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Four qubits can be entangled in nine different ways

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We consider a single copy of a pure four-partite state of qubits and investigate its behavior under the action of stochastic local quantum operations assisted by classical communication (SLOCC). This leads to a complete classification of all different classes of pure states of four qubits. It is shown that there exist nine families of states corresponding to nine different ways of entangling four qubits. The states in the generic family give rise to Greenberger-Horne-Zeilinger-like entanglement. The other ones contain essentially two-or three-qubit entanglement distributed among the four parties. The concept of concurrence and 3-tangle is generalized to the case of mixed states of four qubits, giving rise to a seven-parameter family of entanglement monotones. Finally, the SLOCC operations maximizing all these entanglement monotones are derived, yielding the optimal single-copy distillation protocol.

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One of the open questions in the field of quantum information theory is to understand the different ways in which multipartite systems can be entangled. As the concept of entanglement is related to the nonlocal properties of a state, local quantum operations cannot affect the intrinsic nature of entanglement. It is therefore natural to define equivalence classes of states generated by the group of reversible stochastic local quantum operations assisted by classical communication (SLOCC) operations [1,2]. In this paper we are concerned with SLOCC operations on one copy of a state, which means that we are considering actions under LOCC operations on one copy of a state without imposing that they can be achieved with unit certainty. Two states belonging to the same class are able to perform the same quantum information processing (QIP) tasks, although with a different probability.

In the case of a single copy of an entangled pure state of two qubits, it is well known that it can be converted to the singlet state by SLOCC operations [3]. In the case of three entangled qubits, it was shown [2,4,5] that each state can be converted by SLOCC operations either to the GHZ-state $(|000\rangle+|111\rangle)/\sqrt{2}$, or to the W-state $(|001\rangle+|010\rangle+|100\rangle)/\sqrt{3}$, leading to two inequivalent ways of entangling three qubits. The GHZ state is generally considered as the state with the genuine three-partite entanglement, while the W state has the peculiar property of having the maximal expected amount of two-partite entanglement if one party is traced out [2]. In this paper, we extend these results to the case of four qubits. Furthermore the widely celebrated entanglement measures concurrence [6] and 3-tangle [2,7], characterizing the amount of genuine two-and three-qubit entanglement, are generalized to the case of four qubits, giving rise to a seven-parameter family of entanglement monotones. The SLOCC filtering operations maximizing all these entanglement monotones are derived, and it is shown that these are the unique operations [8] (up to local unitaries) bringing a state into a locally stochastic form (i.e., bringing all local-density operators equal to the identity). Following Gisin and Bechmann-Pasquinucci [9], we claim that these operations maximize the true four-partite entanglement.

Interestingly, we found that there exist eight families of pure four-qubit states that cannot be brought into local stochastic form by finite SLOCC operations. These states do have the peculiar property that they have the maximal amount of two-and/or three-qubit entanglement shared between all four parties. In some sense their entanglement is maximally robust against the loss of one or two qubits.

An interesting feature about entanglement that emerges out of the results of this paper is the fact that a quantum state has only a finite susceptibility for entanglement. This will be illustrated by the fact that the operations maximizing the true four-partite entanglement are precisely the operations that destroy all local correlations (i.e., the local-density operators are made stochastic) and that also destroy the three-partite entanglement (i.e., the three-qubit entanglement of the states obtained by tracing out one party becomes equal to zero). The states having maximal two- or three-partite entanglement shared among the four parties on the other hand are exactly the states having zero genuine four-partite entanglement (i.e., the four concurrences are all equal to zero).

Before developing the mathematical formalism, it should be noted that the study of states of four qubits is particularly interesting as the current experimental state of the art allows us to entangle four photons [10–12] or ions [13]. Furthermore SLOCC operations can relatively easily be implemented on photons, and it is therefore of interest to implement the optimal SLOCC operations such as to yield a state with maximal four-partite entanglement.

