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Normal forms, entanglement monotones and optimal filtering of multipartite quantum systems

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We consider one single copy of a mixed state of a multipartite system of arbitrary dimension and bring those states in a normal form by a class of local operations and classical communication (SLOCC). These normal forms are uniquely defined up to local unitary transformations and are a generalization of the Bell diagonal states in that any partial trace over all systems but one yields a multiple of the identity matrix. We next investigate how the entanglement of mixed multipartite systems changes under the action of SLOCC operations and introduce a whole new class of entanglement monotones. All these entanglement monotones are maximized under the SLOCC operations that bring the state into its normal form. As specific examples we show that the concurrence and the 3-tangle belong to the introduced class of entanglement measures, and we propose generalizations to systems of arbitrary dimension. Finally, the local operations that bring a state into normal form are shown to be the optimal filtering operations in the sense that they maximize all the introduced entanglement monotones.

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One of the major challenges in the field of quantum information theory is to get a deep understanding of how local operations assisted by classical communication (LOCC) performed on a multipartite quantum system can affect the entanglement between the spatially separated systems. In this paper we investigate this problem in the case that only operations on one copy of the system are allowed. This is different from the general setup of entanglement distillation, where global operations on a large (infinite) number of copies are performed to concentrate the entanglement in a few copies. The main motivation of this work was to characterize the optimal SLOCC operations to be performed on one copy of a multipartite system such that, with a non zero chance, a state with maximal possible entanglement is obtained. In other words, we want to design the optimal filtering operations for a given state, such that with a certain chance we prepare the optimal attainable one.

In the case of a pure state of two qubits, this optimal filtering procedure is commonly known as the Procrustean method [1]. Following the work of Gisin [2], Horodecki [3],Linden et al. [4] and Kent et al. [5,6], the optimal filtering procedure for mixed states of two qubits was recently derived in Verstraete et al. [7]. In this paper we extend these ideas to pure and mixed multipartite systems of qudits of arbitrary dimension.

The optimal filtering operations in Verstraete et al. [7] were derived by proving the existence of a decomposition of a mixed state of two qubits as a unique Bell diagonal state multiplied left and right by a tensor product representing local operations. A Bell diagonal state is special in the sense that one party alone cannot acquire any information at all about the state: its local density operator is equal to the identity. This can readily be generalized to multipartite systems of arbitrary dimensions, and the existence of local operations transforming a generic state to a state with all local density operators equal to the identity will be proved in the first part of this paper. This decomposition is unique, and in the case of pure states it leads to a transparent method of deriving essentially different states such as GHZ- and W-states [8].

In a second part we show that all quantities exhibiting some kind of invariance under the considered SLOCC operations are entanglement monotones [9]. It is shown that the concurrence and the 3-tangle, introduced by Wootters et al. [10,11], belong to this class of entanglement measures. Therefore a natural generalization of these measures is obtained to systems of arbitrary dimensions and an arbitrary number of parties.

The third part of the paper is concerned with finding the optimal filtering operations for a given multipartite state. It is shown that the SLOCC operations bringing a state into its normal form maximize all the introduced entanglement monotones. This was expected in the light of the work by Nielsen about majorization [12]: the notion of local disorder is intimately connected to the existence of entanglement.

I. NORMAL FORMS

We consider a multipartite system of arbitrary dimension and stochastic local quantum operations assisted by classical communication (SLOCC) of the kind [13]

$$\rho' \simeq (A_1 \otimes A_2 \otimes \cdots \otimes A_n)\, \rho\, (A_1 \otimes A_2 \otimes \cdots \otimes A_n)^\dagger \quad (1)$$

where the $\{A_i\}$ are arbitrary local operators of the dimension of the respective subsystems and where ρ' is not normalized. Here we will restrict the $\{A_i\}$ to be full rank and thus invertible, as lower rank operators can only destroy entanglement, and we will therefore impose the condition $\forall i: \det(A_i) = 1$. The most general local operations involve also mixing, but as this also destroys entanglement we will not consider this. All density matrices connected by the SLOCC operations (1) form a class, and it's therefore a natural question to find some kind of representative state of each class. Inspired by the results on two qubit systems [7], it is natural to define the representative state as the one for which all local correlations are washed out: we expect this state to have the largest amount of entanglement of the whole class as it is well-known that there is some kind of trade-off between local correlations and entanglement in a system. Let us now state a first theorem:

Theorem 1 *Consider a $N_1 \times N_2 \times \cdots N_p$ pure or mixed multipartite state. Then this state can be brought into a normal form by determinant 1 SLOCC operations (1), where the normal form is uniquely defined up to local unitary operations and has all partial traces over all but one party proportional to the identity. Moreover the trace of the normal form is the minimal one that can be obtained by determinant 1 SLOCC operations.*

