ELSEVIER

Systems & Control Letters 46 (2002) 265–270

SYSTEMS & CONTROL LETTERS

www.elsevier.com/locate/sysconle

Subspace angles between ARMA models

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Received 23 January 2001; received in revised form 11 February 2002; accepted 27 February 2002

Abstract

We define a notion of subspace angles between two linear, autoregressive moving average, single-input–single-output models by considering the principal angles between subspaces that are derived from these models. We show how a recently defined metric for these models, which is based on their cepstra, relates to the subspace angles between the models. © 2002 Elsevier Science B.V. All rights reserved.

Keywords: Principal angles between subspaces; ARMA models; Linear systems; Stochastic realization; Distance measure; Time series; Cepstrum

1. Introduction

The concept of principal angles between subspaces of linear vector spaces is due to Jordan [9] in the 19th century. This notion was translated by Hotelling [8] into the statistical quantities of canonical correlations, which are widely applied (see e.g. [5]). In the area of systems and control, the principal angles between two subspaces are used in subspace identification methods [13] and also in model updating [3] and damage location [4]. In the latter two applications, one starts from a finite element model and measurements of a certain mechanical structure and one tries to find the subset of parameters of the model that should be adapted to explain the measurements, which is done by computing the principal angles between a certain measurement space and the parameterized space. In that way, damage to the structure can be located. The subspace-based fault detection algorithm of Basseville et al. [1], on the other hand, is based on linear dynamical models, the type of models that we deal with. Changes in the eigenmodes of the observed system are determined by monitoring the difference between the column spaces of the observability matrix of the nominal linear dynamical model and the observability matrix of the model that can be identified from the measurements. The difference between the column spaces can be quantified by the principal angles between the subspaces. As will become clear in Section 4, these are the angles that we will define as the subspace angles between two autoregressive (AR) models. A generalization to the subspace angles between autoregressive moving average (ARMA) models, which also take into account the zeros of the models, is given in Section 5. Furthermore, we show how the subspace angles between two ARMA models are related to the cepstral metric for ARMA models defined by Martin [11]. The statistical properties and the applicability of these concepts are topics of future research.

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PII: S0167-6911(02)00135-4

The paper is organized as follows. In Section 2, we briefly recall the definition of principal angles between and corresponding principal directions in two subspaces. In Section 3, we discuss a cepstral distance measure for ARMA models that has recently been defined by Martin [11]. Our definition of the subspace angles between AR models and their relation to the distance measure of Martin is given in Section 4. In Section 5, the definition of subspace angles is extended to the ARMA model class. Finally, in Section 6 we give the conclusions and point out possible further developments of our work.

2. Principal angles between subspaces

In this section we discuss the notion of principal angles between and principal directions in two subspaces. We start with the definition in Section 2.1, and in Section 2.2 we show how the angles and directions can be characterized by solving a generalized eigenvalue problem.

2.1. Definition

Let $A \in \mathbb{R}^{m\times p}$ and $B \in \mathbb{R}^{m\times q}$ be given real matrices with the same number of rows and assume for convenience that A and B have full column rank and that $p \geqslant q$. We denote the range (column space) of a matrix A by range(A).

Definition 1. The q principal angles $\theta_k \in [0, \pi/2]$, between range(A) and range(B) and the corresponding principal directions Ax_k and By_k in range(A), respectively range(B), are recursively defined for $k = 1, 2, \ldots, q$ as

$$\cos\theta_1 = \max_{\substack{x\in\mathbb{R}^p\\ y\in\mathbb{R}^q}} \frac{|x^{\mathrm{T}}A^{\mathrm{T}}By|}{\|Ax\|_2\|By\|_2} = \frac{|x_1^{\mathrm{T}}A^{\mathrm{T}}By_1|}{\|Ax_1\|_2\|By_1\|_2},$$