This paper is organized as follows. First we derive a simple way of determining whether two pure four-qubit states are connected by local unitary operations. Next some advanced linear algebra is used to determine the orbits generated by SLOCC operations. This leads to nine different families of states, corresponding to nine essentially different ways of entangling four qubits, although only one family is generic. This analysis gives rise to seven independent entanglement monotones characterizing the four-partite en-

tanglement. Finally, the optimal SLOCC operations are derived such as to maximize all these entanglement monotones.

Let us now first consider the problem to determine whether two pure four-qubit states are equivalent up to local unitary operations. Therefore, the following accident in Lie-group theory can be exploited:

$$\mathrm{SU}(2)\otimes\mathrm{SU}(2)\simeq\mathrm{SO}(4).$$

Here, SO(4) denotes the family of real orthogonal matrices with determinant equal to one. More specifically, it holds that $\forall U_1,\ U_2\in\mathrm{SU}(2)$: $T(U_1\otimes U_2)T^\dagger\in\mathrm{SO}(4)$ where

$$T=\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0&1\\0&i&i&0\\0&-1&1&0\\i&0&0&-i\end{pmatrix}. \tag{1}$$

A pure state of four qubits is parametrized by a four index tensor $\psi_{i_1i_2i_3i_4}$ with $i_j\in\{1,2\}$. This tensor can be rewritten as a 4×4 matrix $\tilde{\psi}$ by concatenating the indices (i_1,i_2) and (i_3,i_4). Next we define the matrix R as

$$R=T\tilde{\psi}T^\dagger. \tag{2}$$

It is then straightforward to show that a local unitary transformation $|\psi'\rangle=U_1\otimes U_2\otimes U_3\otimes U_4|\psi\rangle$ results in a transformation $R'=O_1RO_2$ with $O_1,\ O_2\in\mathrm{SO}(4)$ and $O_1=T(U_1\otimes U_2)T^\dagger$, $O_2=T(U_3\otimes U_4)^TT^\dagger$. A normal form under local unitary operations can now be imposed as follows: make the (1, 1) entry of R real by multiplying the whole matrix with the appropriate phase, and use O_1 and O_2 to diagonalize the real part of R through the unique real singular value decomposition. This procedure eliminates all 13 degrees of freedom of the local unitary operations, and two states are therefore equivalent up to local unitary operations if they have the same normal form.

Next we move to the central problem of this paper, namely, characterizing the local orbits generated by SLOCC operations of the form

$$|\psi'\rangle=A_1\otimes A_2\otimes A_3\otimes A_4|\psi\rangle \tag{3}$$

with $\{A_i\}$ full rank and therefore invertible 2×2 matrices. There is no restriction in choosing $\{A_i\}\in\mathrm{SL}(2,\mathbb{C})$, and then a useful accident arises

$$\mathrm{SL}(2,\mathbb{C})\otimes\mathrm{SL}(2,\mathbb{C})\simeq\mathrm{SO}(4,\mathbb{C}). \tag{4}$$

SO(4, $\mathbb{C}$) denotes the noncompact group of complex orthogonal matrices $O^TO=I_4$. Again it holds that $\forall\ A,\ B\in\mathrm{SL}(2,\mathbb{C})$: $T(A_1\otimes A_2)T^\dagger\in\mathrm{SO}(4,\mathbb{C})$ with T given in Eq. (1), and SLOCC-operations therefore correspond to left and right multiplication of R (2) with complex orthogonal matrices. The challenge is now to exploit the two times 12 degrees of freedom of these complex orthogonal matrices to bring A into an unique normal form with maximal 8 real degrees of freedom left. This will be possible using some advanced techniques of linear algebra.

We will now state a technical theorem that is a generalization of the singular value decomposition to complex orthogonal matrices.

