ON FREQUENCY WEIGHTED BALANCED TRUNCATION: A CONSTRUCTIVE COUNTEREXAMPLE TO ENNS' CONJECTURE

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Abstract

The concept of frequency weighted balancing, as proposed by Enns, is a generalization of internally balanced model truncation. Internally balanced truncation is simple to apply and additionally attractive because of the existence of an upper H_∞ error bound that is a function of the neglected Hankel singular values. A conjecture on the generalization of this upper error bound for the case of frequency weighted balanced truncation was formulated by Enns, but the proof has not been found. In this paper, Enns' conjecture is refuted and it is shown that there does not exist a frequency weighted upper error bound that depends only on neglected frequency weighted Hankel singular values. It is explained that this is due to cross terms which appear in the frequency weighted error bound. However, these cross terms are inherent in the frequency weighted balancing technique proposed by Enns.

1 Introduction

Popular methods for model reduction are internally balanced truncation and optimal Hankel norm approximation [5]. Their main advantage apart from simplicity of application is that there exists an a priori lower and upper error bound based on the Hankel singular values of the full-order system. However, these model reduction techniques use a uniform weighting on the whole frequency range. In system identification it may be interesting to frequency weight the error in order to reduce the model truncation error in certain frequency ranges of interest [9, 14]. Frequency weighting has also been applied in order to enhance the robustness of the controller [2, 3, 16, 17]. Therefore, Enns introduced the concept of frequency weighted balancing [3, 4] as a generalization of frequency weighted balancing so as to take these frequency dependencies of the admissible truncation error into account.

However, the generalization of the a priori balanced model reduction upper error bound in terms of the so-called frequency weighted Hankel singular values, has not been found yet [1, 3, 7, 14, 15, 17]. In this paper we explain that certain types of generalizations of the upper bound, which we will call Enns' Conjecture [3] in Section 2, cannot serve as an upper error bound. It can be shown that this is due to a cross term that can become unbounded in terms of the frequency weighted Hankel singular values. This cross term is inherent in frequency weighted balancing. We give numerical counterexamples to Enns' conjecture using a constructive algorithm that generates counterexamples.

This paper is organized as follows. Frequency weighted balancing and Enns' conjecture are reviewed in Section 2. The conjecture is refuted in Section 3. This paper is a companion paper of [13].

2 Internally Balanced Model Reduction

Consider a stable, continuous time linear system of order n with transfer matrix $G(s) = C(sI \quad A)^{\ 1}B + D$ and with realization (A, B, C, D). The system is assumed to be minimal which means that the controllability Gramian $P = \lim_{t\to\infty} \int_0^t \exp(A\tau)BB^T \exp(A^T\tau)\, d\tau$ and the observability Gramian $Q = \lim_{t\to\infty} \int_0^t \exp(A^T\tau)C^T C \exp(A\tau)\, d\tau$ are positive definite. These Gramians can be computed by solving linear matrix equations. A similarity transformation T on $(A, B, C, D) \xrightarrow{T} (TAT^{\ 1}, TB, CT^{\ 1}, D)$ can be found that diagonalizes the Gramians in the corresponding contragredient transformation $(P, Q) \xrightarrow{T} (TPT^T, T^{\ T}QT^{\ 1}) = (\Sigma_n, \Sigma_n)$, while preserving the eigenvalues of $P \cdot Q = \Sigma_n^2$. The system $G(s)$ with realization $(TAT^{\ 1}, TB, CT^{\ 1}, D)$ is then called

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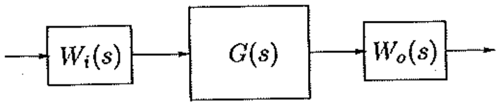

Figure 1: Series connection $W_o(s) \cdot G(s) \cdot W_i(s)$ of the input weighting $W_i(s)$, the system $G(s)$ and the output weighting $W_o(s)$ in order to perform frequency weighted balancing.

