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An algebraic description of realizations of partial covariance sequences

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Abstract

The solutions of the partial realization problem have to satisfy a finite number of interpolation conditions at ∞. The minimal degree of an interpolating deterministic system is called the algebraic degree or McMillan degree of the partial covariance sequence and is easy to compute. The solutions of the partial stochastic realization problem have to satisfy the same interpolation conditions and have to fulfill a positive realness constraint. The minimal degree of a stochastic realization is called the positive degree. The interpolating deterministic solutions can be parameterized by the Kimura-Georgiou parameterization. In the literature, the solutions of the partial stochastic realization problem are then described by checking the positive realness constraints for each interpolating deterministic system. In this paper, an alternative parameterization for the deterministic solutions of the interpolation problem is presented. Both the solutions of the partial and partial stochastic realization problem are analyzed in this alternative parameterization. Based on the structure of the parameterization, a lower bound for the positive degree is obtained.

1 Introduction

In signal processing, speech processing and system identification applications, one can often model signals as a stationary random sequence that is generated by passing white noise through a filter or system with a stable transfer function and letting the system come to a statistical steady state [5, 7, 16, 24, 25]. However, the generating filter or system is not known a priori in many real life situations. The system then has to be estimated from given observations, after which one can use the filter and the corresponding spectral density e.g. in design processes. In practice, only a finite number of samples of the signal is available and a finite covariance sequence can be computed.

For such a given covariance sequence, the systems that are solutions of the *partial realization problem* match the first correlation coefficients of the given covariance sequence. This problem of matching the first correlation coefficients is solved in deterministic realization theory [10, 11, 13, 15] and corresponds to interpolation theory [1, 19, 21, 22, 23] through the interpolation conditions at ∞. The McMillan or algebraic degree are defined as the minimal degree of an interpolating deterministic system. An interpolating solution with degree equal to the McMillan degree is often preferred, e.g. for reasons of computational efficiency.

The solutions of the *partial stochastic realization problem* have to fulfill the same interpolation conditions of the partial realization problem and the solutions have to be (stable) stochastic systems. The latter condition is called the positive real condition: the solutions have to be stable and must have a spectral density which positive on the unit circle. Following [4, 6], the minimal degree of such an interpolating stochastic system will be called the positive degree in this paper. This positive degree is bounded from below by the algebraic degree and from above by the length of the covariance sequence. However, the methods described in the literature propose to calculate the value of the positive degree by using a trial and error approach.

In this paper, an alternative parameterization [20, 21, 23] for the partial realization problem is presented. This parameterization was derived by one of the authors for interpolation conditions at finite points in and for mixed types of interpolation conditions in [20] and [21], respectively. In [23] the partial realization problem was studied in this parameterization. Since this parameterization gives all the solutions of the partial realization problem, the solutions of the partial stochastic realization problem are obtained by additionally checking the positive realness constraints for each such interpolating solution. Both the partial realization problem and the partial stochastic realization problem are discussed in this alternative parameterization. The main focus of this analysis is the "open problem" of determining the value of the positive degree without using trial and error methods. By analyzing the structure of the parameterization, a lower bound for the positive degree is obtained.

2 Preliminaries

Some theory of stochastic systems is reviewed in subsection 2.1. In the literature [2, 4, 8, 14], the so-called Kimura-Georgiou parameterization is used to parameterize the (deterministic) solutions of the partial realization problem. In this set of solutions, the solutions of the corresponding partial stochastic interpolation problem are obtained by checking the positive real constraints for each interpolating solution. This is refreshed in subsection 2.2. In the last subsection, the minimality of the solutions is discussed and the definitions of algebraic and positive degree are reviewed.

