Available at
www.ComputerScienceWeb.com
POWERED BY SCIENCE DIRECT

Signal Processing 83 (2003) 439–443

SIGNAL PROCESSING

www.elsevier.com/locate/sigpro

Short communication

On a cepstral norm for an ARMA model and the polar plot of the logarithm of its transfer function

Katrien De Cock[a,*,1], Bernard Hanzon[b], Bart De Moor[a,2]

[a] Department of Electrical Engineering (ESAT – SCD), Research Group SISTA, Katholieke Universiteit Leuven, Kasteelpark Arenberg 10, B-3001 Leuven-Heverlee, Belgium
[b] Centrum Voor Wiskunde en Informatica (CWI), P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

Received 11 April 2002

Abstract

In this paper, we show that a recently defined cepstral norm for ARMA models equals, up to a constant factor, the square root of the area enclosed by the polar plot of the logarithm of the transfer function.

Keywords: ARMA models; Cepstrum; Distance measure; HSH norm; Polar plot

1. Introduction

Cepstral analysis is used in a variety of applications such as speech processing, radar and sonar and fault detection in rotating machines. Another area in which cepstra show up is that of distance measures between models and/or signals. In order to quantify the distance between two stochastic processes, one usually relies on their second-order statistical properties only. For requirements of invariance with respect to the measurement scale, it is desirable that the distance is a function of the ratio between the spectra of the processes, i.e., of the difference between the cepstra [1]. One such a cepstral distance for ARMA models was defined in [4]. It has some nice system theoretical properties, such as its formulation in terms of the poles and zeros [4] and its relation to the principal angles between certain subspaces derived from the models of the stochastic processes [2].

In this paper, we show that this distance between two single input single output (SISO) ARMA models with respective transfer functions $G_1(z)$ and $G_2(z)$ has another beautiful interpretation as the area enclosed by the image of the unit circle produced by $\log G_1(z)/G_2(z)$. The proof of this equality is completely analogous to the one of Hanzon [3], who showed that the Hilbert–Schmidt–Hankel norm of a stable system is equal, up to a constant factor, to the square root of the area enclosed by the Nyquist diagram of the transfer function.

In this paper, we consider SISO systems that are discrete-time, linear time invariant, stable and minimum phase, i.e. the poles and zeros all lie inside the

* Corresponding author. Tel.: +32-16-32-17-09; fax: +32-16-32-19-70.
E-mail address: katrien.decock@esat.kuleuven.ac.be (K. De Cock).
[1] Dr. Katrien De Cock is a research assistant at the K.U.Leuven.
[2] Dr. Bart De Moor is full professor at the K.U.Leuven.

0165-1684/03/$ - see front matter
PII: S0165-1684(02)00425-5

Nomenclature

$c(k)$	complex cepstrum of a model
$\|\log G\|_{\mathrm{cep}}$	cepstral norm of the model with transfer function G
$\delta(\theta)$	curve $\log G(\mathrm{e}^{\mathrm{i}\theta})$, where θ is running from 0 to 2π
$\gamma(\theta)$	curve $G(\mathrm{e}^{\mathrm{i}\theta})$, where θ is running from 0 to 2π
$A(\delta)$	area enclosed by $\delta(\theta)$
$\|H\|_{\mathrm{HS}}$	Hilbert–Schmidt norm of the Hankel matrix H
$\operatorname{tr} A$	trace of the matrix A
$\|G(z)\|_{\mathrm{HSH}}$	Hilbert–Schmidt–Hankel norm of the transfer function $G(z)$

unit circle. Furthermore, we assume that their transfer function is of the form

$$G(z) = \frac{z^n + b_1 z^{n-1} + \cdots + b_{n-1} z + b_n}{z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n}, \tag{1}$$

where $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$ and n is the order of the model. The results in this paper can be extended to a distance measure for a set of equivalence classes in which all transfer functions that differ only in a constant factor, are considered to be equivalent.

The paper's outline is as follows. In Section 2 we discuss the cepstral distance between ARMA models defined by Martin [4]. In Section 3 we show that the related system norm is equal to the Hilbert–Schmidt norm of a Hankel matrix and hence can be related to the area enclosed by a curve in the complex plane. An analogous interpretation for the distance between two models is made in Section 4.