Proof: We will give a constructive proof of this theorem. The idea is that the local determinant 1 operators A_i bringing ρ into its normal form can be iteratively determined by a procedure where at each step one party minimizes the trace of ρ'. Consider therefore the partial trace $\rho_1 = \mathrm{Tr}_{2\cdots p}(\rho)$. If ρ_1 is full rank, there exists an operator X with determinant 1 such that $\rho_1' = X\rho_1 X^\dagger \sim I_{N_1}$. Indeed, $X = |\det(\rho_1)|^{2/N_1}(\sqrt{\rho_1})^{-1}$ does the job, and we have $\rho_1' = \det(\rho_1)^{1/N_1} I_{N_1}$. We therefore have the relation:

$$\mathrm{Tr}(\rho') = N_1 \det(\rho_1)^{1/N_1} \leq \mathrm{Tr}(\rho_1), \qquad (2)$$

where $\rho' = (X \otimes I \cdots \otimes I)\rho(X \otimes I \cdots \otimes I)^\dagger$. This inequality follows from the fact that the geometric mean is always smaller then the arithmetic mean, with equality iff ρ_1 is proportional to the identity. Therefore the trace of ρ decreases after this. We can now repeat this procedure with the other parties, and then repeat everything iteratively. After each iteration, the trace of ρ will decrease unless all partial traces are equal to the identity. Because the trace of a positive definite operator is bounded from below, we know that the decrements become arbitrarily small and following equation (2) this implies that all partial traces converge to operators arbitrarily close to the identity. We still have to consider the case where ρ_i is not full rank. Then there exists an X tending to infinity with determinant 1 such that $X\rho_i X^\dagger = 0$, leading to a normal form identical to zero, clearly the positive operator with minimal possible trace. This ends the proof of the existence of the normal form.

The proof of the unicity (up to local unitary operations) is obtained as follows. If the normal form were not unique, there would exist a normal σ_q and σ_r and diagonal matrices $\{D_i\}$ with determinant 1 such that $\sigma_q = (D_1 \otimes \cdots \otimes D_p)\sigma_r(D_1 \otimes \cdots \otimes D_p)^\dagger$: if the D_i were not diagonal we could always make them diagonal through local unitary transformations. Writing the diagonal elements of the normal σ_q, σ_r in the $N_1 \times N_2 \times \cdots \times N_p$ tensors T_q, T_r, it is readily observed that both tensors are stochastic in all directions: the sum of whatever column in whatever direction results in 1 and all elements are greater or equal to zero. Moreover both tensors are connected by diagonal $\{D_i\}$ working on the respective indices. For the sake of simplicity we now reason on the bipartite case where T_q and T_r are doubly stochastic. Because the column sums $D_1 T_q$ lie between the minimal and maximal element of D_1 (corresponding to nonzero elements in T_q), D_2 can only compensate for the effect of D_1 if both D_1 and D_2 are a multiple of the identity, implying $\sigma_q = \sigma_r$. The only exception arises when there are zeros in T_q. It turns out that these cases correspond to block doubly stochastic matrices [1] and even then we have $T_q = T_r$. This easily generalizes to the multipartite case. It follows that $D_1 \otimes \cdots \otimes D_p$ has elements of value 1 on the indices where the (equal) diagonal elements of σ_q and σ_r do not vanish. Due to the positiveness of σ it follows that $\sigma_q = \sigma_r$. Therefore the unicity of the normal form is proven.

Due to the constructive proof of the normal form and its unicity, it follows that the trace of the normal form is the minimal one under all determinant 1 SLOCC operations, which ends the proof. □

It should be noted that the group of matrices having a determinant equal to 1 is not compact. There indeed exist low rank states that can only be brought into their respective normal form by infinite transformations, although the class of states with this property is clearly of measure zero. Note also that for these states the given proof of the uniqueness is not longer valid, although we are strongly convinced that the property of uniqueness is still there. As an example consider the W-state [8] $|\psi\rangle = |001\rangle + |010\rangle + |100\rangle$. The following identity is easily checked:

$$\lim_{t\to\infty} \begin{pmatrix} 1/t & . \\ . & t \end{pmatrix}^{\otimes 3} |W\rangle = 0.$$

[1] This happens for example in the case of the EPR and GHZ state: there exists local non-trivial operations that map the state onto itself.

The normal form corresponding to the W-state is therefore equal to zero, clearly the state with the minimal possible trace. This is interesting, as it will be shown that a normal form is zero iff a whole class of entanglement monotones is equal to zero. Therefore the states with normal form equal to zero are fundamentally different from those with finite normal form, and this leads to the generalization of the W-class to arbitrary dimensions. It thus happens that some states have normal form equal to 0. This also happens if the state does not have full support on the Hilbert space in that one partial trace ρ_i is rank deficient.