2.1 Stochastic Systems

Let the observed discrete time signal y_t be generated by passing normalized white noise e_t through a linear filter with stable rational transfer function

$$w(z) = w_0 + w_1 z^{-1} + w_2 z^{-2} + \dots,$$

with $w_k \in \mathbb{R}$. The stationary (real) process $\{y_t\}_{t \in \mathbb{Z}}$ has a rational spectral density

$$\Phi(z) = w(z)w(z^{-1}),$$

which is assumed to be positive on the unit circle. The stable transfer function $w(z)$ is called a stable spectral factor of $\Phi(z)$, which is taken minimum phase. This means that the rational function

$$w(z) = \frac{\sigma(z)}{a(z)}$$

has all its poles and zeros in the open unit disc and $w_0 = w(\infty) \neq 0$.

Hence, the output $\{y_t\}_{t \in \mathbb{Z}}$ is the output of a shaping filter $w(z)$ driven by a normalized white noise input e_t, as is depicted by Figure 1. The Fourier transform of the spectral density $\Phi(z)$ is then equal to

$$\Phi(z) = c_0 + \sum_{i=1}^{\infty} c_i \left(z^i + z^{-i}\right),$$

where $c_0, c_1, c_2, \dots$ is the covariance sequence defined as

$$c_k = \mathrm{E}\{y_{t+k} y_t\},$$

$l = 0, 1, 2, \dots$. Without loss of generality, we will work with normalized sequences, i.e., with $c_0 = 1$ in the sequel of this paper. One can then construct the partial covariance sequences

$$C_n = \{1, c_1, \dots, c_n\},$$

which have the property that the Toeplitz matrices

$$\mathcal{T}(C_n) = \begin{bmatrix} 1 & c_1 & c_2 & \cdots & c_n \\ c_1 & 1 & c_1 & \cdots & c_{n-1} \\ c_2 & c_1 & 1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n-1} & \cdots & \cdots & 1 \end{bmatrix}$$

are positive definite[1] for $0 \leq n$. If the Toeplitz matrix is not positive definite, the partial covariance sequence is not generated by a stochastic system. Hence, there exists no solution of the stochastic realization problem.

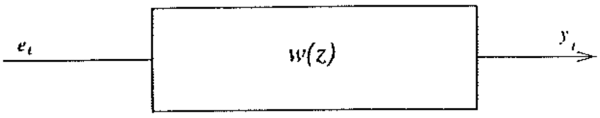

Figure 1: Shaping filter.

The partial stochastic realization problem now can be stated as follows: *Given* a partial covariance sequence $C_N = \{1, c_1, \dots, c_N\}$. *Find* an infinite rational extension $\{c_{N+1}, c_{N+2}, \dots\}$ s.t. $\mathcal{T}(C_n) \geq 0$ for all $n \geq 0$ or find a positive real part $v(z)$ of $\Phi(z) = v(z) + v(z^{-1})$,

$$v(z) = \frac{1}{2}\frac{b(z)}{a(z)} = \frac{1}{2} + \sum_{k=1}^{\infty} c_k z^{-k},$$

which is rational with

$$\begin{aligned} a(z) &= z^n + a_1 z^{n-1} + \dots + a_n \\ b(z) &= z^n + b_1 z^{n-1} + \dots + b_n. \end{aligned}$$

The monic polynomials $a(z)$ and $b(z)$ have to satisfy the following conditions:

1. Interpolation conditions at ∞:

$$\Pi^0_{-N} \frac{b(z)}{a(z)} = \Pi^0_{-N} F(z), \tag{1}$$

with

$$F(z) = \frac{1}{2} + c_1 z^{-1} + \dots + c_N z^{-N}$$

and the projection operator

$$\Pi^\beta_\alpha \sum_k f_k z^k = \sum_{k=\alpha}^{\beta} f_k z^k.$$

2. Positive real conditions: if $a(z)$ and $b(z)$ are relatively co-prime, then the rational function $v(z)$ is positive real iff

   (i) $v(z)$ is analytic on $|z| > 1$ or $a(z)$ has all its zeros in $|z| < 1$.

   (ii) $v(z) + v(z^{-1}) > 0$ on $|z| = 1$ or

   $$a(z)b(z^{-1}) + a(z^{-1})b(z) > 0$$

   on $|z| = 1$.