internally balanced with $P = Q = \Sigma_n = \text{diag}(\sigma_1, \ldots, \sigma_n)$ [5, 6, 10] The diagonal elements $\sigma_1, \ldots, \sigma_n$ are called the Hankel singular values and are ordered in a non-increasing order. Provided that $\sigma_{r+1} < \sigma_r$, the reduced order model $G_r(s) = C_1(sI \quad A_{11}) \;{}^{1}B_1 + D$ with order r is then obtained by truncating the partitioned balanced system

$$TAT \;{}^{1} = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right],$$

$$TB = \left[\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right], \quad CT \;{}^{1} = \left[\begin{array}{c|c} C_1 & C_2 \end{array}\right],$$

with $A_{11} \in \mathbb{R}^{r \times r}$. The reduced order system is stable [11] and there exists an a priori upper bound on the H_∞ error [3, 5]:

$$\begin{aligned} E_\infty &= \|E(j\omega)\|_\infty = \|G(j\omega) \quad G_r(j\omega)\|_\infty \\ &\leq 2 \sum_{k=r+1}^{n} \sigma_k. \end{aligned} \quad (1)$$

The concept of frequency weighted balanced truncation is a generalization of internally balanced truncation and was introduced by Enns [3, 4] in order to tune the approximation error in certain frequency ranges. Given both an input weighting filter $W_i(s) = C_i(sI \quad A_i) \;{}^{1}B_i + D_i$ and an output weighting filter $W_o(s) = C_o(sI \quad A_o) \;{}^{1}B_o + D_o$, the frequency weighting is obtained by making the series connection $W_o(s) \cdot G(s) \cdot W_i(s)$ of the input filter $W_i(s)$, the original system $G(s)$ and the output filter $W_o(s)$, as is depicted in Figure 1. By constructing the state space realizations of the augmented systems $G(s) \cdot W_i(s)$ and $W_o(s) \cdot G(s)$, respectively:

$$\bar{A}_i = \begin{bmatrix} A & BC_i \\ 0 & A_i \end{bmatrix}, \bar{B}_i = \begin{bmatrix} BD_i \\ B_i \end{bmatrix}, \bar{C}_i = \begin{bmatrix} C & 0 \end{bmatrix},$$

$$\bar{A}_o = \begin{bmatrix} A & 0 \\ B_oC & A_o \end{bmatrix}, \bar{B}_o = \begin{bmatrix} B \\ 0 \end{bmatrix}, \bar{C}_o = \begin{bmatrix} D_oC & C_o \end{bmatrix},$$

the extended Gramians

$$\bar{P}_i = \begin{bmatrix} P & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}$$

and

$$\bar{Q}_o = \begin{bmatrix} Q & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix}$$

are obtained as the solutions to the following Lyapunov equations:

$$\bar{A}_i\bar{P}_i + \bar{P}_i\bar{A}_i^T + \bar{B}_i\bar{B}_i^T = 0 \quad (2)$$

$$\bar{A}_o^T\bar{Q}_o + \bar{Q}_o\bar{A}_o + \bar{C}_o^T\bar{C}_o = 0, \quad (3)$$

respectively. The system $G(s)$ with realization (A, B, C, D) is called *frequency weighted balanced in the sense of Enns* (with respect to the input and output weighting transfer functions $W_i(s)$ and $W_o(s)$) iff the input and output frequency weighted Gramians, P and Q, are diagonal and equal: $P = Q = \Sigma_n = \text{diag}(\sigma_1, \ldots, \sigma_n)$. The values $\sigma_1, \ldots, \sigma_n$ are now called the frequency weighted Hankel singular values and are ordered in a non-increasing order. For a given realization, there exists a similarity transformation T that balances the system in the frequency weighted sense [3]. Motivated by the upper error bound (1) for the internally balanced model truncation error, Enns formulated the following conjecture about an upper error bound for the frequency weighted balanced model truncation error [3] (p. 105):

Conjecture 1 (Enns' Conjecture) *When truncating a frequency weighted balanced system, the infinity norm E_∞ of the weighted difference between the original system $G(s) = C(sI \quad A) \;{}^{1}B$ of order n and the reduced system $G_r(s) = C_1(sI \quad A_{11}) \;{}^{1}B_1$ of order r can be upper bounded by $2(1+\alpha)$ times the sum of the neglected weighted singular values:*

$$\begin{aligned} E_\infty &= \|W_o(j\omega)\,(G(j\omega) \quad G_r(j\omega))\,W_i(j\omega)\|_\infty \\ &\leq 2(1+\alpha) \sum_{k=r+1}^{n} \sigma_k, \end{aligned} \quad (4)$$

with $\alpha < 1$ when $E_\infty < 1$.

