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The QR Decomposition and the Singular Value Decomposition in the Symmetrized Max-Plus Algebra Revisited*

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Abstract. This paper is an updated and extended version of the paper "The QR Decomposition and the Singular Value Decomposition in the Symmetrized Max-Plus Algebra" (B. De Schutter and B. De Moor, *SIAM J. Matrix Anal. Appl.*, 19 (1998), pp. 378–406). The max-plus algebra, which has maximization and addition as its basic operations, can be used to describe and analyze certain classes of discrete-event systems, such as flexible manufacturing systems, railway networks, and parallel processor systems. In contrast to conventional algebra and conventional (linear) system theory, the max-plus algebra and the max-plus-algebraic system theory for discrete-event systems are at present far from fully developed, and many fundamental problems still have to be solved. Currently, much research is going on to deal with these problems and to further extend the max-plus algebra and to develop a complete max-plus-algebraic system theory for discrete-event systems.

In this paper we address one of the remaining gaps in the max-plus algebra by considering matrix decompositions in the symmetrized max-plus algebra. The symmetrized max-plus algebra is an extension of the max-plus algebra obtained by introducing a max-plus-algebraic analogue of the −-operator. We show that we can use well-known linear algebra algorithms to prove the existence of max-plus-algebraic analogues of basic matrix decompositions from linear algebra such as the QR decomposition, the singular value decomposition, the Hessenberg decomposition, the LU decomposition, and so on. These max-plus-algebraic matrix decompositions could play an important role in the max-plus-algebraic system theory for discrete-event systems.

Key words. max-plus algebra, matrix decompositions, QR decomposition, singular value decomposition, discrete-event systems

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1. Introduction.

In recent years both industry and the academic world have become more and more interested in techniques to model, analyze, and control complex systems such as flexible manufacturing systems, telecommunication networks,

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multiprocessor operating systems, railway networks, traffic control systems, logistic systems, intelligent transportation systems, computer networks, multilevel monitoring and control systems, and so on. These systems are typical examples of *discrete-event systems*, the subject of an emerging discipline in system and control theory. The class of the discrete-event systems essentially contains man-made systems that consist of a finite number of resources (e.g., machines, communications channels, or processors) that are shared by several users (e.g., product types, information packets, or jobs), all of which contribute to the achievement of some common goal (e.g., the assembly of products, the end-to-end transmission of a set of information packets, or a parallel computation) [1]. There exist many different modeling and analysis frameworks for discrete-event systems such as Petri nets, finite state machines, queuing networks, automata, semi-Markov processes, max-plus algebra, formal languages, temporal logic, perturbation analysis, process algebra, and computer models [1, 5, 24, 37, 38, 39, 57, 64].

Although in general discrete-event systems lead to a nonlinear description in conventional algebra, there exists a subclass of discrete-event systems for which this model becomes "linear" when we formulate it in the max-plus algebra [1, 8, 10], which has maximization and addition as its basic operations. Discrete-event systems in which only synchronization and no concurrency or choice occur can be modeled using the operations maximization (corresponding to synchronization: a new operation starts as soon as all preceding operations have been finished) and addition (corresponding to durations: the finishing time of an operation equals the starting time plus the duration). This leads to a description that is "linear" in the max-plus algebra. Therefore, discrete-event systems with synchronization but no concurrency are called *max-plus-linear discrete-event systems*.

There exists a remarkable analogy between the basic operations of the max-plus algebra (maximization and addition) on the one hand, and the basic operations of conventional algebra (addition and multiplication) on the other hand. As a consequence, many concepts and properties of conventional algebra (such as the Cayley–Hamilton theorem, eigenvectors, eigenvalues, and Cramer's rule) also have a max-plus-algebraic analogue [1, 10, 25, 55]. This analogy also allows us to translate many concepts, properties, and techniques from conventional linear system theory to system theory for max-plus-linear discrete-event systems. However, there are also some major differences that prevent a straightforward translation of properties, concepts, and algorithms from conventional linear algebra and linear system theory to max-plus algebra and max-plus-algebraic system theory for discrete-event systems.

