exists, where the balanced representation is obtained by solving a set of Riccatti equations rather than Lyapunov equations. See [15] for more details on forward innovation models and their properties.

5 Experimental results

The model reduction technique described in this paper was used to detect spurious modes in a stochastic

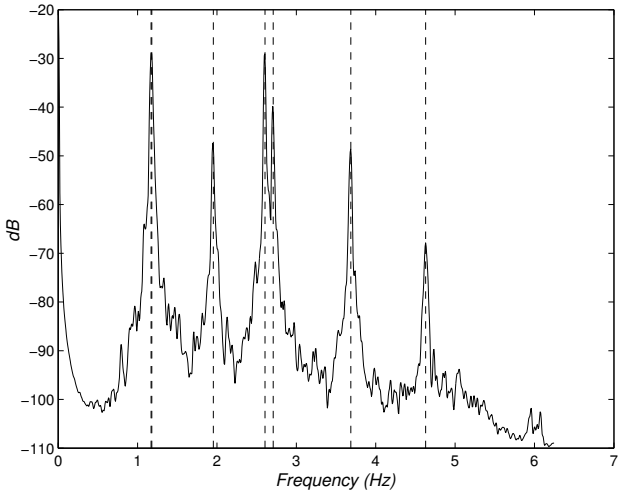

Figure 1: *Spectrum of a steel mast*

damp.	*freq.*	L^2	L_∞
0.57%	**2.709Hz**	**12.07**	**132.44**
1.30%	**1.179Hz**	**10.07**	**109.81**
0.56%	**2.605Hz**	**8.09**	**88.29**
1.17%	**1.171Hz**	**6.58**	**71.06**
1.10%	**1.954Hz**	**7.17**	**63.21**
0.44%	**4.629Hz**	**6.63**	**58.34**
0.54%	**3.686Hz**	**5.96**	**51.94**
0.65%	6.169Hz	3.71	16.25
1.18%	4.661Hz	3.91	15.34
5.27%	5.067Hz	5.33	12.27
3.66%	5.071Hz	4.15	9.73
14.06%	4.820Hz	5.65	8.58
16.27%	2.643Hz	4.11	6.36
6.84%	5.114Hz	3.87	6.21
20.65%	1.948Hz	3.74	5.19
15.96%	4.979Hz	4.21	4.99
9.73%	5.501Hz	3.67	4.55
62.56%	2.935Hz	3.50	2.42
23.94%	5.530Hz	3.30	2.30

Table 1: *Modes extracted from measurements on a steel mast and corresponding L^2 and L_∞ norms. Ranking as a function of decreasing L_∞ norm.*

subspace model from measurements on a steel transmitter mast for cellular phone networks [12]. Nine accelerometers were placed on the mast and the mast's response on the wind turbulence was measured. A 40^{th}-order stochastic subspace model was thereafter created for the mast using the N4SID procedures described in [13]. For each couple of complex conjugated poles λ_i, $\bar{\lambda}_i$ in the model, the following error norms were calculated:

$$d_2 = \|H_i\|_{L^2}, \quad d_\infty = \|H_i\|_\infty, \tag{62}$$

with $H_i(z)$ as defined in the previous Section.
Before proceeding we have two caveats:

First of all, in an experiment, not all accelerometers are excited by all the modes present in the system. Furthermore, some accelerometers may be excited more by the system as a whole than others, due to scaling differences in the measuring equipment or simply due to the locations of the accelerometers on the structure. The measures d_2 and d_∞ are strongly dependent on the scaling of one or more outputs. Hence, in case a certain mode is only present in a single accelerometer, and this accelerometer happens to be poorly excited by the structure as a whole, the obtained measures d_∞ and d_2 will inevitably be very small and the mode will always be identified as spurious. Hence, it is recommended to scale all outputs so as to have a comparable amount of total energy stored in each output to avoid this problem.

Secondly it is known that subspace identification methods for stochastic systems can fail due to problems with stability or positive realness of the obtained model [14]. Several algorithms have been proposed in the literature to deal with these problems [15, 16, 17, 18, 19]. In the examples reported here we have used the methods described in [15] and [18] to obtain stable and positive real models.

The results of the analysis are given in Table 1. For each mode found in the model, the corresponding damping value and resonance frequency are given in the first two columns. The last two columns contain the results of the L^2 and L_∞ criteria, given in (62). Modes that were found to be physical in an independent user interactive procedure described in [12] are displayed in bold. Note that all emphasized modes in the table correspond to high L^2 and L_∞ norms, with the results for the L_∞ norm apparently being the most useful for the detection of spurious modes.