$$\cos\theta_k = \max_{\substack{x\in\mathbb{R}^p\\ y\in\mathbb{R}^q}} \frac{|x^{\mathrm{T}}A^{\mathrm{T}}By|}{\|Ax\|_2\|By\|_2} = \frac{|x_k^{\mathrm{T}}A^{\mathrm{T}}By_k|}{\|Ax_k\|_2\|By_k\|_2}$$

for $k = 2, \ldots, q$

s.t. $x_i^{\mathrm{T}}A^{\mathrm{T}}Ax = 0$ and $y_i^{\mathrm{T}}B^{\mathrm{T}}By = 0$

for $i = 1, 2, \ldots, k-1$.

Note that the principal angles satisfy $0 \leqslant \theta_1 \leqslant \cdots \leqslant \theta_q \leqslant \pi/2$. Following the notation in [13], the ordered set of q principal angles between the ranges of the matrices A and B is denoted as

$$(\theta_1, \theta_2, \ldots, \theta_q) = [A \measuredangle B].$$

2.2. The principal angles and directions as the solution of a generalized eigenvalue problem

It can be shown (see e.g. [7]) that the principal angles and the principal directions between range(A) and range(B) follow from the symmetric generalized eigenvalue problem:

$$\begin{pmatrix} 0 & A^{\mathrm{T}}B \\ B^{\mathrm{T}}A & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A^{\mathrm{T}}A & 0 \\ 0 & B^{\mathrm{T}}B \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\lambda, \tag{1}$$

s.t. $x^{\mathrm{T}}A^{\mathrm{T}}Ax = 1$ and $y^{\mathrm{T}}B^{\mathrm{T}}By = 1$.

The link between the generalized eigenvalue problem in (1) and Definition 1 goes via the so-called variational characterization of the eigenvalue problem.

Assume again that $A \in \mathbb{R}^{m\times p}$, $B \in \mathbb{R}^{m\times q}$ and $p \geqslant q$ and that the $p+q$ (real) generalized eigenvalues λ_i are sorted in non-increasing order as $\lambda_1 \geqslant \cdots \geqslant \lambda_{p+q}$, then one can show that

$$\lambda_1 = \cos\theta_1, \ldots, \lambda_q = \cos\theta_q \geqslant 0, \tag{2a}$$

$$\lambda_{q+1} = \lambda_{q+2} = \cdots = \lambda_p = 0, \tag{2b}$$

$$\lambda_{p+1} = -\cos\theta_q, \ldots, \lambda_{p+q} = -\cos\theta_1. \tag{2c}$$

The vectors Ax_i and By_i, for $i = 1, \ldots, q$ where x_i and y_i satisfy (1) with $\lambda = \lambda_i$, are the principal directions corresponding to the principal angle θ_i.

Note 1: When considering the principal angles between equidimensional subspaces ($p = q$), Eq. (2b) does not come into play. The squared cosines of the principal angles between the ranges of A and B are then equal to the eigenvalues of $(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}B(B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}A$, or equivalently the largest p eigenvalues of $\Pi_A\Pi_B = (A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}})(B(B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}})$, where Π_A and Π_B are the orthogonal projectors into the range of A, respectively B.

The above-mentioned method for the computation of the principal angles and vectors based on the generalized eigenvalue decomposition, is given for theoretical purposes only (we will use the characterization in the proof of Theorem 4). Numerically stable methods to compute the principal angles and vectors via a singular value decomposition have been proposed in [2,7] and can also be found in [6].

3. A metric for the set of single-input–single-output ARMA models

In [11] Martin defines a new metric for the set of single-input–single-output (SISO) ARMA models, which is based on the cepstrum of the model. Further on in the paper we will show that this metric is related to the principal angles between specific subspaces derived from the ARMA models. For the sake of completeness, we repeat in this section some results that have been reported in [11].