Compared to linear algebra and linear system theory, the max-plus algebra and the max-plus-algebraic system theory for discrete-event systems is at present far from fully developed, and much research on this topic is still needed in order to get a complete system theory. The main goal of this paper is to fill one of the gaps in the theory of the max-plus algebra by showing that there exist max-plus-algebraic analogues of many fundamental matrix decompositions from linear algebra such as the QR decomposition and the singular value decomposition. These matrix decompositions are important tools in many linear algebra algorithms (see [31, 40, 41, 59] and the references cited therein) and in many contemporary algorithms for the identification of linear systems (see [44, 45, 50, 60, 61, 62, 63] and the references cited therein). We conjecture that the max-plus-algebraic analogues of these decompositions will also play an important role in the max-plus-algebraic system theory for discrete-event systems. For an overview of ongoing work in connection with the max-plus algebra and with modeling, identification, and control of max-plus-linear discrete-event systems in particular, we refer the interested reader to [1, 3, 4, 9, 22, 25, 26, 27, 28, 29, 35, 36, 46] and the references therein.

In [55], Olsder and Roos used asymptotic equivalences to show that every matrix has at least one max-plus-algebraic eigenvalue and to prove max-plus-algebraic versions of Cramer's rule and of the Cayley–Hamilton theorem. We shall use an extended and formalized version of their technique to prove the existence of the QR decomposition and the singular value decomposition in the symmetrized max-plus algebra. The symmetrized max-plus algebra is an extension of the max-plus algebra obtained by introducing a max-plus-algebraic analogue of the $-$-operator (see section 3.2). In our existence proof we shall use algorithms from linear algebra. This proof technique can easily be adapted to prove the existence of max-plus-algebraic analogues of many other matrix decompositions from linear algebra such as the Hessenberg decomposition, the LU decomposition, the eigenvalue decomposition, the Schur decomposition, and so on.

This paper is an updated and extended version of [19]. To make the paper more accessible, we have added extra examples and included some additional background material and references to the (recent) literature. Furthermore, some recent results in connection with algorithms to compute max-plus-algebraic matrix factorizations have been added.

The paper is organized as follows. After introducing some concepts and definitions in section 2, we give a short introduction to the max-plus algebra and the symmetrized max-plus algebra in section 3. Next, we establish a link between a ring of real functions (with conventional addition and multiplication as basic operations) and the symmetrized max-plus algebra. In section 5 we use this link to define the QR decomposition and the singular value decomposition of a matrix in the symmetrized max-plus algebra and to prove the existence of these decompositions. In section 6 we discuss some methods to compute max-plus-algebraic matrix decompositions. We conclude with a worked example.

2. Notations and Definitions.

In this section we give some definitions that will be needed in the following sections.

2.1. Matrices and Vectors.

The set of all reals except for 0 is represented by $\mathbb{R}_0$ ($\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$). The set of the nonnegative real numbers is denoted by $\mathbb{R}^+$, and the set of the nonpositive real numbers is denoted by $\mathbb{R}^-$. We have $\mathbb{R}_0^+ = \mathbb{R}^+ \setminus \{0\}$. The set of the integers is denoted by $\mathbb{Z}$ and the set of the nonnegative integers by $\mathbb{N}$. We have $\mathbb{N}_0 = \mathbb{N} \setminus \{0\}$.

We shall use "vector" as a synonym for "n-tuple." Furthermore, all vectors are assumed to be column vectors. If a is a vector, then a_i is the ith component of a. If A is a matrix, then a_{ij} or $(A)_{ij}$ is the entry on the ith row and the jth column of A. The transpose of the matrix A is denoted by A^T. The n by n identity matrix is denoted by I_n and the m by n zero matrix is denoted by $O_{m\times n}$.