The results given in Table 1 can intuitively be understood by looking at the true spectrum given in figure 1. Since the true modes of the system have

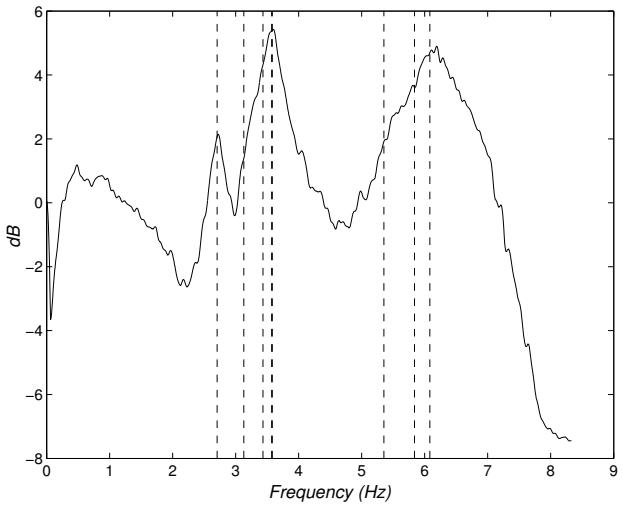

Figure 2: *Spectrum of an airplane*

damp.	*ang. freq.*	L^2	L_∞
2.65%	**3.571Hz**	**10.45**	**55.23**
4.96%	**3.433Hz**	**10.85**	**42.80**
4.03%	**6.082Hz**	**8.94**	**29.47**
3.72%	**3.582Hz**	**3.62**	**16.16**
2.83%	**5.838Hz**	**3.52**	**14.12**
2.58%	**2.707Hz**	**1.43**	**8.83**
2.99%	**3.129Hz**	**1.55**	**8.26**
3.75%	**5.353Hz**	**1.90**	**6.92**
5.77%	3.866Hz	2.42	6.59
2.01%	3.365Hz	1.26	6.22
4.39%	2.091Hz	1.14	4.85
5.73%	3.697Hz	1.57	4.41
45.32%	0.182Hz	0.84	3.59
3.14%	3.983Hz	0.74	2.69
1.53%	4.179Hz	0.22	1.14
6.75%	4.146Hz	0.29	0.71
8.17%	2.816Hz	0.16	0.43

Table 2: *Modes extracted from measurements on an airplane and corresponding L^2 and L_∞ norms. Ranking as a function of decreasing L_∞ norm.*

relatively low damping values and their frequencies are relatively far apart, one can expect sharp and easily detectable peaks, which is precisely what can be seen in figure 1. For such clearly separated peaks the L_∞ and L^2 norms can intuitively be interpreted as giving a measure for the height, respectively the surface beneath the peaks in the output spectrum, up to a certain constant determined by the variance of the feeding white noise. Hence, it comes as no surprise that the model reduction approach described in this paper retains those poles that correspond to the peaks in this spectrum. The relation with the spectrum is not always as straightforward though, as will be seen in the following example where the individual peaks in the spectrum are hardly distinguishable.

Measurements on an airplane

A second example involves in-flight measurements on an airplane. A noisy dataset with 1 input and 10 outputs was analyzed using a 40^{th} order deterministic subspace model and the measures given in (62). Results are given in table 2, mostly in the same way as for the prior example, with as modes in bold the ones that where found to be physical in an independent analysis done by the airplane manufacturer. Analysis of the MAC-values, relating the mode shapes of different modes [20], easily revealed that the modes with angular frequencies 1.347 and 1.351 are in fact one and the same mode and could be grouped together, for instance by applying balanced model reduction on the corresponding fourth order subsystem. Notice that most true modes can not be distinguished in the output spectrum of one of the accelerometers, displayed in figure 2, which is quite representable for the output spectra that were obtained for the other accelerometers in this experiment. Using model reduction in the model form however, a separation between true and spurious modes was again obtained, at least in L_∞ norm, although the results are not as convincing as in the prior experiment. It is therefore important to note that although the proposed technique gives a good indication in many cases as to whether a mode is spurious or not, its use should be combined with other techniques with similar performance, for instance in an approach involving many criteria like stabilization in the stabilization diagram and pole/zero cancellations [11], to get a good overall performance.

6 Conclusions

In this paper it was shown that by applying ideas from the theory of balanced model reduction on a model in the modal form, it is in many cases possible to detect spurious modes. The proposed technique could serve as an extra indication in combined approaches that take a set of parameters into account, ranging from stabilization in the stabilization diagram to pole/zero cancellations.

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