2. A cepstral distance between ARMA models

The definition of Martin [4] for the distance between two ARMA models is based on the power cepstrum of the models, which is the inverse Z-transform of the logarithm of the power spectrum. In order to fully exploit the properties of the Z-transform, we prefer to use the complex cepstrum instead of the power cepstrum. Since for stable and minimum phase models the complex cepstrum coefficients $c(k)$ for $k \geqslant 0$ are equal to the corresponding power cepstrum coefficients and since the distance measure defined in [4] is based only on the power cepstrum coefficients for $k > 0$, we can equally well formulate the distance measure in terms of the complex cepstrum.

The complex cepstrum $c(k)$ of a model with transfer function $G(z)$ is the inverse Z-transform of the logarithm of the transfer function:

$$\sum_{k=-\infty}^{\infty} c(k) z^{-k} = \log G(z),$$

where the complex logarithm of $G(z)$ is appropriately defined [5, pp. 495–497]. The complex cepstrum is a real-valued function, despite its name. For stable and minimum phase systems the complex cepstrum is causal: $c(k) = 0$, for $k < 0$.

We obtain the following definition for the distance between two stable and minimum phase ARMA models, which is equivalent to the definition by Martin [4].

Definition 1. Given two stable and minimum phase ARMA models of order n_1 and n_2 with transfer functions G_1 and G_2 as in (1) and complex cepstra $c_1(k)$ and $c_2(k)$, respectively, the distance between the models is defined as

$$\mathrm{d}^2(G_1, G_2) = \sum_{k=1}^{\infty} k |c_1(k) - c_2(k)|^2. \tag{2}$$

From the results of Martin [4] it follows that the sum in (2) converges for stable and minimum phase models. Observe that for ARMA models with transfer function as in (1), $c(0) = 0$, which ensures (2) to be a metric on that class of models.

The associated cepstral norm for an ARMA model is defined as follows.

Definition 2. Given a stable and minimum phase ARMA model with transfer function $G(z)$ as in (1) and cepstrum $c(k)$, the cepstral norm of this model is defined as

$$\|\log G\|_{\mathrm{cep}}^2 = \sum_{k=1}^{\infty} k c(k)^2. \tag{3}$$

3. The cepstral system norm and the polar plot of $\log G(z)$

In this section, we obtain the relation between the system norm $\|\log G\|_{\mathrm{cep}}$, defined in (3), and the area enclosed by the curve $\delta(\theta) = \log G(e^{i\theta})$ where θ is running from 0 to 2π. This curve is indeed closed in the complex plane due to the appropriate choice of the complex logarithm. By the area enclosed by $\delta(\theta)$ we mean the following integral:

$$A(\delta) = \int_{x+iy\in\delta} y\,dx. \tag{4}$$

In Section 3.1 we first show that the cepstral norm $\|\log G\|_{\mathrm{cep}}$ is equal to the Hilbert–Schmidt norm of the Hankel matrix with the cepstrum coefficients. In Section 3.2 we give the relation between $\|\log G\|_{\mathrm{cep}}$ and the area enclosed by $\delta(\theta)$. An example is given in Section 3.3.

3.1. The Hilbert–Schmidt–Hankel norm for cepstra

Let $c(k)$ denote the cepstrum coefficients of the model with transfer function $G(z)$ and consider the double infinite Hankel matrix H

$$H = \begin{pmatrix} c(1) & c(2) & c(3) & \cdots \\ c(2) & c(3) & c(4) & \cdots \\ c(3) & c(4) & c(5) & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \in \mathbb{R}^{\infty\times\infty}.$$

The Hilbert–Schmidt norm of H is given by

$$\|H\|_{\mathrm{HS}}^2 = \mathrm{tr}\, HH^{\mathrm{T}} = \sum_{k=1}^{\infty} kc(k)^2.$$