We recall that the cepstrum $c(k),\ k\in\mathbb{Z}$ of a linear SISO model with transfer function $H(z)$ is the inverse z-transform of the logarithm of its spectrum:

$$\log P(z) = \log H(z)H(z^{-1}) = \sum_{k\in\mathbb{Z}} c(k)z^{-k}. \tag{3}$$

Let M_1 and M_2 be stable and minimum phase (i.e. all poles and zeros lie inside the unit circle) ARMA models with cepstrum $c_1(k)$ and $c_2(k),\ k\in\mathbb{Z}$, respectively.

Definition 2 (Martin [11]). The distance between M_1 and M_2 is defined as

$$d(M_1,M_2) = \sqrt{\sum_{k=0}^{\infty} k|c_1(k)-c_2(k)|^2}.$$

Martin subsequently shows that for stable AR models M_1 with order n_1 and poles α_i $(i=1,\dots,n_1)$ and M_2 with order n_2 and poles β_i $(i=1,\dots,n_2)$ the following equality holds:

$$d(M_1,M_2)^2 = \log \frac{\prod_{i=1}^{n_1}\prod_{j=1}^{n_2}|1-\bar{\alpha}_i\beta_j|^2}{\prod_{i,j=1}^{n_1}(1-\bar{\alpha}_i\alpha_j)\prod_{i,j=1}^{n_2}(1-\bar{\beta}_i\beta_j)}, \tag{4}$$

where $\bar{x}$ denotes the complex conjugate of $x\in\mathbb{C}$.

This equality basically follows from the expression that relates the cepstrum coefficients of an AR model to its poles (see e.g. [10; 12, p. 502]):

$$c_1(k) = \frac{1}{k}\sum_{i=1}^{n_1}\alpha_i^k \quad\text{and}\quad c_2(k) = \frac{1}{k}\sum_{i=1}^{n_2}\beta_i^k \quad \text{for } k>0.$$

As an example, if M_1 and M_2 are two first order stable AR models, their squared distance equals

$$d(M_1,M_2)^2 = \log\frac{(1-\alpha\beta)^2}{(1-\alpha^2)(1-\beta^2)} = \log\frac{1}{\cos^2\theta},$$

where θ is the angle between the vectors

$$(1 \quad \alpha \quad \alpha^2 \quad \cdots)\in\mathbb{R}^\infty$$

and

$$(1 \quad \beta \quad \beta^2 \quad \cdots)\in\mathbb{R}^\infty.$$

It will become apparent in Section 4.2 that for higher order models, the squared distance as defined by Martin [11] can be expressed as the logarithm of a product of $1/\cos^2\theta_i$ (see Theorem 4). The angles θ_i will be called the subspace angles between the models.

4. Subspace angles between AR models

In this section we start the discussion of our new concept of angles between models, by considering AR models. The definition of the subspace angles between two AR models is given in Section 4.1. In Section 4.2 we show how the subspace angles between two AR models are related to the cepstral distance of the models as defined in [11] (see also Definition 2).

For reasons of conciseness, we only consider AR models that have the same model order. The definitions can, however, be extended to models with distinct orders.

4.1. Definition

Let two stable and observable nth order AR models M_1 and M_2 be characterized in state space terms by their system matrix A_1 and A_2 and output matrix C_1 and C_2, respectively.

Their infinite observability matrix,

$$(C_i^{\mathrm{T}} \quad A_i^{\mathrm{T}}C_i^{\mathrm{T}} \quad A_i^{2^{\mathrm{T}}}C_i^{\mathrm{T}} \quad \cdots)^{\mathrm{T}}\in\mathbb{R}^{\infty\times n},$$

is denoted as $\mathcal{O}_\infty(M_i)$ for $i=1,2$.