Observe that the condition $\alpha < 1$ in the conjecture can be omitted. Indeed, by scaling $G(s)$ with a factor $\lambda \in \mathbb{R}^+$, the model reduction algorithm returns $\lambda G_r(s)$ and $\lambda\sigma_i$ $(i = 1, \ldots, n)$, instead of $G_r(s)$ and σ_i. The error E_∞ is also scaled by a factor λ and accordingly there will be a value of λ for which the error will be less than one. Since α is independent of the scaling process, one can always find a scaling such that $\alpha < 1$. Were α to be zero, the bound would be equal to the bound for internally balanced truncation. In other words, α is introduced to extend the result of (1) to the frequency weighted case.

However, the conjecture has not been proven and no value for α has been reported [1, 3, 7, 14, 17]. In [7] a (conservative) upper bound was derived. This bound is not an a priori error bound and depends on the Hankel singular values, P_{12}, the weighting and the system.

Also note that stability is not guaranteed when non-constant input and output weightings are both present [12]. This is due to the cross terms $BC_iP_{12} + P_{12}^TC_i^TB^T$ and $Q_{12}B_oC + C^TB_o^TQ_{12}^T$ in the Lyapunov equations (2) and (3), which when one truncates the equation may result in $A_{11}\Sigma_r + \Sigma_rA_{11} \not\leq 0$. Of itself, this does not imply that A_{11} is unstable, but simply that, in contrast to the unweighted case, stability does not follow from a truncated Lyapunov equation. In [8], Lin and Chiu propose an alternative method of frequency weighted balancing, with guaranteed stability of the reduced order model. Stability is obtained by removing the cross terms by block diagonalizing the extended Gramians $\bar{P}_i$ and $\bar{Q}_o$. The Gramian

$\bar{P}_i$ is block diagonalizes via the contragredient transformation $\bar{P} \xrightarrow{\bar{T}} \bar{T}\bar{P}\bar{T}^T$:

$$\begin{bmatrix} P & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \xrightarrow{\bar{T}} \begin{bmatrix} I & X_i^T \\ 0 & I \end{bmatrix} \begin{bmatrix} P & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ X_i & I \end{bmatrix}$$
$$= \begin{bmatrix} P - P_{12}P_{22}^{-1}P_{12}^T & 0 \\ 0 & P_{22} \end{bmatrix}, \qquad (5)$$

with $X_i = -P_{22}^{-1}P_{12}^T$. The upper left block of (2) then becomes

$$P_iA^T + AP_i + BD_iD_i^TB^T + BD_iB_i^TX_i$$
$$+X_i^TB_iD_i^TB^T + X_i^TB_iB_i^TX_i = 0, \qquad (6)$$

with $P_i = P - P_{12}P_{22}^{-1}P_{12}^T$. A similar transformation can be applied to block diagonalize the extended output Gramian of the output weighted frequency Lyapunov equation (3), resulting in $Q \to Q - Q_{12}Q_{22}^{-1}Q_{12}^T = Q_o$.

The system is now called frequency weighted balanced in the sense of Lin and Chiu iff $P_i = Q_o = \Sigma_n$, with $\Sigma_n = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$ diagonal and $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$. Since we have [8, 12]

$$BD_iD_i^TB^T + BD_iB_i^TX_i + X_i^TB_iD_i^TB^T + X_i^TB_iB_i^TX_i$$
$$= (BD_i + X_i^TB_i)(BD_i + X_i^TB_i)^T \geq 0,$$

stability can be proven following [11].

3 Enns' Conjecture refuted

Enns' Conjecture is refuted by means of a constructive counterexample. Numerical counterexamples are given in Example 1. It is explained that the conjecture does not hold because of the cross terms that appear in the extended Lyapunov equations. These cross terms are inherent in frequency weighted balancing in the sense of Enns. Removing these cross terms like in frequency weighted balanced truncation in the sense of Lin and Chiu does not yield an upper error bound of the same type of (4).