The matrix $A \in \mathbb{R}^{n\times n}$ is called orthogonal if $A^T A = I_n$. The Frobenius norm of the matrix $A \in \mathbb{R}^{m\times n}$ is represented by

$$\|A\|_{\mathrm{F}} = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2}.$$

The 2-norm of the vector a is defined by $\|a\|_2 = \sqrt{a^T a}$, and the 2-norm of the matrix A is defined by

$$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2 .$$

THEOREM 2.1 (QR decomposition). *If $A \in \mathbb{R}^{m\times n}$, then there exist an orthogonal matrix $Q \in \mathbb{R}^{m\times m}$ and an upper triangular matrix $R \in \mathbb{R}^{m\times n}$ such that $A = QR$. We say that QR is a* QR decomposition *(QRD) of A.*

THEOREM 2.2 (singular value decomposition). *Let $A \in \mathbb{R}^{m\times n}$ and let $r = \min(m,n)$. Then there exist a diagonal matrix $\Sigma \in \mathbb{R}^{m\times n}$ and two orthogonal matrices $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ such that*

$$A = U\,\Sigma\,V^T \tag{1}$$

with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0$, where $\sigma_i = (\Sigma)_{ii}$ for $i = 1,2,\ldots,r$. Factorization (1) *is called a* singular value decomposition *(SVD) of A.*

Let $U\Sigma V^T$ be an SVD of the matrix $A \in \mathbb{R}^{m\times n}$. The diagonal entries of Σ are the singular values of A. We have $\sigma_1 = \|A\|_2$. The columns of U are the left singular vectors of A and the columns of V are the right singular vectors of A. For more information on the QRD and the SVD the interested reader is referred to [31, 40, 41, 58, 59].

2.2. Functions.

We use f, $f(\cdot)$, or $x \mapsto f(x)$ to represent a function. The domain of definition of the function f is denoted by $\operatorname{dom} f$, and the value of f at $x \in \operatorname{dom} f$ is denoted by $f(x)$.

DEFINITION 2.3 (analytic function). *Let f be a real function and let $\alpha \in \mathbb{R}$ be an interior point of* $\operatorname{dom} f$. *Then f is analytic in α if the Taylor series of f with center α exists and if there is a neighborhood of α where this Taylor series converges to f.*

A real function f is analytic in an interval $[\alpha,\beta] \subseteq \operatorname{dom} f$ if it is analytic in every point of that interval.

A real matrix-valued function $\tilde{F}$ is analytic in $[\alpha,\beta] \subseteq \operatorname{dom} \tilde{F}$ if all its entries are analytic in $[\alpha,\beta]$.

DEFINITION 2.4 (asymptotic equivalence in the neighborhood of ∞). *Let f and g be real functions such that ∞ is an accumulation point of* $\operatorname{dom} f$ *and* $\operatorname{dom} g$. *If there is no real number K such that g is identically zero in $[K,\infty)$, then we say that f is asymptotically equivalent to g in the neighborhood of ∞, denoted by $f(x) \sim g(x)$, $x \to \infty$, if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.*

If there exists a real number L such that both f and g are identically zero in $[L,\infty)$, then we also say that $f(x) \sim g(x)$, $x \to \infty$.

Let $\tilde{F}$ and $\tilde{G}$ be real m by n matrix-valued functions such that ∞ is an accumulation point of $\operatorname{dom}\tilde{F}$ *and* $\operatorname{dom}\tilde{G}$. *Then $\tilde{F}(x) \sim \tilde{G}(x)$, $x \to \infty$, if $\tilde{f}_{ij}(x) \sim \tilde{g}_{ij}(x)$, $x \to \infty$ for $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$.*

The main difference between this definition and the conventional definition of asymptotic equivalence is that Definition 2.4 also allows us to say that a function is asymptotically equivalent to 0 in the neighborhood of ∞: $f(x) \sim 0$, $x \to \infty$, if there exists a real number L such that $f(x) = 0$ for all $x \geq L$.

3. The Max-Plus Algebra and the Symmetrized Max-Plus Algebra.

In this section we give a short introduction to the max-plus algebra and the symmetrized max-plus algebra. A complete overview of the max-plus algebra can be found in [1, 10, 25].