Definition 3. We define the subspace angles between M_1 and M_2 as the principal angles between the ranges of their infinite observability matrices:

$$[M_1 \measuredangle M_2] = [\mathcal{O}_\infty(M_1) \measuredangle \mathcal{O}_\infty(M_2)]. \tag{5}$$

The existence of the subspace angles is guaranteed by the stability of the models. Indeed, the matrices $Q_{kl}=\mathcal{O}_\infty(M_k)^{\mathrm{T}}\mathcal{O}_\infty(M_l)$ $(k,l=1,2)$, which are needed in the generalized eigenvalue problem (1) for the computation of the angles, can be obtained by solving the Lyapunov equation

$$\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} = \begin{pmatrix} A_1^{\mathrm{T}} & 0 \\ 0 & A_2^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} + \begin{pmatrix} C_1^{\mathrm{T}} \\ C_2^{\mathrm{T}} \end{pmatrix} (C_1 \quad C_2),$$

the solution of which exists and is unique due to the stability of M_1 and M_2.

4.2. Relation of Martin's metric and the subspace angles between two AR models

The subspace angles between two AR models are related to the distance between AR models as defined in [11] (see Definition 2) in the following way.

Theorem 4. *For the stable and observable AR models M_1 and M_2 of order n,*

$$d(M_1,M_2)^2 = -\log \prod_{i=1}^{n} \cos^2\theta_i,$$

where $(\theta_1,\theta_2,\ldots,\theta_n)$ are the subspace angles between M_1 and M_2.

Proof. Assume M_1 has poles $\alpha_1,\ldots,\alpha_n$ and M_2 has poles $\beta_1,\ldots,\beta_n$. One can show that the subspaces range$(\mathcal{O}_\infty(M_1))$ and range$(\mathcal{O}_\infty(M_2))$ only depend on the poles of the corresponding AR model. More specifically, if the system matrices A_1 and A_2 are diagonalizable, then

$$\mathrm{range}(\mathcal{O}_\infty(M_1)) = \mathrm{range}(\Gamma_1),$$
$$\mathrm{range}(\mathcal{O}_\infty(M_2)) = \mathrm{range}(\Gamma_2), \tag{6}$$

where

$$\Gamma_1 = \begin{pmatrix} 1 & \cdots & 1 \\ \alpha_1 & \cdots & \alpha_n \\ \alpha_1^2 & \cdots & \alpha_n^2 \\ \vdots & & \vdots \end{pmatrix} \in \mathbb{C}^{\infty\times n} \quad \text{and}$$

$$\Gamma_2 = \begin{pmatrix} 1 & \cdots & 1 \\ \beta_1 & \cdots & \beta_n \\ \beta_1^2 & \cdots & \beta_n^2 \\ \vdots & & \vdots \end{pmatrix} \in \mathbb{C}^{\infty\times n}. \tag{7}$$

From Definition 3 and Note 1 it follows that the squared cosines of the subspace angles between M_1 and M_2 are equal to the n eigenvalues of $R_{11}^{-1}R_{12}R_{22}^{-1}R_{21}$, where $R_{kl}=\Gamma_k^{\mathrm{H}}\Gamma_l$ $(k,l=1,2)$ (the superscript $\cdot^{\mathrm{H}}$ denotes the complex conjugate transpose).

The product of the squared cosines of the subspace angles between M_1 and M_2 is therefore equal to

$$\prod_{i=1}^{n} \cos^2\theta_i = \det(R_{11}^{-1}R_{12}R_{22}^{-1}R_{21}) = \frac{\det R_{12}\det R_{21}}{\det R_{11}\det R_{22}}. \tag{8}$$

The elements of the matrices R_{kl} $(k,l=1,2)$ can be computed from the poles of the models M_1 and M_2 by applying $\sum_{k=0}^{\infty} x^k = 1/(1-x)$ for $|x|<1$. For R_{12}, e.g. one obtains

$$R_{12}(i,j) = \frac{1}{1-\bar{\alpha}_i\beta_j} \quad (i,j=1,\ldots,n), \tag{9}$$

where $R_{12}(i,j)$ is the element of the matrix R_{12} on the ith row and the jth column.