Condition (i) corresponds to the stability of the modeling filter, whereas condition (ii) is equivalent to $v(z)$ being pseudo-positive real $Re(v(z)) > 0$ on the unit circle [17].

The solutions of the partial deterministic realization problem only have to satisfy the interpolation conditions at ∞, i.e. Condition 1.

[1] When $\mathcal{T}(C_n) \geq 0$ and $\det \mathcal{T}(C_n) = 0$, there exists a unique positive real solution. In this case, C_n is called singularly nonnegative.

2.2 Kimura-Georgiou Parameterization

Based on the covariance sequence C_N, the Szegő polynomials of the first and second kind can be calculated:

$$\begin{aligned} \varphi_t(z) &= z^t + \varphi_{t1} z^{t-1} + \ldots + \varphi_{tt}, & (2) \\ \psi_t(z) &= z^t + \psi_{t1} z^{t-1} + \ldots + \psi_{tt}. & (3) \end{aligned}$$

These polynomials are calculated by the recursions:

$$\begin{aligned} \varphi_{t+1} &= z\varphi_t(z) - \gamma_t \varphi_t^*(z), \\ & \qquad t = 0, 1, \ldots, N-1, \quad \varphi_0(z) = 1, \\ \varphi_{t+1}^* &= z\varphi_t^*(z) - \gamma_t \varphi_t(z), \\ & \qquad t = 0, 1, \ldots, N-1, \quad \varphi_0^*(z) = 1, \end{aligned}$$

and

$$\begin{aligned} \psi_{t+1} &= z\psi_t(z) + \gamma_t \psi_t^*(z), \\ & \qquad t = 0, 1, \ldots, N-1, \quad \psi_0(z) = 1, \\ \psi_{t+1}^* &= z\psi_t^*(z) + \gamma_t \psi_t(z), \\ & \qquad t = 0, 1, \ldots, N-1, \quad \psi_0^*(z) = 1, \end{aligned}$$

with the reversed polynomials $\psi_t^*(z) = z^t \psi_t(1/z)$ and $\varphi_t^*(z) = z^t \varphi_t(1/z)$. The Schur parameters are defined by:

$$\begin{aligned} \gamma_t &= \frac{1}{r_t} \sum_{i=0}^{t} \varphi_{t,t-i} c_{i+1}, \qquad t = 0, 1, 2, \ldots, N-1, \qquad (4) \\ r_{t+1} &= (1 - \gamma_t^2) r_t, \qquad t = 0, 1, 2, \ldots, N-1;\ r_0 = 1. \end{aligned}$$

The well known Maximum Entropy solution [26] constructed from the Nth order polynomials satisfies the interpolation condition at ∞:

$$\frac{1}{2} \frac{\psi_N(z)}{\varphi_N(z)} = \frac{1}{2} + c_1 z^{-1} + c_2 z^{-2} + \ldots + c_N z^{-N} + \ldots$$

and is an all pole realization:

$$w(z) = \frac{\sqrt{r_N} z^N}{\varphi_N(z)}.$$

However, in many applications finite spectral zeros are desired [5, 7]. Based on the Szegő polynomials, all solutions (up to order N) of the interpolation problem (1) are described in the Kimura-Georgiou parameterization [8, 14]:

$$v(z) = \frac{1}{2} \frac{\psi_N(z) + \alpha_1 \psi_{N-1}(z) + \ldots + \alpha_N \psi_0(z)}{\varphi_N(z) + \alpha_1 \varphi_{N-1}(z) + \ldots + \alpha_N \varphi_0(z)}. \qquad (5)$$

Notice that the elements of this parameterization only satisfy the interpolation conditions (1). The set of all positive real solutions still has to be searched for in this set by checking conditions (i) and (ii) for each possible choice of parameters $\{\alpha_i\}_{i=1}^N$. Condition (i) can be checked by some well known stability tests (e.g. the Jury and Schur-Cohn Criterion [1]). In [2] condition (ii) was translated to conditions on the signature of a Hankel matrix. In [9] condition (ii) is checked by repeatedly applying the Schur-Cohn algorithm. In this parameterization, it is not guaranteed that $a(z)$ and $b(z)$ are relatively prime and an additional check is needed. This can e.g. be done by checking whether the resultant of $a(z)$ and $b(z)$ vanishes [12].