Note also that states which do not have full support on the Hilbert space, such as pure states from which one party is fully separable, all have normal form equal to zero.

As a second remark, we should note that the proof was constructive and leads to a very efficient algorithm for actually calculating the normal form. In the case of pure states however, a square root version of the above algorithm can easily be derived that directly acts on the state and not on the density operator, which has clear numerical advantages.

A third remark concerns the continuity of the normal form with relation to the original density matrix. First of all note that the non-uniqueness due to the local unitaries can be circumvented by imposing all A_i to be hermitian. Using techniques of matrix differentiation, it is then possible to prove that a small perturbation of the original density matrix results is a perturbation of the same order of magnitude in the normal form if all $\{A_i\}$ are finite: the normal form is robust against noise.

II. ENTANGLEMENT MONOTONES

The above formalism suggests a very general way of constructing entanglement monotones:

Theorem 2 *Consider a linearly homogeneous positive function of a pure (unnormalized) state $M(\rho = |\psi\rangle\langle\psi|)$ that remains invariant under the determinant 1 SLOCC operations (1). If its definition is extended to mixed states by the convex roof formalism, then $M(\rho)$ is an entanglement monotone where ρ is an arbitrary normalized state.*

Proof: A quantity $M(\rho)$ is an entanglement monotone iff its expected value decreases under the action of every local operation. Due to the convex roof formalism, it is immediately clear that M is decreasing under the action of mixing. It is therefore sufficient to show that for every local $A_1 \leq I_{N_1}$, $\bar{A}_1 = \sqrt{I_{N_1} - A_1^\dagger A_1}$, it holds that

$$M(\rho) \geq \mathrm{Tr}\left((A_1 \otimes I)\rho(A_1 \otimes I)^\dagger\right) M\left(\frac{(A_1 \otimes I)\rho(A_1 \otimes I)^\dagger}{\mathrm{Tr}\left((A_1 \otimes I)\rho(A_1 \otimes I)^\dagger\right)}\right)$$
$$+\mathrm{Tr}\left((\bar{A}_1 \otimes I)\rho(\bar{A}_1 \otimes I)^\dagger\right) M\left(\frac{(\bar{A}_1 \otimes I)\rho(\bar{A}_1 \otimes I)^\dagger}{\mathrm{Tr}\left((\bar{A}_1 \otimes I)\rho(\bar{A}_1 \otimes I)^\dagger\right)}\right)$$

A_1 can be transformed to a determinant 1 matrix by dividing it by $\det(A_1)^{1/N_1}$. Note that the homogeneity of $M(\alpha\rho) = \alpha M(\rho)$ and its invariance for pure states under determinant 1 SLOCC operations implies that if $\{\psi_\lambda\}$ represents an optimal decomposition of ρ in terms of the convex roof formalism, then $\{(\otimes A_i)\psi_\lambda)\}$ represents an optimal decomposition of $(\otimes A_i)\rho(\otimes A_i)^\dagger$. Therefore the previous inequality is equivalent to

$$M(\rho) \geq (|\det(A_1)|^{2/N_1} + |\det(\bar{A}_1)|^{2/N_1})M(\rho).$$

As the arithmetic mean always exceeds the geometric mean, this inequality is always satisfied. The same argument can now be repeated for the other A_i, which ends the proof. □

Entanglement monotones of the above class can readily be constructed using the completely antisymmetric tensor $\epsilon_{i_1\cdots i_N}$. Indeed, it holds that $\sum A_{i_1j_1}A_{i_2j_2}\cdots A_{i_Nj_N}\epsilon_{j_1\cdots j_N} = \det(A)\epsilon_{i_1\cdots i_N}$, and as $\det(A) = 1$ this leads to invariant quantities under determinant 1 SLOCC operations.

Consider for example the case of two qubits. The quantity $|\sum \psi_{i_1j_1}\psi_{i_2j_2}\epsilon_{i_1i_2}\epsilon_{j_1j_2}|$ is clearly of the considered class, and it happens to be identical to the concurrence [10]. In the case of three qubits the simplest non-trivial homogeneous quantity invariant under determinant 1 SLOCC operations is given by

$$|\psi_{i_1j_1k_1}\psi_{i_2j_2k_2}\psi_{i_3j_3k_3}\psi_{i_4j_4k_4}\epsilon_{i_1i_2}\epsilon_{i_3i_4}\epsilon_{j_1j_2}\epsilon_{j_3j_4}\epsilon_{k_1k_3}\epsilon_{k_2k_4}|^{1/2}$$

This happens to the square root of the 3-tangle introduced by Wootters et al. [11], which quantifies the true tripartite entanglement.