Table 3.1 *Some analogies between conventional algebra and the max-plus algebra.*

Conventional algebra		Max-plus algebra
$+$	$\leftrightarrow$	$\oplus$ (=max)
$\times$	$\leftrightarrow$	$\otimes$ (=+)
0	$\leftrightarrow$	ε (=$-\infty$)
1	$\leftrightarrow$	0

3.1. The Max-Plus Algebra. The basic max-plus-algebraic operations are defined as follows:

$$x \oplus y = \max(x, y), \tag{2}$$

$$x \otimes y = x + y \tag{3}$$

for $x, y \in \mathbb{R} \cup \{-\infty\}$ with, by definition, $\max(x, -\infty) = x$ and $x + (-\infty) = -\infty$ for all $x \in \mathbb{R} \cup \{-\infty\}$. The reason for using the symbols $\oplus$ and $\otimes$ to represent maximization and addition is that there is a remarkable analogy between $\oplus$ and addition, and between $\otimes$ and multiplication: many concepts and properties from conventional linear algebra (such as the Cayley–Hamilton theorem, eigenvectors, eigenvalues, and Cramer's rule) can be translated to the (symmetrized) max-plus algebra by replacing $+$ by $\oplus$ and $\times$ by $\otimes$ (see also section 4 and Table 3.1). Therefore, we also call $\oplus$ the max-plus-algebraic addition. Likewise, we call $\otimes$ the max-plus-algebraic multiplication. The resulting algebraic structure $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \oplus, \otimes)$ is called the *max-plus algebra*.

Define $\mathbb{R}_\varepsilon = \mathbb{R} \cup \{-\infty\}$. The zero element for $\oplus$ in $\mathbb{R}_\varepsilon$ is represented by $\varepsilon \overset{\text{def}}{=} -\infty$. So $x \oplus \varepsilon = x = \varepsilon \oplus x$ for all $x \in \mathbb{R}_\varepsilon$. Let $r \in \mathbb{R}$. The rth max-plus-algebraic power of $x \in \mathbb{R}$ is denoted by $x^{\otimes^r}$ and corresponds to rx in conventional algebra. If $x \in \mathbb{R}$, then $x^{\otimes^0} = 0$ and the inverse element of x with respect to (w.r.t.) $\otimes$ is $x^{\otimes^{-1}} = -x$. There is no inverse element for ε since ε is absorbing for $\otimes$. If $r > 0$, then $\varepsilon^{\otimes^r} = \varepsilon$. If $r \leq 0$, then $\varepsilon^{\otimes^r}$ is not defined.

The rules for the order of evaluation of the max-plus-algebraic operators are similar to those of conventional algebra. So max-plus-algebraic power has the highest priority, and max-plus-algebraic multiplication has a higher priority than max-plus-algebraic addition.

Example 3.1. *We have*

$$\begin{aligned}
2 \oplus 3 &= \max(2, 3) = 3, \\
2 \otimes 3 &= 2 + 3 = 5, \\
2^{\otimes^3} &= 3 \cdot 2 = 6, \\
2 \oplus \varepsilon &= \max(2, -\infty) = 2, \\
2 \otimes \varepsilon &= 2 + (-\infty) = -\infty = \varepsilon, \\
3 \otimes (-1) \oplus 2 \otimes \varepsilon &= (3 \otimes (-1)) \oplus (2 \otimes \varepsilon), \\
&= (3 + (-1)) \oplus \varepsilon, \\
&= 2 \oplus \varepsilon, \\
&= 2.
\end{aligned}$$

Consider the finite sequence a_1, a_2, $\ldots$, a_n with $a_i \in \mathbb{R}_\varepsilon$ for all i. We define

$$\bigoplus_{i=1}^{n} a_i = a_1 \oplus a_2 \oplus \cdots \oplus a_n .$$

The matrix E_n is the n by n max-plus-algebraic identity matrix:

$$\begin{aligned} (E_n)_{ii} &= 0 && \text{for } i = 1,2,\ldots,n\,, \\ (E_n)_{ij} &= \varepsilon && \text{for } i = 1,2,\ldots,n \text{ and } j = 1,2,\ldots,n \text{ with } i \neq j\,. \end{aligned}$$

The m by n max-plus-algebraic zero matrix is represented by $\mathcal{E}_{m\times n}$:

$$(\mathcal{E}_{m\times n})_{ij} = \varepsilon \qquad \text{for all } i,j\,.$$

The off-diagonal entries of a max-plus-algebraic diagonal matrix $D \in \mathbb{R}_\varepsilon^{m\times n}$ are equal to ε: $d_{ij} = \varepsilon$ for all i,j with $i \neq j$. A matrix $R \in \mathbb{R}_\varepsilon^{m\times n}$ is a max-plus-algebraic upper triangular matrix if $r_{ij} = \varepsilon$ for all i,j with $i > j$. If we permute the rows or the columns of the max-plus-algebraic identity matrix, we obtain a max-plus-algebraic permutation matrix.