Because the right-hand side is equal to the cepstral system norm defined in (3), we obtain

$$\|\log G\|_{\mathrm{cep}}^2 = \|H\|_{\mathrm{HS}}^2.$$

3.2. Relation to the polar plot of $\log G(z)$

In [3] Hanzon considers the Hankel matrix with the Markov parameters $g(k)$ of a model without direct feed-through term (i.e. $g(0) = 0$). The Markov parameters form a causal sequence whose Z-transform is equal to $G(z)$. The Hilbert–Schmidt norm of that Hankel matrix, i.e. the Hilbert–Schmidt–Hankel norm of $G(z)$, denoted by $\|G(z)\|_{\mathrm{HSH}}$, is related to the Nyquist diagram of the transfer function $G(z)$ in the following way:

$$\|G(z)\|_{\mathrm{HSH}}^2 = \sum_{k=1}^{\infty} kg(k)^2 = \frac{1}{\pi}A(\gamma),$$

where $A(\gamma)$ is the area enclosed by the curve $\gamma(\theta) = G(e^{i\theta})$, $\theta \in [0, 2\pi)$ running from 0 to 2π.

An analogous result holds for the cepstral norm defined in (3).

Theorem 3. *Let $G(z)$ be the transfer function of a stable and minimum phase ARMA model as in* (1). *The cepstral norm of $G(z)$ is then equal to*

$$\|\log G\|_{\mathrm{cep}}^2 = \frac{1}{\pi}A(\delta), \tag{5}$$

where $A(\delta)$ is the area enclosed by $\delta(\theta) = \log G(e^{i\theta})$, $\theta \in [0, 2\pi)$ running from 0 to 2π.

Proof. Due to the causality of $c(k)$ the cepstral norm of $G(z)$ that is defined in (3), is equal to

$$\|\log G\|_{\mathrm{cep}}^2 = \sum_{k=-\infty}^{\infty} kc(k)^2.$$

Applying Parseval's theorem leads to the following line integral over the unit circle

$$\|\log G\|_{\mathrm{cep}}^2 = -\frac{1}{2\pi i}\oint \frac{d}{dz}(\log G(z))\overline{\log G(z)}\,dz = -\frac{1}{2\pi i}\int_{\delta} \overline{\log G(z)}\,d(\log G(z)),$$

where $\delta(\theta)$ is the closed curve $\log G(e^{i\theta})$ and θ is running from 0 to 2π. This can further be written as

$$\|\log G\|_{\mathrm{cep}}^2 = -\frac{1}{2\pi i}\int_{x+iy\in\delta} (x - iy)\,d(x + iy).$$

By Stokes' theorem this is equal to [3]

$$\|\log G\|_{\mathrm{cep}}^2 = \frac{1}{2\pi i}\iint d((x - iy)\,d(x + iy)),$$

[3] The change of sign is due to the clockwise orientation of δ.

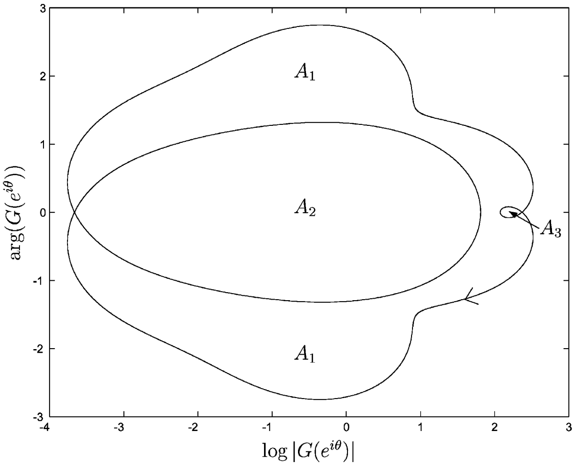

Fig. 1. The curve $\delta(\theta) = \log G(e^{i\theta})$, where θ is running from 0 to 2π. The starting point is $(2.0746, 0)$ and the orientation is indicated by the arrow. The area enclosed by δ is $A_1 + 2A_2 + 2A_3$ and this is equal to 11.1925π.

which leads to

$$\|\log G\|^2_{\text{cep}} = \frac{1}{2\pi i}\iint d(x - iy)\, d(x + iy)$$
$$= \frac{1}{2\pi i}\iint 2i\, dx\, dy$$
$$= \frac{1}{\pi}\iint dx\, dy$$

and this is $(1/\pi)A(\delta)$. □

3.3. An example

As an example we take the fifth-order ARMA model with poles $0.9 \pm 0.1i$, $0.2 \pm 0.8i$ and -0.95 and zeros $-0.5 \pm 0.82i$, $0.1 \pm 0.7i$ and 0.92. Its transfer function is denoted by $G(z)$.