An alternative parameterization can be based on the result of [3, 4]. For each choice of a Schur polynomial $\sigma(z)$, there exists just one positive real realization, i.e., there exists just one corresponding $a(z)$, which is the solution of a convex optimization problem. All solutions (up to order N) are then described by parameterizing the Schur polynomials $\sigma(z)$ and calculating the corresponding $a(z)$.

2.3 Algebraic and Positive Degree

The notion of minimality is an important property as well for the solutions of the partial deterministic as for the solutions of the partial stochastic realization problem. The minimal degree of all deterministic realizations is called the McMillan degree of the covariance sequence and is denoted by δ_a. In [4] this is also called the *algebraic degree*.

The minimal degree of all stochastic realizations of a finite covariance sequence is called the *positive degree* [4] and is denoted by δ^+. The following constraints hold for the positive degree:

$$\delta_a \leq \delta^+ \leq N. \qquad (6)$$

Obviously, each solution of the partial stochastic realization problem is also a solution of the corresponding partial deterministic realization problem. The second inequality holds since the degree of the Maximum Entropy solution is not larger than N.

3 Parameterization for the Interpolation Problem

In this Section, the parameterization [20, 21, 23] of the solutions of the interpolation problem is presented for interpolation conditions all taken at ∞. In subsection 3.1, we introduce some extra notation and the (equivalent) Linearized Rational Interpolation Problem. The parameterization is described in subsection 3.2. Based on this parameterization, some properties of realization theory are derived. A main result is that a lower bound for the positive degree δ^+ is easily obtained. This is done in subsection 3.3. More details can be found in [23].

3.1 Notation

The numerator $n_p(z)$ and the denominator $d_p(z)$ of

$$v(z) = \frac{n_p(z)}{d_p(z)}$$

are combined in the ordered *polynomial couple* $p(z) = (n_p(z), d_p(z)) \neq (0, 0)$. Two rational expressions or polynomial couples $(n_{p1}(z), d_{p1}(z))$ and $(n_{p2}(z), d_{p2}(z))$ are called *equivalent* if

$$n_{p1}(z) d_{p2}(z) = n_{p2}(z) d_{p1}(z).$$

The degree δ of the rational expression $(n_p(z), d_p(z))$ is defined as $\delta = \max\{\deg n_p(z), \deg d_p(z)\}$.

We say that the polynomial couple $p(z) = (n_p(z), d_p(z))$ satisfies the first l interpolation conditions of the Proper Rational Interpolation Problem (PRIP) (1) iff:

$$\frac{n_p(z)}{d_p(z)} = F(z) + O_-(z^{-l}), \quad z \to \infty. \tag{7}$$

The couple $p(z) = (n_p(z), d_p(z))$ with $\delta \geq \deg d_p(z)$ satisfies the first l interpolation conditions of the Linearized Rational Interpolation (LRIP) problem iff:

$$n_p(z) - d_p(z)F(z) = O_-(z^{-l+\delta}), \quad z \to \infty. \tag{8}$$

The entity δ chosen in (8) is called the *potential degree* of the polynomial couple $p(z) = (n_p(z), d_p(z))$. The extra flexibility introduced by choosing $\delta \geq \deg d_p(z)$ will be used in subsection 3.2. We call the highest potential degree coefficients of $p(z)$ the coefficients of $n_p(z)$ and $d_p(z)$ with degree equal to the potential degree δ. Also note that for $l > \delta - \deg d_p(z)$, $\deg d_p(z) = \deg n_p(z) = \deg p(z)$. If $\delta = \deg d_p(z)$, then the PRIP (7) follows from the LRIP (8). Given the PRIP (7), the LRIP (8) follows with $\delta \geq \deg d_p(z)$.