The operations $\oplus$ and $\otimes$ are extended to matrices as follows. If $\alpha \in \mathbb{R}_\varepsilon$, $A, B \in \mathbb{R}_\varepsilon^{m\times n}$, and $C \in \mathbb{R}_\varepsilon^{n\times p}$, then we have

$$\begin{aligned} (\alpha \otimes A)_{ij} &= \alpha \otimes a_{ij} = \alpha + a_{ij} && \text{for } i = 1,2,\ldots,m \text{ and } j = 1,2,\ldots,n, \\ (A \oplus B)_{ij} &= a_{ij} \oplus b_{ij} = \max(a_{ij}, b_{ij}) && \text{for } i = 1,2,\ldots,m \text{ and } j = 1,2,\ldots,n, \end{aligned}$$

and

$$(A \otimes C)_{ij} = \bigoplus_{k=1}^{n} a_{ik} \otimes c_{kj} = \max_{k=1,\ldots,n} \{a_{ik} + c_{kj}\} \quad \text{for } i = 1,\ldots,m \text{ and } j = 1,\ldots,p.$$

EXAMPLE 3.2. *Consider*

$$A = \begin{bmatrix} 3 & 2 \\ 0 & \varepsilon \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} -1 & \varepsilon \\ \varepsilon & 4 \end{bmatrix}.$$

Note that B is a max-plus-algebraic diagonal matrix. We have

$$\begin{aligned} 2 \otimes A &= \begin{bmatrix} 2\otimes 3 & 2 \otimes 2 \\ 2 \otimes 0 & 2 \otimes \varepsilon \end{bmatrix} = \begin{bmatrix} 2+3 & 2+2 \\ 2+0 & \varepsilon \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 2 & \varepsilon \end{bmatrix}, \\ A \oplus B &= \begin{bmatrix} 3 \oplus (-1) & 2 \oplus \varepsilon \\ 0 \oplus \varepsilon & \varepsilon \oplus 4 \end{bmatrix} = \begin{bmatrix} \max(3,-1) & \max(2,-\infty) \\ \max(0,-\infty) & \max(-\infty,4) \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix}, \\ A \otimes B &= \begin{bmatrix} 3 \otimes (-1) \oplus 2 \otimes \varepsilon & 3 \otimes \varepsilon \oplus 2 \otimes 4 \\ 0 \otimes (-1) \oplus \varepsilon \otimes \varepsilon & 0 \otimes \varepsilon \oplus \varepsilon \otimes 4 \end{bmatrix} \\ &= \begin{bmatrix} 2 \oplus \varepsilon & \varepsilon \oplus 6 \\ -1 \oplus \varepsilon & \varepsilon \oplus \varepsilon \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ -1 & \varepsilon \end{bmatrix}. \end{aligned}$$

The matrix

$$P = \begin{bmatrix} \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & 0 \\ 0 & \varepsilon & \varepsilon \end{bmatrix}$$

is a max-plus-algebraic permutation matrix. We have

$$P \otimes \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 1 & 2 & 3 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \otimes P = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 4 & 5 \\ 9 & 7 & 8 \end{bmatrix}.$$

3.2. The Symmetrized Max-Plus Algebra. One of the major differences between conventional algebra and the max-plus algebra is that there exist no inverse elements w.r.t. $\oplus$ in $\mathbb{R}_\varepsilon$: if $x \in \mathbb{R}_\varepsilon$, then there does not exist an element $y_x \in \mathbb{R}_\varepsilon$ such that $x \oplus y_x = \varepsilon = y_x \oplus x$, except when x is equal to ε. So $(\mathbb{R}_\varepsilon, \oplus)$ is not a group. Therefore, we now introduce $\mathbb{S}_{\max}$ [1, 25, 49], which is a kind of symmetrization of the max-plus algebra. This can be compared with the extension of $(\mathbb{N}, +, \times)$ to $(\mathbb{Z}, +, \times)$. In section 4 we shall show that $\mathbb{R}_{\max}$ corresponds to a set of nonnegative real functions with addition and multiplication as basic operations and that $\mathbb{S}_{\max}$ corresponds to a set of real functions with addition and multiplication as basic operations. Since the $\oplus$ operation is idempotent, we cannot use the conventional symmetrization technique since every idempotent group reduces to a trivial group [1, 49]. Nevertheless, it is possible to adapt the method of the construction of $\mathbb{Z}$ from $\mathbb{N}$ to obtain "balancing" elements rather than inverse elements.