A constructive counterexample to Enns' Conjecture
In the next Theorem, Enns' Conjecture is disproved; moreover, it is shown that there does not exists an α such that (4) holds for all possible systems and weightings.

Theorem 1 (Enns' Conjecture disproved) *Let $W_i(s) = C_i(sI - A_i)^{-1}B_i + D_i$ and $W_o(s) = C_o(sI - A_o)^{-1}B_o + D_o$ be stable, minimum phase transfer functions for the input and output weighting. Let the asymptotically stable system $G(s) = C(sI - A)^{-1}B + D$ be frequency weighted balanced with respect to the input and output weightings $W_i(s)$ and $W_o(s)$ and let the Gramians be given by $P = Q = \Sigma_n = \text{diag}(\sigma_1, \ldots, \sigma_r, \sigma_{r+1}, \ldots, \sigma_n)$. There exists no finite α such that a frequency error bound*

$$||W_o(s)\left(C(sI - A)^{-1}B - C_1(sI - A_{11})^{-1}B_1\right)W_i(s)||_\infty$$
$$\leq 2(1+\alpha)\sum_{k=r+1}^{n} \sigma_k \qquad (7)$$

holds for all possible weightings and all possible transfer functions.

In order to disproving Enns' Conjecture, it suffices to produce a counterexample for each $\alpha \in \mathbb{R}^+$. A counterexample is constructed for the case of a truncating a first order system $g(s) = c(s - a)^{-1}b$, input weighted with $w_i(s) = c_i(s - a_i)^{-1}b_i + d_i$, while no output weigting is applied, i.e., $w_o(s) = 1$. The model $g(s) = c(s - a)^{-1}b$ is a strictly proper, stable first order SISO system[1], while the input weighting $w_i(s) = c_i(s - a_i)^{-1}b_i + d_i$ is a stable, minimum phase first order SISO system. The reduced order model is $g_r(s) = 0$, since we do not introduce a feed-through term d, following the approach of [3]. The proof requires quite some algebra and we refer to [13] for the details. The outline of the proof is as follows. The Lyapunov equations (2) and (3) become:

$$2a\sigma + 2bc_ip_{12} + (bd_i)^2 = 0 \qquad (8)$$
$$(a + a_i)p_{12} + bc_ip_{22} + bd_ib_i = 0 \qquad (9)$$
$$2a_ip_{22} + b_i^2 = 0 \qquad (10)$$
$$2a\sigma + c^2 = 0. \qquad (11)$$

Following the approach of [3, 7], the H_∞ error

$$E_\infty = ||(g(j\omega) - 0) \cdot w_i(j\omega)||_\infty$$
$$= \sup_\omega \left(g(j\omega)w_i(j\omega)w_i(-j\omega)g(-j\omega)\right)^{\frac{1}{2}}$$

can be rewritten as follows [13]

$$E_\infty = \sup_\omega \left(\theta_{noFW}(j\omega) + \theta_{FW}(j\omega)\right)^{\frac{1}{2}}, \qquad (12)$$

where

$$\theta_{noFW}(j\omega) = \sigma^2(1 + \phi\phi^{-H})(1 + \phi^H\phi^{-1}), \qquad (13)$$
$$\theta_{FW}(j\omega) = -4\sigma^2\left(\frac{bc_ip_{12}}{\sigma a_i}\right)\left(\frac{a_i^2a^2 - a_ia\omega^2}{(\omega^2 + a_i^2)(\omega^2 + a^2)}\right), \qquad (14)$$

with $\phi = (j\omega - a)^{-1}$ and $\phi_i = (j\omega - a_i)^{-1}$. The term $\theta_{noFW}(j\omega)$ is equal to the expression for the error bound (1) obtained in [3]. In the case of frequency weighting an additional term $\theta_{noFW}(j\omega)$ appears in the expression for the error due to the cross terms in the Lyaponov equation (2). Evaluating both terms at $\omega = 0$, we will show that for each given $\alpha \in \mathbb{R}^+$, there exists a stable $g(s)$ and a stable minimum phase input weighting filter $w_i(s)$ such that the error $E_\infty > 2(1+\alpha)\sigma$ or

$$E_\infty^2 = \sup_\omega \left(\theta_{noFW}(j\omega) + \theta_{FW}(j\omega)\right)$$
$$> 4\left(1 + 2\alpha + \alpha^2\right)\sigma^2. \qquad (15)$$