The *vector space* of all polynomial couples with degree not greater than δ and satisfying the first l interpolation conditions (8) is denoted by $S_{l,\delta}$. The following important relations hold between various "neighboring" vector spaces of $S_{l,\delta}$:

$$\begin{aligned} S_{l,\delta} &\subseteq S_{l,\delta+1} \\ S_{l+1,\delta} &\subseteq S_{l,\delta} \\ S_{l,\delta} &\subseteq S_{l+1,\delta+1} \end{aligned}$$

We also have

$$\{p(z) \in S_{l+1,\delta+1} | \text{ highest pot. deg. coeff. of } p(z) \text{ are zero}\} = S_{l,\delta}. \tag{9}$$

The *residual* of the polynomial couple $p(z)$ with potential degree δ satisfying the first l interpolation conditions at the $(l+1)$th interpolation condition is denoted by $\mathcal{R}^{\delta}_{l+1}p(z)$ and is given by:

$$z^{-l}\mathcal{R}^{\delta}_{l+1}p(z) = \Pi^{-l}_{-l}(z^{-\delta}n_p(z) - z^{-\delta}F(z)d_p(z)). \tag{10}$$

For two polynomial couples $u(z) \in S_{l-1,\delta_u}$, $w(z) \in S_{l-1,\delta_w}$ and two polynomials $x(z)$, $y(z)$, we have that the residual of a linear combination is equal to

$$R^{\delta}_l(x(z)u(z) + y(z)w(z)) = x_h R^{\delta_u}_l u(z) + y_h R^{\delta_w}_l w(z), \tag{11}$$

if the potential degree δ of the linear combination is equal to

$$\begin{aligned} \delta &= \deg x(z) + \text{potdeg } u(z) \\ &= \deg y(z) + \text{potdeg } w(z), \end{aligned}$$

with x_h and y_h being the highest degree coefficients of the polynomials $x(z)$ and $y(z)$ respectively. From the above definitions it is easy to see that a polynomial couple $p(z) \in S_{l,\delta}$ iff

- the residuals $R^{\delta}_i p$ are equal to zero for $i = 1, \ldots, l$
- the degree of $p(z)$ is smaller than or equal to the potential degree δ.

The last notation that is needed is that of a *set of polynomial couples*: given a $\delta \in \mathbb{N}$ and polynomial couple $p(z)$ with *potential* degree δ_p, the set of polynomial couples $\{p(z)\}^{\delta}$ is defined as follows:

$$\begin{aligned} \{p(z)\}^{\delta} &= \emptyset, \\ &\quad \text{if} \quad \delta < \delta_p, \end{aligned} \tag{12}$$

$$\begin{aligned} \{p(z)\}^{\delta} &= \{p(z), zp(z), \ldots, z^{\delta-\delta_p}p(z)\}, \\ &\quad \text{if} \quad \delta \geq \delta_p. \end{aligned} \tag{13}$$

3.2 Parameterization

A parameterization for all solutions of the LRIP (8) is given in Theorem 1. The parameterization for the corresponding PRIP (7) is described by Property 2.

Theorem 1 *[21, 23] There exist two polynomial couples $u_l(z)$ and $w_l(z)$:*

$$\begin{aligned} u_l(z) &= (n_{l,u}(z), d_{l,u}(z)) \\ w_l(z) &= (n_{l,w}(z), d_{l,w}(z)), \end{aligned}$$

with potential degree $\delta_{l,u}$ and $\delta_{l,w}$ respectively, such that for each $\delta < \infty$, a basis $BS_{l,\delta}$ for the vector space $S_{l,\delta}$ is given by

$$BS_{l,\delta} = \{u_l(z)\}^{\delta} \cup \{w_l(z)\}^{\delta}.$$

A proof by induction can be found in [23]. The Algorithm to construct the polynomial couples $u_{N+1}(z)$ and $w_{N+1}(z)$ with corresponding potential degrees $\delta_{N+1,u}$ and $\delta_{N+1,w}$ is given in Algorithm 1. In the sequel of the paper, the ordering of Algorithm 1 is assumed, i.e. $u_l(z)$ is the polynomial couple with lowest potential degree.