We shall restrict ourselves to a short introduction to the most important features of $\mathbb{S}_{\max}$. This introduction is based on [1, 25, 49].

3.2.1. The Algebra of Pairs. We consider the set of ordered pairs $\mathcal{P}_\varepsilon \overset{\text{def}}{=} \mathbb{R}_\varepsilon \times \mathbb{R}_\varepsilon$ with operations $\oplus$ and $\otimes$ that are defined as follows:

$$(x, y) \oplus (w, z) = (x \oplus w,\ y \oplus z), \tag{4}$$

$$(x, y) \otimes (w, z) = (x \otimes w \oplus y \otimes z,\ x \otimes z \oplus y \otimes w) \tag{5}$$

for (x, y), $(w, z) \in \mathcal{P}_\varepsilon$, where the operations $\oplus$ and $\otimes$ on the right-hand side correspond to maximization and addition as defined in (2) and (3). The reason for also using $\oplus$ and $\otimes$ on the left-hand side is that these operations correspond to $\oplus$ and $\otimes$ as defined in $\mathbb{R}_{\max}$. Indeed, if $x, y \in \mathbb{R}_\varepsilon$, then we have

$$(x, -\infty) \oplus (y, -\infty) = (x \oplus y,\ -\infty),$$
$$(x, -\infty) \otimes (y, -\infty) = (x \otimes y,\ -\infty).$$

So the operations $\oplus$ and $\otimes$ of the algebra of pairs as defined by (4)–(5) correspond to the operations $\oplus$ and $\otimes$ of the max-plus algebra as defined by (2)–(3).

It is easy to verify that in $\mathcal{P}_\varepsilon$ the $\oplus$ operation is associative, commutative, and idempotent, and its zero element is $(\varepsilon, \varepsilon)$; that the $\otimes$ operation is associative, commutative, and distributive w.r.t. $\oplus$; that the identity element of $\otimes$ is $(0, \varepsilon)$; and that the zero element $(\varepsilon, \varepsilon)$ is absorbing for $\otimes$. We call the structure $(\mathcal{P}_\varepsilon, \oplus, \otimes)$ the *algebra of pairs*.

Example 3.3. *We have*

$$(3, 0) \oplus (2, 5) = (3 \oplus 2, 0 \oplus 5) = (3, 5),$$
$$(3, 0) \otimes (2, 5) = (3 \otimes 2 \oplus 0 \otimes 5, 3 \otimes 5 \oplus 0 \otimes 2) = (5 \oplus 5, 8 \oplus 2) = (5, 8).$$

If $u = (x, y) \in \mathcal{P}_\varepsilon$, then we define the max-plus-absolute value of u as $|u|_\oplus = x \oplus y$ and we introduce two unary operators: $\ominus$ (the max-plus-algebraic minus operator) and $(\cdot)^\bullet$ (the balance operator) such that $\ominus u = (y, x)$ and $u^\bullet = u \oplus (\ominus u) = (|u|_\oplus, |u|_\oplus)$. We have

$$u^\bullet = (\ominus u)^\bullet = (u^\bullet)^\bullet, \tag{6}$$

$$u \otimes v^\bullet = (u \otimes v)^\bullet, \tag{7}$$

$$\ominus(\ominus u) = u, \tag{8}$$

$$\ominus(u \oplus v) = (\ominus u) \oplus (\ominus v), \tag{9}$$

$$\ominus(u \otimes v) = (\ominus u) \otimes v \tag{10}$$

for all $u, v \in \mathcal{P}_\varepsilon$. The last three properties allow us to write $u \ominus v$ instead of $u \oplus (\ominus v)$. Since the properties (8)–(10) resemble properties of the $-$-operator in conventional algebra, we could say that the $\ominus$-operator of the algebra of pairs can be considered as the analogue of the $-$-operator of conventional algebra (see also section 4). As for the order of evaluation of the max-plus-algebraic operators, the max-plus-algebraic minus operator has the same, i.e., the lowest, priority as the max-plus-algebraic addition operator.