Computing the determinants of the matrices R_{kl} $(k,l=1,2)$ in (8), e.g. via the formula for the determinant of a Cauchy matrix (see Appendix A), leads to

$$-\log \prod_{i=1}^{n} \cos^2\theta_i = \log \frac{\prod_{i,j=1}^{n} |1-\bar{\alpha}_i\beta_j|^2}{\prod_{i,j=1}^{n}(1-\bar{\alpha}_i\alpha_j)(1-\bar{\beta}_i\beta_j)}. \tag{10}$$

The right-hand side of (10) is equal to the squared cepstral distance between M_1 and M_2 (see Eq. (4)).

Consequently,

$$-\log \prod_{i=1}^{n} \cos^2 \theta_i = d(M_1, M_2)^2. \qquad \square$$

5. Distance and angles between ARMA models

As mentioned in Section 3, Martin defined a metric, not only for AR models, but more generally for ARMA models [11]. On the basis of this definition (Definition 2) and a property of this metric that is given in Section 5.1, we define in Section 5.2 the subspace angles between two ARMA models.

5.1. A property of the metric

Since the cepstrum is the inverse z-transform of the logarithm of the spectrum (see Eq. (3)), the following property holds [11]:

$$d(H_1H_3, H_2H_3) = d(H_1, H_2),$$

where H_i is the transfer function of the ARMA model M_i for $i = 1, 2$ and H_3 is an arbitrary stable minimum phase transfer function. This implies that in order to compute the distance between ARMA models, it is sufficient to consider AR models. Indeed, for $H_1(z) = b_1(z)/a_1(z)$ and $H_2(z) = b_2(z)/a_2(z)$ of order n_1 and n_2, respectively, take $H_3(z) = z^{n_1+n_2}/b_1(z)b_2(z)$, so that

$$d\left(\frac{b_1(z)}{a_1(z)}, \frac{b_2(z)}{a_2(z)}\right) = d\left(\frac{z^{n_1+n_2}}{a_1(z)b_2(z)}, \frac{z^{n_1+n_2}}{a_2(z)b_1(z)}\right). \tag{11}$$

Because M_1 and M_2 are stable and minimum phase, the two resulting AR models in (11) are stable.

We now propose the following definition of the subspace angles between ARMA models.

5.2. Subspace angles between ARMA models

Let M_1 of order n_1 and M_2 of order n_2 be stable, minimum phase ARMA models with transfer function $H_1(z) = b_1(z)/a_1(z)$ and $H_2(z) = b_2(z)/a_2(z)$, respectively. Assume that the AR models with transfer function $z^{n_1+n_2}/a_1(z)b_2(z)$ and $z^{n_1+n_2}/a_2(z)b_1(z)$ are observable.

Definition 5. We define the subspace angles between M_1 and M_2 as the subspace angles between the AR models with transfer function $z^{n_1+n_2}/a_1(z)b_2(z)$ and $z^{n_1+n_2}/a_2(z)b_1(z)$, respectively.

Consequently, the $n_1 + n_2$ subspace angles between M_1 and M_2 are equal to the principal angles between the ranges of $(\mathcal{O}_\infty(M_1) \quad \mathcal{O}_\infty(M_2^{-1}))$ and $(\mathcal{O}_\infty(M_2) \quad \mathcal{O}_\infty(M_1^{-1}))$. Analogous to (6) and (7), the range of the observability matrix of the inverse model M^{-1} is only dependent on the zeros of M.

From Definition 5 and Eq. (11) it is clear that Theorem 4, which was given for AR models, is also valid for ARMA models.

Theorem 6. *For the stable and minimum phase ARMA models M_1 of order n_1 and M_2 of order n_2,*

$$d(M_1, M_2)^2 = -\log \prod_{i=1}^{n_1+n_2} \cos^2 \theta_i,$$

where $(\theta_1, \theta_2, \ldots, \theta_{n_1+n_2})$ are the subspace angles between the ARMA models M_1 and M_2.