Algorithm 1 *Construction of a basis for the Linearized Rational Interpolation Problem*

Initialization

$$\begin{bmatrix} u_0(z) & w_0(z) \\ \delta_{0,u} & \delta_{0,u} \end{bmatrix} = \begin{bmatrix} n_{0,u}(z) & n_{0,w}(z) \\ d_{0,u}(z) & d_{0,w}(z) \\ \delta_{0,u} & \delta_{0,w} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

for $l = 0, 1, \ldots, N$,

if $R^{\delta_{l,u}}_{l+1} u_l = 0$

$$U_{l+1}(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$[\delta_{l+1,u} \quad \delta_{l+1,w}] = [\delta_{l,u} \quad \delta_{l,w}+1]$
else
$U'_{l+1}(z) = \begin{bmatrix} -hz^{\delta_{l,w}-\delta_{l,u}} & 1 \\ 1 & 0 \end{bmatrix}$
with $h = R_{l+1}^{\delta_{l,w}} / R_{l+1}^{\delta_{l,u}}$
$[\delta_{l+1,u'} \quad \delta_{l+1,w'}] = [\delta_{l,w} \quad \delta_{l,u}+1]$
if $\delta_{l+1,u'} \leq \delta_{l+1,w'}$
$U_{l+1}(z) = U'_{l+1}(z)$
$[\delta_{l+1,u} \quad \delta_{l+1,w}] = [\delta_{l+1,u'} \quad \delta_{l+1,w'}]$
else
$U_{l+1}(z) = U'_{l+1}(z) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
$[\delta_{l+1,u} \quad \delta_{l+1,w}] = [\delta_{l+1,u'} \quad \delta_{l+1,w'}] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
endif
endif
endfor

$$\begin{bmatrix} n_{N+1,u}(z) & n_{N+1,w}(z) \\ d_{N+1,u}(z) & d_{N+1,w}(z) \end{bmatrix} = U_0(z) \cdot U_1(z) \cdots U_{N+1}(z).$$

Based on the construction of $u_l(z)$ and $w_l(z)$ in Algorithm 1, the following important properties concerning the minimality of $u_l(z) = (n_{l,u}(z), d_{l,u}(z))$ and $w_l(z) = (n_{l,w}(z), d_{l,w}(z))$ hold [21]:

Property 1 *Both polynomial couples are co-prime, i.e., $n_{l,u}(z)$ and $d_{l,u}(z)$ have no common factors and $n_{l,w}(z)$ and $d_{l,w}(z)$ have no common factors.*

Property 2 *The parameterization of all solutions of the PRIP (7) with degree δ satisfying the first l interpolation conditions is given by*

$$p_{x,y}(z) = x(z)u_l(z) + y(z)w_l(z), \tag{14}$$

with the following restrictions on the polynomial parameters $x(z)$ and $y(z)$:

$$\deg x(z) \leq \delta - \delta_{l,u}, \tag{15}$$
$$\deg y(z) \leq \delta - \delta_{l,w}, \tag{16}$$
$$\deg p_{x,y}(z) = \delta. \tag{17}$$

The last condition (17) means that the linear combination $(x(z), y(z))$ may not result in a zero highest degree coefficient. Two solutions of the rational interpolation problem p_{x_1,y_1} and p_{x_2,y_2} are equivalent iff $(x_1(z), y_1(z)) = k(x_2(z), y_2(z))$.