Theorem 1. Given a complex $n\times n$ matrix R, then there always exist complex square orthogonal matrices O_1 and O_2 such that $R'=O_1RO_2$ is a unique direct sum of blocks of the form

(1) $m\times m$ blocks of the form $(\lambda_jI_m+S_m)$ being symmetric Jordan blocks (see, for example, [14] 4.4.9), and λ_j is a complex parameter (note that the case $m=1$ corresponds to the scalar case);

(2) $m\times m$ blocks consisting of an upper left $(m_1+1)\times m_1$ part being the matrix obtained by taking the odd rows and even columns of an $(2m_1+1)\times(2m_1+1)$ symmetric Jordan block, and a lower right $(m-m_1-1)\times(m-m_1)$ part being the transpose of the matrix obtained by taking the odd rows and even columns of a $[2(m-m_1)-1]\times[2(m-m_1)-1]$ symmetric Jordan block.

Proof. Consider the $2n\times2n$ complex symmetric matrix

$$P=\begin{pmatrix}0&R\\R^T&0\end{pmatrix}. \tag{5}$$

Due to Theorem 5 in Chap. XI of [15], there exists a complex orthogonal Q such that $P=QP'Q^T$ with P' a direct sum of symmetric $m\times m$ Jordan blocks J_i with eigenvalue λ_i. Next we observe that whenever $[v_1;v_2]$ (v_1 and v_2 both have n rows such that $[v_1;v_2]$ has $2n$ rows) is the eigenspace of P corresponding to a symmetric Jordan block J_i, then $[v_1;-v_2]$ is the eigenspace of P corresponding to a Jordan block $-J_i$. Due to the uniqueness of the Jordan canonical decomposition, these eigenspaces will be either linearly independent (this holds, for example, for sure if the corresponding eigenvalue is different from zero), or equal to each other (which implies that the corresponding eigenvalue is equal to zero). If the first case applies, both v_1 and v_2 are orthogonal matrices.

The second degenerated case however is more difficult. In this case, it holds that $[v_1;v_2]=[v_1;-v_2]Q$ for some orthogonal Q. Let us first calculate the standard nonsymmetric Jordan canonical form $\tilde{J}$ of the symmetric Jordan block with eigenvalue 0: $J=U^\dagger\tilde{J}U$ with U unitary and symmetric. If we define $[x_1;x_2]=[v_1;v_2]U^\dagger$ and $\tilde{Q}=U^\dagger QU$, the following identities hold: $\tilde{Q}^TA_{Sip}\tilde{Q}=A_{Sip}$, $\tilde{Q}\tilde{J}=-\tilde{J}\tilde{Q}$ and $[x_1;x_2]^T[x_1;x_2]=A_{Sip}$ (the matrix A_{Sip} is defined as the matrix permuting all vectors $[x_1,x_2\cdots x_n]$ to $[x_n,x_{n-1}\cdots x_1]$). The conditions on $\tilde{Q}$ imply that $\tilde{Q}$ is equal to the matrix $\tilde{Q}_{ij}=\pm(-1)^i\delta_{ij}$. Therefore $[x_1;x_2]$ is either of the form

$$\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}a_1&0&b_1&0&c_1&\cdots\\0&a_2&0&b_2&0&\cdots\end{pmatrix} \tag{6}$$

or

$$\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}0&a_1&0&b_1&0&\cdots\\a_2&0&b_2&0&c_2&\cdots\end{pmatrix}. \tag{7}$$

Due to the constraint $[x_1;x_2]^T[x_1;x_2]=A_{Sip}$, the row dimension of $[x_1;x_2]$ and therefore of J has to be odd, as otherwise the upper rightmost entry cannot be equal to one. Retransforming to the original picture with the unitary U, it holds that this structure is preserved, and the eigenspace $[v_1;v_2]$ is of a form (6) or (7).