Evaluated at $\omega = 0$, this corresponds to [13]

$$-4\left(\frac{bc_ip_{12}}{\sigma a_i}\right) > 8\alpha + 4\alpha^2. \qquad (16)$$

[1] In the remainder of this paper, we will assume $D = 0$, because D does not influence the balanced truncation error.

α	a_i	b_i	c_i	d_i	E_∞	$2(1+\alpha)\sigma$
0.1	-0.86	1.311	0.19	1.27	2.21	2.2
1	-0.06	0.510	0.19	1.27	4.08	4
10	-1.5e-03	0.116	0.19	1.27	22.6	22
100	-1.7e-05	0.013	0.19	1.27	207	202

Table 1: Counterexamples to Enns' conjecture for different values of α. For different values of α, an input weighting $w_i(s) = c_i(s \quad a_i)^{1} b_i + d_i$ is designed for the system $g(s) = \sqrt(2)(s \quad 1)^{1}$ using the constructive algorithm derived in the proof of Theorem 1. The corresponding E_∞ is reported in the 6th column, while the $2(1+\alpha)\sigma$ error bound (4) from Enns' conjecture is reported in the last column, refuting the conjecture. See Example 1 for details.

Given α, we now choose $a < 0$, b and σ and we calculate c from (11). By choosing

$$d_i^2 < \frac{2a\sigma}{b^2}, \tag{17}$$

a stable a_i can be found from [13]

$$\frac{1}{a_i}\left(\frac{b^2 d_i^2}{2\sigma} + a\right) > 2\alpha + \alpha^2. \tag{18}$$

Choosing $p_{12} = 1$ we obtain c_i from (8). Substituting $p_{22} > 0$ in (9) gives a quadratic equation in b_i with real solutions

$$\begin{aligned} b_i &= \frac{bd_i \pm \sqrt{(bd_i)^2 + 2\frac{bc_i p_{12}(a+a_i)}{a_i}}}{\frac{bc_i}{a_i}} \\ &= \frac{bd_i \pm \sqrt{\frac{a}{a_i}\left((bd_i)^2 + 2\sigma(a+a_i)\right)}}{\frac{bc_i}{a_i}}. \end{aligned} \tag{19}$$

The zero z_0 of $w_i(s)$ is given by:

$$\begin{aligned} z_0 &= a_i \quad \frac{c_i b_i}{d_i} \\ &= \not{d}_i \quad \not{d}_i \mp \frac{a_i}{d_i b}\sqrt{(bd_i)^2 + 2\frac{bc_i p_{12}(a+a_i)}{a_i}}. \end{aligned} \tag{20}$$

Hence, there exists always a minimum phase filter $w_i(s)$, by an appropriate choice of the sign in (19). We refer to [13] for all further details.

A practical algorithm to generate counterexamples consists of the following steps:

1. Select a given value for α for which one wants to refute the conjecture.
2. Choose $a < 0$, b and $\sigma > 0$. Determine c from (11), i.e.,
$$c = \sqrt{\quad 2a\sigma}.$$
3. Calculate
$$d_i = 0.9\sqrt{\quad 2\sigma a/b^2}$$
and
$$a_i = 0.95\frac{a + (b^2 di^2)/(2\sigma)}{2\alpha + \alpha^2}.$$
4. Calculate
$$c_i = \frac{b^2 di^2 \quad 2a\sigma}{2b}.$$
5. Evaluate the error $E(j\omega)$ at $\omega = 0$:
$$|E(0)| = |\frac{cb}{a}(\frac{c_i b_i}{a_i} + d_i)|.$$

The values 0.9 and 0.95 for the calculation of d_i and a_i are chosen smaller than 1 such that (17) and (18) hold. Also observe that the constructive algorithm is designed to generate counterexamples such that E_∞ is only a little bit larger than the upper error bound $2(1+\alpha)\sigma$ from (4). One can generate "stronger" counterexamples by choosing other values instead of 0.9 and 0.95. An alternative way to obtain such a "stronger" counterexample for given α is to choose, e.g., $\alpha^* = 2\alpha$ and generate the counterexample for α^* following the constructive algorithm.