The proof follows from the parameterization of the LRIP (8) and the equivalence with the PRIP (7). The two rational expressions $p_{x_1,y_1}(z) = (n_{x_1,y_1}(z), d_{x_1,y_1}(z))$ and $p_{x_2,y_2}(z) = (n_{x_2,y_2}(z), d_{x_2,y_2}(z))$ are equivalent iff

$$\det \begin{bmatrix} n_{x_1,y_1}(z) & n_{x_2,y_2}(z) \\ d_{x_1,y_1}(z) & d_{x_2,y_2}(z) \end{bmatrix} = 0 \tag{18}$$

$$\Leftrightarrow \det \begin{bmatrix} x_1(z) & x_2(z) \\ y_1(z) & y_2(z) \end{bmatrix} = 0,$$

$$\text{since} \quad \det \begin{bmatrix} n_{l,u}(z) & n_{l,w}(z) \\ d_{l,u}(z) & d_{l,w}(z) \end{bmatrix} \neq 0.$$

From the minimality of $p_{x_1,y_1}(z)$ and $p_{x_2,y_2}(z)$, both $x_1(z)$ and $y_1(z)$ have no nontrivial common factors and neither have $x_2(z)$ and $y_2(z)$. Hence, $(x_1(z), y_1(z)) = k(x_2(z), y_2(z))$, with k a constant and not a polynomial.

3.3 Properties

The two following properties from deterministic realization theory are now easy to prove.

Property 3 *If the degree of $u_{N+1}(z)$ is equal to the potential degree, then the McMillan degree or algebraic degree is this potential degree $\delta_a = \delta_{N+1,u}$. In the other case, $\delta_a = \delta_{N+1,w}$.*

The question when the solution with algebraic degree δ_a is unique was solved in [18] by imposing a rank condition on a finite Hankel matrix, formed with the elements of the covariance sequence. In the above parameterization, the condition becomes:

Property 4 *There exists only one solution with the algebraic degree δ_a iff $\delta_a = \delta_{N+1,u} < \delta_{N+1,w}$.*

The McMillan or algebraic degree of a polynomial couple satisfying the interpolation conditions of the covariance sequence C_N is given by Property 3. However this realization is not guaranteed to be positive real. Therefore, the McMillan degree δ_a is not necessarily equal to the positive degree δ^+. The following relations hold:

Property 5 *Based on the properties of the interpolation conditions, the following relations can be derived with respect to the algebraic and positive degree:*

(a) The algebraic degree δ_a of all solutions is given by Property 3. Iff the corresponding solution is positive real, the positive degree is equal to this algebraic degree.

(b) If $\delta_a = \deg u(z)$ and $u(z)$ is not positive real, a lower bound for δ^+ is $\delta_{N+1,w}$.

(c) If $\delta_a = \deg w(z)$, the lower bound for δ^+ remains δ_a.

Part (a) is trivial. Part (b) follows from the parameterization for the partial realization problem given by Property 2.

From inequalities (15)-(16) it follows that the minimal degree for a solution different from $u(z)$ is $\delta_{N+1,w}$. Hence, since $u(z)$ is not positive real, a lower bound for δ^+ is $\delta_{N+1,w}$. Part (c) is easily proven by similar reasoning.

Combination of parts (b) and (c) with Property 4 leads to the following interesting insight. When there exists only one solution (Property 4) for the interpolation problem, the next degree that is possible is equal to $\delta_{N+1,w}$. Hence, when the solution with the McMillan degree is not positive real, the minimal solution of the partial stochastic realization problem should at least have degree $\delta_{N+1,w}$. The number of free parameters is equal to $\delta_{N+1,w} - \delta_a + 1$.

Besides the lower bound for the positive degree, an advantage of the parameterization of Property 2 is that the search for the positive degree can be started from the lower bound. Especially for large covariance sequences, this has substantial benefits with respect to the Kimura-Georgiou parameterization (5). Another advantage is that for non-trivial choices of $x(z)$ and $y(z)$, $n_{x,y}(z)$ and $d_{x,y}(z)$ are relatively prime by Property 2, which has to checked in the Kimura-Georgiou parameterization. Finally we mention that this parameterization also allows for solutions with degree higher than N. An algebraic description of all solutions for the partial stochastic realization problem is given in Algorithm 2.

Algorithm 2 *Algebraic description of the rational solutions of the covariance extension problem*

Step 0 *Check if* $\mathcal{T}(C_N) \geq 0$, *iff not* Stop.