More generally, as the considered entanglement monotones are invariant under the determinant 1 SLOCC operations, the number of independent entanglement monotones is equal to the degrees of freedom of the normal form obtained in the case of a pure state minus the degrees of freedom induced by the local unitary operations. Indeed, this is the amount of invariants of the whole class of states connected by SLOCC operations. It is then easily proven that a normal form is equal to zero if and only if all the considered entanglement monotones are equal to zero: the entanglement monotones are homogeneous functions of the normal form, and if the normal form is not equal to zero there always exists an SLOCC invariant quantity that is different from zero.

In the case of 4 qubits for example, parameter counting leads to $(2 \cdot 2^4 - 1) - 4 \cdot 6 = 7$ independent entanglement monotones. The simplest monotone is given by

$$|\psi_{i_1j_1k_1l_1}\psi_{i_2j_2k_2l_2}\epsilon_{i_1i_2}\epsilon_{j_1j_2}\epsilon_{k_1k_2}\epsilon_{l_1l_2}|^{1/2},$$

and the other 6 entanglement monotones can be obtained by including more factors. These are clearly generalizations of the concurrence and the 3-tangle to four parties.

If the subsystems happen to be of unequal dimension, it is easy to prove that the corresponding normal forms for pure states are equal to zero: a 2×4 pure state for example effectively lives in a 2×2 Hilbert space. This implies that all possible invariants will also be equal to zero. Therefore the considered entanglement monotones make only sense if all the subsystems have the same dimension. Note also that all considered entanglement monotones will be zero for pure states for which one party is completely disentangled from the other ones, as the normal form of such a state is again zero.

Let us finally give a non-trivial example of an entanglement monotone of the considered class in the case of three qutrits:

$$|\sum \psi_{i_1j_1k_1}\psi_{i_2j_2k_2}\psi_{i_3j_3k_3}\psi_{i_4j_4k_4}\psi_{i_5j_5k_5}\psi_{i_6j_6k_6} \epsilon_{i_1i_2i_3}\epsilon_{i_4i_5i_6}\epsilon_{j_1j_2j_4}\epsilon_{j_3j_5j_6}\epsilon_{k_1k_5k_6}\epsilon_{k_2k_3k_4}|^{1/3}.$$

ϵ is now the $3\times 3\times 3$ completely antisymmetric tensor, and the other $(2\cdot 3^3-1)-(3\cdot 16)-1=4$ independent entanglement monotones can be constructed by including more factors.

III. OPTIMAL FILTERING

The main motivation of this paper was to characterize the optimal SLOCC operations to be performed on one copy of a multipartite system such that, with a non zero chance, a state with maximal possible true multipartite entanglement is obtained. This question is of importance for experimentalists as in general they are not able to perform joint operations on multiple copies of the system. Therefore the procedure outlined here often represents the best entanglement distillation procedure that is practically achievable.

In the previous section a whole class of entanglement monotones that measure the amount of multipartite entanglement were introduced. Let us now state the following theorem:

Theorem 3 *Consider a pure or mixed multipartite state, then the local filtering operations that maximize all entanglement monotones introduced in theorem 2 are represented by operators proportional to the determinant 1 SLOCC operations that transform the state into its normal form.*

Proof: The proof of this theorem is surprisingly simple. Indeed, all the quantities introduced in theorem 2 are invariant under determinant 1 SLOCC operations if the states do not get normalized. The entanglement monotones themselves however are only defined on normalized states, and due to the linear homogeneity the following identity holds:

$$M\left(\frac{(\otimes_iA_i)\rho(\otimes_iA_i)^\dagger}{\mathrm{Tr}\left((\otimes_iA_i)\rho(\otimes_iA_i)^\dagger\right)}\right)=\frac{M(\rho)}{\mathrm{Tr}\left((\otimes_iA_i)\rho(\otimes_iA_i)^\dagger\right)}$$

The optimal filtering operators are then obtained by the $\{A_i\}$ minimizing $\mathrm{Tr}\left((\otimes_iA_i)\rho(\otimes_iA_i)^\dagger\right)$. But this problem was solved in theorem 1, where it was proved that the $\{A_i\}$ bringing the state into its unique normal form minimize this trace. □

It is therefore proved that the (reversible) procedure of washing out the local correlations maximizes the multipartite entanglement in the system. This is in complete accordance with the results of Nielsen [12] where it was shown that the notion of local disorder is intimately connected to the existence of entanglement.

The previous theorem allows to find the optimal filtering operators for a given pure or mixed state such that the value of all the considered entanglement monotones for the filtered state are maximized. This is remarkable as there is no direct way of actually calculating the value of these monotones for mixed states due to the difficulty involved in finding the optimal decomposition in the convex roof formalism!

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