In Fig. 1 the curve $\delta(\theta) = \log G(e^{i\theta})$ corresponding to the ARMA model is drawn. The curve starts and ends in $(2.0746, 0)$ and its orientation is indicated by the arrow. The area enclosed by δ is the sum

$$A(\delta) = A_1 + 2A_2 + 2A_3$$

as shown in Fig. 1. The area was obtained by computing the integral in (4) numerically in Matlab, which multiplied by $1/\pi$ resulted in 11.1925, and the formula for the cepstral norm as a function of the poles and zeros, described in [4], gave the same result. Note that we do *not* need to identify A_1, A_2 and A_3 in order to calculate the area.

4. The cepstral distance between two ARMA models as an area in the complex plane

Consider two stable and minimum phase ARMA models of order n_1 and n_2 with transfer functions $G_1(z)$ and $G_2(z)$ and cepstra $c_1(k)$ and $c_2(k)$, respectively. The cepstral distance between the two models is defined in (2) as

$$d^2(G_1, G_2) = \sum_{k=1}^{\infty} k|c_1(k) - c_2(k)|^2.$$

The sequence $c_1(k) - c_2(k)$ $(k \in \mathbb{Z})$ is the inverse Z-transform of $\log G_1(z) - \log G_2(z) = \log G_1(z)/G_2(z)$. Hence, $c_1(k) - c_2(k)$ is the cepstrum of the stable and minimum phase ARMA model with transfer function $G_1(z)/G_2(z)$, of order $\leqslant n_1 + n_2$. Consequently, the distance between G_1 and G_2 is

$$d^2(G_1, G_2) = \left\|\log \frac{G_1}{G_2}\right\|^2_{\text{cep}}.$$

By applying (5), the distance is related to an area in the complex plane as follows:

$$d^2(G_1, G_2) = \frac{1}{\pi} A(\delta_{1,2}),$$

where $A(\delta_{1,2})$ is the area enclosed by $\delta_{1,2}(\theta) = \log G_1(e^{i\theta})/G_2(e^{i\theta})$, $\theta \in [0, 2\pi)$ running from 0 to 2π.

5. Conclusions

In this paper, we have shown that the cepstral distance between two stable and minimum phase ARMA models that was defined by Martin [4] is equal, up to a factor $1/\sqrt{\pi}$, to the square root of the area enclosed by the polar plot of the logarithm of the ratio of their transfer functions.

Acknowledgements

Our research is supported by grants from several funding agencies and sources: Research Council KUL: GOA-Mefisto 666, several Ph.D./postdoc & fellow grants; Flemish Government: FWO: Ph.D./postdoc grants, projects, G.0240.99 (multilinear algebra), G.0407.02 (support vector machines), G.0197.02 (power islands), G.0141.03 (Identification and cryptography), G.0491.03 (control for intensive care glycemia), G.0120.03 (QIT), research communities (ICCoS, ANMMM); AWI: Bil. Int. Collaboration Hungary/Poland; IWT: Ph.D. Grants, Soft4s (softsensors); Belgian Federal Government: DWTC (IUAP IV-02 (1996–2001) and IUAP V-22 (2002-2006), PODO-II (CP/40: TMS and Sustainability); EU: CAGE; ERNSI; Eureka 2063-IMPACT; Eureka 2419-FliTE; Contract Research/agreements: Data4s, Electrabel, Elia, LMS, IPCOS, VIB.

References

[1] M. Basseville, Distance measures for signal processing and pattern recognition, Signal Processing 18 (4) (December 1989) 349–369.

[2] K. De Cock, B. De Moor, Subspace angles between ARMA models, Systems Control Lett. 46 (4) (July 2002) 265–270.

[3] B. Hanzon, The area enclosed by the (oriented) Nyquist diagram and the Hilbert–Schmidt–Hankel norm of a linear system, IEEE Trans. Automat. Control 37 (6) (June 1992) 835–839.

[4] R.J. Martin, A metric for ARMA processes, IEEE Trans. Signal Process. 48 (4) (April 2000) 1164–1170.

[5] A.V. Oppenheim, R.W. Schafer, Digital Signal Processing, Prentice Hall International, London, 1975.