One might well imagine that a relaxation or reformulation of the conjecture would be true. For a fairly broad relaxation, as the next theorem shows, this is not the case.

Theorem 2 *Under the same conditions for $W_i(s)$, $W_o(s)$ and $G(s)$ as in Theorem 1, there exists no upper error bound of the type*

$$\begin{aligned} &\|W_o(s) \quad C(sI \quad A)^{1}B \quad C_r(sI \quad A_r)^{1}B_r) W_i(s)\|_\infty \\ &\qquad \leq f(\sigma_{r+1}, \ldots, \sigma_n, C, A, B)\sum_{r+1}^{n}\sigma_i, \end{aligned} \tag{21}$$

with $f(x)$ depending only on x.

Indeed, the outline of the proof of Theorem 1 shows that when σ, a, b and c are fixed, there is still enough freedom in the choice of the other parameters making up the weight to make $E_\infty > 2(1+\alpha)\sigma$ for an arbitrary choice of α. The proof of this theorem then follows by choosing $\alpha = f(\sigma, a, b, c)/2 \quad 1$.

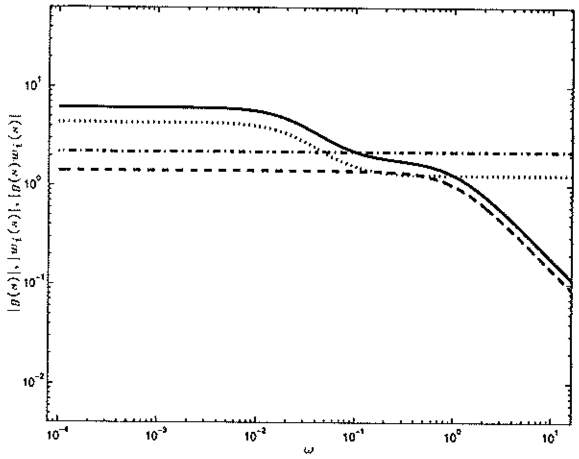

Figure 2: Evolution of the amplitude of the different transfer functions as a function of ω. The amplitudes of $|g(s)|$, $|w_i(s)|$ and $|g(s)w_i(s)|$ are denoted in dashed, dotted and full line, respectively. The upper error bound $2(1+\alpha)\sigma$ from Enns' Conjecture is denoted by the dashed-dotted line for $\alpha = 1$, while the example was generated using the constructive algorithm choosing $\alpha = 2$.

Example 1 Consider the system $g(s)$ with frequency weighted balanced realisation $(\;1, 1, \sqrt{2}, \cdot)$ and Hankel singular value $\sigma = 1$. In Table 1, some counterexamples to Enns' Conjecture are given for different values of α, i.e., $\alpha = 0.1, 1, 10$ and 100, respectively. These counterexamples are constructed following the constructive algorithm derived in the proof of Theorem 1 [13].

Also note that the error E_∞ for the third input weighting $W_i(s)$ with realisation $(\;0.0015, 0.1163, 0.1900, 1.2728)$ in Table 1, for $G(s)$ with realisation $(1/100000, 1, \sqrt{2}/100000, \cdot)$ is $E_\infty = 22.5669$, which is nearly 2σ, with $\sigma = 11.2763$. This illustrates that for a given $W_i(s)$, Enns' Conjecture is not violated for all choices of $G(s)$ as one would expect.

In Figure 2, a counterexample was generated for $\alpha^* = 2$ (yielding $a_i = \;0.0226, b_i = 0.3643, c_i = 0.1900$ and $d_i = 1.2728$) in order to refute the conjecture for a given $\alpha = 1$. The amplitude of the transfer functions $g(s), w_(s)$ and the error $w_i(s)g(s)$ are depicted as a function of ω by the dashed, dotted and full line, respectively. The corresponding upper error bound (4) from Enns' conjecture is depicted by the dashed-dotted line. For low frequencies, this example illustrates that the conjecture does not hold.