Step 1 *Calculate the polynomial couples* $u_{N+1}(z)$, $w_{N+1}(z)$ *with their potential degrees* $\delta_{N+1,u}$, $\delta_{N+1,w}$ *(Algorithm 1).*
Calculate the algebraic degree δ_a *based on Property 3.*

Step 2 *For each* δ, $\delta_a \leq \delta$, *for each possible choice of the polynomials* $x(z)$ *and* $y(z)$ *in (14) of Property 2:*

Step 2a *Check if* $a(z) = p_d(z)$ *is stable (e.g. Schur-Cohn Criterion).*

Step 2b *Check if* $p(z)$ *is pseudo-positive real, [2, 9].*

4 Example

Let us start from the Laurent series [4, 6] of $v(z) = \frac{1}{2}\frac{z+1+\epsilon}{z+1-\epsilon}$, with $\epsilon > 0$. By an appropriate (small) choice of $\epsilon > 0$, N can be made arbitrarily large such that $\mathcal{T}(C_N) > 0$ and $\mathcal{T}(C_{N+1}) \not> 0$, since $v(z)$ has a zero outside the unit disc. The corresponding partial covariance sequence is given by

$$C_N = \{\frac{1}{2}, c_1, \ldots, c_N\},$$

with $c_i = \epsilon(-1)^{i+1}(1-\epsilon)^{i-1}$ for $i \geq 1$. The corresponding solutions of Algorithm 1 for $N \geq 2$ are

$$\begin{bmatrix} n_u(z) & n_w(z) \\ d_u(z) & d_w(z) \\ \delta_u & \delta_w \end{bmatrix} = \begin{bmatrix} z+1+\epsilon & z+2\epsilon \\ 2(z+1-\epsilon) & 2z \\ 1 & N \end{bmatrix}.$$

This means that for $N \geq 2$ and $\mathcal{T}(C_N) > 0$, the positive degree $\delta^+ = N$. For example, for $\epsilon = 0.1$, $C_{15} = \{1, 0.1, -0.09, 0.081, -0.0729, 0.0656, -0.0590, 0.0531, -0.0478, 0.0430, -0.0387, 0.0349, -0.0314, 0.0282, -0.0254, 0.0229\}$ with $\mathcal{T}(C_{15}) > 0$, while $\mathcal{T}(C_{16}) \not> 0$. The solutions of Algorithm 1 for C_{15} are:

$$\begin{bmatrix} n_u(z) & n_w(z) \\ d_u(z) & d_w(z) \\ \delta_u & \delta_w \end{bmatrix} = \begin{bmatrix} 10z+11 & 5z+1 \\ 2(10z+9) & 10z \\ 1 & 15 \end{bmatrix}.$$

Hence, since $u(z)$ is not positive real, the positive degree $\delta^+ = 15$. Also note that as an intermediate result, $u_3(z) = (n_u(z), d_u(z))$ and $w_3(z) = (n_w(z), d_w(z))$ with potential degrees $\delta_{3,u} = 1$ and $\delta_{3,w} = 2$ respectively, are obtained. This solution corresponds to the "deterministic" system that generated the valid partial covariance sequence. For all next interpolation points $R_{l+1}^{\delta_{l,u}} u_l = 0$ for $l = 3, 4, \ldots$ and the potential degree δ_w is increased by one, resulting into the solution (19). For this example, the lower bound of Property 5 is strict. A positive real solution with minimal degree is e.g. the Maximum Entropy solution.

5 Conclusions

An alternative parameterization [20, 21, 23] is presented to parameterize the (deterministic) solutions of the partial realization problem. Minimality of the solutions parameterized by this solution is satisfied. This implies that he minimal degree of an interpolating solution is easily obtained. This degree is called the McMillan or algebraic degree. From the structure of the parameterization one can directly check whether the minimal solution is unique. A description of all solutions of the corresponding partial stochastic realization problem is obtained by checking the positive real constraints for all deterministic solutions. When compared to the Kimura-Georgiou parameterization, we now obtain a non-trivial lower bound for the minimal degree of an interpolating stochastic solution. This degree is the so-called positive degree. The results are illustrated by means of an example.

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