As the dimension of a J_i giving rise to the degenerated case has to be odd, it is compulsory that there is an even number of degenerated cases (indeed, the nondegenerate cases give rise to two times a similar block and the total dimension is even). More precisely, for each $[v_1;v_2]_j$ of the form (6), there has to exist a $[v_1;v_2]_k$ of the form (7) (eventually of different dimension). The eigenstructure of such pairs of degenerate cases can then be brought into the form

$$\begin{pmatrix} a_1^i & b_1^i \cdots a_1^k & b_1^k & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \cdots & 0\ 0 & \cdots & 0 & 0 & \cdots a_2^i & b_2^i \cdots & a_2^k & b_2^k & \cdots \end{pmatrix}$$

by right multiplication with a permutation matrix W. The effect on J_i and J_k is to transform them as

$$W^T\begin{pmatrix} J_i & 0 \\ 0 & J_k \end{pmatrix}W=\begin{pmatrix} 0 & 0 & K_i & 0 \\ 0 & 0 & 0 & K_k^T \\ K_i^T & 0 & 0 & 0 \\ 0 & K_k & 0 & 0 \end{pmatrix}$$

where K_ν represents the matrix obtained by taking the odd rows and even columns of the symmetric Jordan block J_ν.

Collecting all the pieces, it is now easily verified that the canonical form obtained is exactly of the form stated in the theorem. This completes the proof. ■

Due to the equivalence of SL(2,C)⊗SL(2,C), and SO(4,C), the normal forms arising in the above lemma will immediately yield a natural representative state for each class of four-qubit states connected by SLOCC operations. The normal form encodes the genuine non-local properties of the state, while the SLOCC operators needed to bring the state into normal form characterize the local information. The following classification is obtained:

Theorem 2. A pure state of four qubits can, up to permutations of the qubits, be transformed into one of the following nine families of states by determinant one SLOCC operations (3):

$$G_{abcd}=\frac{a+d}{2}(|0000\rangle+|1111\rangle)+\frac{a-d}{2}(|0011\rangle+|1100\rangle)+\frac{b+c}{2}(|0101\rangle+|1010\rangle)+\frac{b-c}{2}(|0110\rangle+|1001\rangle),$$

$$L_{abc_2}=\frac{a+b}{2}(|0000\rangle+|1111\rangle)+\frac{a-b}{2}(|0011\rangle+|1100\rangle)+c(|0101\rangle+|1010\rangle)+|0110\rangle,$$

$$L_{a_2b_2}=a(|0000\rangle+|1111\rangle)+b(|0101\rangle+|1010\rangle)+|0110\rangle+|0011\rangle,$$

$$L_{ab_3}=a(|0000\rangle+|1111\rangle)+\frac{a+b}{2}(|0101\rangle+|1010\rangle)+\frac{a-b}{2}(|0110\rangle+|1001\rangle)+\frac{i}{\sqrt{2}}(|0001\rangle+|0010\rangle+|0111\rangle+|1011\rangle),$$

$$L_{a_4}=a(|0000\rangle+|0101\rangle+|1010\rangle+|1111\rangle)+(i|0001\rangle+|0110\rangle-i|1011\rangle),$$

$$L_{a_20_{3\oplus\bar{1}}}=a(|0000\rangle+|1111\rangle)+(|0011\rangle+|0101\rangle+|0110\rangle),$$

$$L_{0_{5\oplus\bar{3}}}=|0000\rangle+|0101\rangle+|1000\rangle+|1110\rangle,$$

$$L_{0_{7\oplus\bar{1}}}=|0000\rangle+|1011\rangle+|1101\rangle+|1110\rangle,$$

$$L_{0_{3\oplus\bar{1}}0_{3\oplus\bar{1}}}=|0000\rangle+|0111\rangle.$$

The complex parameters a, b, c, d are the unique eigenvalues of P (5) with nonnegative real part, and the indices $L_{\alpha\beta\ldots}$ are representative for the Jordan block structure of P (e.g., $L_{a_20_{3\oplus\bar{1}}}$ means that the eigenstructure of P consists of two 2×2 Jordan blocks with eigenvalues a and $-a$, and a degenerated pair of dimension, respectively, 3 and 1).