Remark 1 (Influence of the cross term) The "problem" in the frequency weighted error is the cross term θ_{FW} due to a non-zero P_{12} in (2). The upper bound [7] is also based on the error bound for the cross term. Stability of the reduced order model cannot be guaranteed in the case of both input and output weighting, due to a non-zero P_{12} and Q_{12} in (2) and (3), respectively.

However, it can be shown [13] that the condition $P_{12} = 0$ in (2) corresponds to

$$BW_i(s)W_i^T(\;s)B^T = BD_iD_i^TB^T, \tag{22}$$

meaning that $W_i(s) = W_{i1}(s) + W_{i2}(s)$, where $W_{i1}(s)W_{i1}^T(\;s) = D_iD_i^T$ and $BW_{i2}(s) = 0$. In other words, the input weighting $W_i(s)$ is composed of a first part which is constant on the frequency axis and a second part that is perpendicular to B. Since the second part does not contribute to the frequency weighting, it is equivalent to having zero weight. Unsurprisingly, the error bound (1) remains valid under input (and output) weighting with a constant matrix[2]. The input weighting with a constant matrix corresponds to taking linear combinations of the columns of the B matrix. Similar conditions can be derived for the output weighting filter.

Remark 2 (Error bounds after stability repair) In the previous Sections, it is explained that because of the cross terms that appear in Lyapunov equations (2) and (3), Enns' Conjecture does not hold and the reduced order model may become unstable. In contrast, whereas the stability problem is solved by frequency weighted balanced truncation in the sense of Lin and Chiu [8], the error formula $||W_o(s)\cdot(G(s)\;\;G_r(s))\cdot W_i(s)||_\infty$ is not simplified by applying the transform (5), see e.g. [12]. Since $\lambda_i(P\cdot Q) \geq \lambda_i((P\;\;P_{12}P_i^{\;1}P_{12}^T)\cdot(Q\;\;Q_{12}^TQ_o^TQ_{12}))$ (Lemma 3.1, [12]), the frequency weighted singular values in the sense of Lin and Chiu are not greater than the frequency weighted Hankel singular values in the sense of Enns. Hence, Enns' Conjecture does not hold when applying frequency weighted balanced truncation in the sense of Lin and Chiu. In the proof of Theorem 1, it is shown that $2(1+\alpha)\sigma$ cannot serve as an upper error bound for the truncation error, with $\sigma = \sqrt{pq}$ and the scalars p and q given by (2) and (3). Because of the stability repair, p is reduced by $p_{12}^2p_i^{\;1} > 0$. In other words, σ now equals $\sigma = \sqrt{(p\;\;p_{12}^2p^{\;1})q} \leq \sqrt{pq}$. Since the model truncation error $||g(s)w_i(s)||_\infty$ of not changed by the alternative, Enns' conjecture does not hold for the frequency weighted balanced model truncation error in the sense of Lin and Chiu.

4 Conclusions

Frequency weighted balanced model truncation is a generalization of internally balanced model truncation, where an a priori H_∞ upper error bound on the frequency response exists. This upper error bound is two times the sum of the neglected Hankel singular values [3, 5]. Although a conjecture was formulated by Enns about an error bound for the frequency weighted case, no a priori error bound based on the frequency weighted Hankel singular values has been found yet, as mentioned frequently

[2] For the sake of completeness, it is mentioned that the matrix may be multiplied with an all pass transfer function $Z(s)$: $D_iZ(s)Z^T(\;s)D_i^T = D_iID_i^T = D_iD_i^T$.

in the literature [1, 3, 7, 14, 17]. In this paper, Enns' conjecture is refuted and it is shown, by means of a constructive algorithm generating numerical counterexamples, that there does not exists an error bound depending only on the sum of the neglected frequency weighted Hankel singular values. This is due to a cross term, which is inherent to the frequency weighted balancing [13]. By removing the cross terms in the Lyapunov equations [8], stability of the reduced order model is guaranteed, but an upper error bound that only depends on the frequency weighted Hankel singular values cannot be derived.

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