6. Conclusions

In this paper we have proposed a definition for the subspace angles between two ARMA models and we have shown a relation between these angles and a recently defined distance measure for ARMA models [11].

In the near future, these new notions of distance and angles between models will be applied to several engineering applications, such as signal classification, fault detection, calculation of the so-called stabilization diagrams in vibrational analysis, etc.

Many questions remain to be tackled. Future developments will comprise the extension to multiple-input–multiple-output (MIMO) models and to deterministic systems. Furthermore, the apparent relation with the notion of mutual information will be explored.

Acknowledgements

Our research is supported by grants from several funding agencies and sources: Research Council

KUL: Concerted Research Action GOA-Mefisto 666 (Mathematical Engineering), IDO (Oncology, Genetic networks), several Ph.D./postdoc and fellow grants; Flemish Government: Fund for Scientific Research Flanders (several Ph.D./postdoc grants, projects G.0256.97 (subspace), G.0115.01 (bio-i and microarrays), G.0240.99 (multilinear algebra), G.0197.02 (power islands), G.0407.02 (support vector machines), research communities ICCoS, ANMMM), AWI (Bil. Int. Collaboration Hungary/Poland), IWT (Soft4s (softsensors), STWW-Genprom (gene promotor prediction), GBOU-McKnow (Knowledge management algorithms), Eureka-Impact (MPC-control), Eureka-FLiTE (flutter modeling)); Belgian Federal Government: DWTC (IUAP IV-02 (1996–2001) and IUAP V-10-29 (2002–2006): Dynamical Systems and Control: Computation, Identification & Modelling), Program Sustainable Development PODO-II (Traffic Management Systems); Direct contract research: Verhaert, Electrabel, Elia, Data4s, IPCOS.

Appendix A. Computing the determinant of the matrices R_{kl} ($k, l = 1, 2$)

The matrices R_{kl} $(k,l = 1,2)$ have a structure (see Eq. (9)) reminiscent to that of a Cauchy matrix, which has the form

$$C(i,j) = \frac{1}{x_i - y_j} \quad (i,j = 1,\ldots,n).$$

A formula for the determinant of a Cauchy matrix was found by Cauchy and can be proven by induction:

$$\det C = \frac{\prod_{i>j}^{n}(x_i - x_j)\prod_{i<j}^{n}(y_i - y_j)}{\prod_{i,j=1}^{n}(x_i - y_j)}. \tag{A.1}$$

The matrices R_{kl} $(k,l = 1,2)$ can be written as the product of a diagonal matrix D_k and a Cauchy matrix C_{kl}:

$$R_{kl} = D_k C_{kl}. \tag{A.2}$$

For R_{12}, e.g. these matrices are equal to

$$D_1 = \operatorname{diag}\left(\frac{1}{\bar{\alpha}_i}\right),$$

where $\operatorname{diag}(1/\bar{\alpha}_i)$ is the diagonal matrix with elements $1/\bar{\alpha}_1,\ldots,1/\bar{\alpha}_n$, and

$$C_{12}(i,j) = \frac{1}{(1/\bar{\alpha}_i) - \beta_j} \quad (i,j = 1,\ldots,n).$$

Substituting (A.2) into (8) gives

$$\prod_{i=1}^{n}\cos^2\theta_i = \frac{\det C_{12}\det C_{21}}{\det C_{11}\det C_{22}},$$

which becomes after applying Cauchy's formula (A.1)

$$\prod_{i=1}^{n}\cos^2\theta_i = \frac{\prod_{i,j=1}^{n}(1 - \bar{\alpha}_i\alpha_j)(1 - \bar{\beta}_i\beta_j)}{\prod_{i,j=1}^{n}|1 - \bar{\alpha}_i\beta_j|^2}.$$

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