Proof. If Theorem 1 is applied to a 4×4 R, it is easily checked that 12 different families arise where a family is defined as having Jordan and degenerated Jordan blocks of specific dimension. Note however that the orthogonal matrices obtained by application of the theorem can have determinant equal to -1, while the SLOCC operations correspond to an orthogonal matrix with determinant $+1$; this is however not a problem as these operations correspond to SLOCC operations followed by a permutation of the qubits $(1\leftrightarrow2)$ or $(3\leftrightarrow4)$. One can proceed by checking that permutations of qubits $(2\leftrightarrow3)$ or $(1\leftrightarrow4)$ transform different families into each other. It is indeed true that $R=J_1(a)\oplus J_1(b)\oplus K_{3\oplus\bar{1}}$ transforms into $R'=J_2(a)\oplus J_2(b)$ if qubit 2 and 3 are permuted. This also happens in the case $J_1(a)\oplus K_{5\oplus\bar{1}}\rightarrow J_4(a)$. Moreover it can be shown that $J_1(a)\oplus K_{3\oplus\bar{3}}$ is equivalent to $J_1(a)\oplus J_3(0)$. Therefore only nine essentially different normal forms are retained. ■

A generic pure state of four qubits can always be transformed to the G_{abcd} state. This state is peculiar in the sense that all local-density operators, obtained by tracing out all parties but one, are proportional to the identity. As shown in [8], this is the unique state (up to local unitary operations) with this property of all states connected by SLOCC operations. In the light of the results of Gisin and Bechmann-Pasquinuca [9] and Nielsen and Nielsen and Kenpe about majorization [16,17], we claim that this is the state with maximal four-partite entanglement on the complete orbit generated by SLOCC operations: the more entanglement, the more local entropy. In a later section, this argument will be made hard by showing that a whole class of entanglement monotones are indeed maximized for the locally stochastic state.

It is interesting to note that the 3-tangle [2] of the mixed states obtained by tracing out one party of this G_{abcd} state is always equal to zero. Indeed, if the right-unitary matrix U

$$U=\frac{1}{\sqrt{2(1+|\beta|^2)}}\begin{pmatrix}1 & \beta & 1 & -\beta\\ \beta & 1 & -\beta & 1\end{pmatrix},$$

$$\beta=\sqrt{-q+\sqrt{q^2-r}},$$

$$q=8a^2d^2+8b^2c^2-4a^2b^2-4a^2c^2-4d^2b^2-4d^2c^2,$$

$$r=(a^2-d^2)(b^2-c^2),$$

is applied to the 8×2 matrix

$$\begin{pmatrix}a+d & 0 & 0 & a-d & 0 & b+c & b-c & 0\\ 0 & b-c & b+c & 0 & a-d & 0 & 0 & a+d\end{pmatrix}^T$$

being the square root of the density operator obtained by tracing out the first qubit, four three-qubit W states are obtained. If we define the mixed 3-tangle as the convex roof of the square root of the three-qubit entanglement, this quantity is clearly equal to zero. Therefore the SLOCC operations maximizing the four-partite entanglement result in a loss of all true three-partite entanglement. This is reminiscent to the case of three qubits where the two-qubit state obtained by tracing out one particle of a GHZ state is separable.

Let us next discuss some specific examples. A completely separable state belongs to the family L_{abc_2} with $a=b=c=0$. If only two qubits are entangled, an Einstein-Podolsky-Rosen (EPR) state arises belonging to the family $L_{a_2b_2}$ with $a=b=0$. A state consisting of two EPR pairs belongs to G_{abcd} with $(a=1;b=c=d=0)$ or $a=b=c=d$, depending on the permutation. The class $L_{0_{3\oplus\bar{1}}0_{3\oplus\bar{1}}}$ consists of all three-qubit GHZ states accompanied with a separable qubit, while the three-qubit W state belongs to the family $L_{a_20_{3\oplus\bar{1}}}$ with $a=0$.

The four-qubit $|\Phi_4\rangle$ state [18] belongs to the generic family, while the four qubit W state $(|0001\rangle+|0010\rangle+|0100\rangle+|1000\rangle)/2$ belongs to the family L_{ab_3} with $a=b=0$. This can be shown to have a mixed 3-tangle equal to zero, but has a concurrence of 1/2 when whatever two qubits are traced out. On the contrary the state $L_{O_{r\oplus\bar{1}}}$ has all concurrences equal to zero if two qubits are traced out. This state is completely symmetric in the permutation of the qubits 2, 3, and 4. It has the property of having a mixed 3-tangle equal to 1/2 if particle 2, 3, or 4 is traced out. This can be proven by considering the 8×2 "square root"

$$\frac{1}{2}\begin{pmatrix}1 & . & . & . & . & . & . & 1\\ . & . & . & 1 & . & . & 1 & .\end{pmatrix}^T.$$

Some straightforward calculations show that the average square root of the three-qubit entanglement of the vectors obtained by multiplying this matrix with whatever $2\times n$ right-unitary matrix is equal to 1/2. Similar arguments show that only three-qubit W-type entanglement ($\tau=0$) is retained if the first qubit is traced out.

The state $L_{0_{5\oplus\bar{3}}}$ is somehow a hybrid of both the four-qubit W state and $L_{O_{7\oplus\bar{1}}}$. Again a mixed 3-tangle of 1/2 is obtained if qubit 2, 3, or 4 is traced out, a mixed 3-tangle equal to zero if qubit 1 is traced out, but now the mixed state obtained by tracing out qubit 1 and (3 or 4) has a concurrence equal to 1/2, while the other concurrences vanish.

Another interesting state belongs to the family L_{a_4} with $a=0$: $|\psi\rangle=(|0001\rangle+|0110\rangle+|1000\rangle)/\sqrt{3}$. Its mixed 3-tangle equals 2/3 in the case of tracing out qubit 1 or 4 and vanishes otherwise. Moreover the concurrence vanishes everywhere if two qubits are traced out except in the case of tracing out qubits 2 and 3, resulting in a concurrence of 2/3.

After this zoological survey, let us next move on to the topic of entanglement monotones. The complex eigenvalues of P (5), given by $\pm(a,b,c,d)$, are the only invariants under all determinant one SLOCC operations (note that an eigenvalue zero is associated to the degenerated Jordan blocks). In [8] it was proven that all real positive functions of the parameters of a pure state that are linearly homogeneous in ρ and remain invariant under determinant one SLOCC operations, are entanglement monotones (in the case of mixed states they are defined by the convex roof formalism). Therefore all real positive homogeneous functions of (a^2,b^2,c^2,d^2) are entanglement monotones, such as

$$M_\alpha(\psi)=|a^\alpha+b^\alpha+c^\alpha+d^\alpha|^{2/\alpha}.$$

Taking into account one degree of freedom due to the phase, this gives rise to a seven-parameter family of entanglement monotones. All these entanglement monotones are maximized by the operations making the density matrix locally stochastic [8] (meaning that the identity is obtained when all qubits but one are traced out). The optimal single-copy distillation procedure for a generic pure state is therefore to implement the SLOCC operations bringing it into its normal form G_{abcd}. This is in complete accordance with the results of Nielsen on majorization [16]. Note that all the other normal forms can only be brought into the local stochastic normal form by a filtering procedure whose probability of success tends to zero [8].

In summary, we have identified all different families of pure states of four qubits generated by SLOCC operations. Only one family is generic, and all states in it can be made locally stochastic by SLOCC operations. The same SLOCC operations represent the optimal single-copy distillation protocol. The eight other families correspond to states having some kind of degenerated four-partite entanglement and are the four-partite generalizations of the three-partite W state.

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