Example 3.4. *We have*

$$\begin{aligned} \ominus(3,0) &= (0,3), \\ |(3,0)|_\oplus &= 3 \oplus 0 = 3, \\ (3,0)^\bullet &= (3,3). \end{aligned}$$

Furthermore, as an illustration of (9)*, we have*

$$\begin{aligned} \ominus\big((3,0) \oplus (2,5)\big) &= \ominus(3,5) = (5,3) = (0 \oplus 5, 3 \oplus 2) = (0,3) \oplus (5,2) \\ &= \big(\ominus(3,0)\big) \oplus \big(\ominus(2,5)\big). \end{aligned}$$

In conventional algebra we have $x - x = 0$ for all $x \in \mathbb{R}$, but in the algebra of pairs we have $u \ominus u = u \oplus (\ominus u) = u^\bullet \neq (\varepsilon, \varepsilon)$ for all $u \in \mathcal{P}_\varepsilon$ unless u is equal to $(\varepsilon, \varepsilon)$, the zero element for $\oplus$ in $\mathcal{P}_\varepsilon$. Therefore, we introduce the following new relation.

Definition 3.5 (balance relation). *Consider $u = (x, y)$, $v = (w, z) \in \mathcal{P}_\varepsilon$. We say that u balances v, denoted by $u \nabla v$, if $x \oplus z = y \oplus w$.*

We have $u \ominus u = u^\bullet = (|u|_\oplus, |u|_\oplus) \nabla (\varepsilon, \varepsilon)$ for all $u \in \mathcal{P}_\varepsilon$. The balance relation is reflexive and symmetric, but it is not transitive, as is shown by the following example.

Example 3.6. *We have $(3,0) \nabla (3,3)$ since $3 \oplus 3 = 3 = 0 \oplus 3$. Furthermore, $(3,3) \nabla (1,3)$. However, $(3,0) \not\nabla (1,3)$ since $3 \oplus 3 = 3 \neq 1 = 0 \oplus 1$.*

So the balance relation is not an equivalence relation and we cannot use it to define the quotient set of $\mathcal{P}_\varepsilon$ by ∇ (as opposed to conventional algebra, where $(\mathbb{N} \times \mathbb{N})/\!=$ yields $\mathbb{Z}$). Therefore, we introduce another relation that is closely related to the balance relation and that is defined as follows: if (x, y), $(w, z) \in \mathcal{P}_\varepsilon$, then

$$(x,y)\mathcal{B}(w,z) \quad \text{if} \quad \begin{cases} (x,y) \nabla (w,z) & \text{if } x \neq y \text{ and } w \neq z\,, \\ (x,y) = (w,z) & \text{otherwise.} \end{cases}$$

Note that, referring to Example 3.6, we have $(3,0) \not\mathcal{B} (3,3)$ and $(3,3) \not\mathcal{B} (1,3)$. If $u \in \mathcal{P}_\varepsilon$, then $u \ominus u \not\mathcal{B} (\varepsilon, \varepsilon)$ unless u is equal to $(\varepsilon, \varepsilon)$. It is easy to verify that $\mathcal{B}$ is an equivalence relation that is compatible with $\oplus$ and $\otimes$, with the balance relation ∇, and with the $\ominus$, $|\cdot|_\oplus$, and $(\,\cdot\,)^\bullet$ operators. We can distinguish among three kinds of equivalence classes generated by $\mathcal{B}$:

1. $\overline{(w, -\infty)} = \{\,(w, x) \in \mathcal{P}_\varepsilon \,|\, x < w\,\}$, called max-plus-positive;
2. $\overline{(-\infty, w)} = \{\,(x, w) \in \mathcal{P}_\varepsilon \,|\, x < w\,\}$, called max-plus-negative;
3. $\overline{(w, w)} = \{\,(w, w) \in \mathcal{P}_\varepsilon\,\}$, called balanced.

The class $\overline{(\varepsilon, \varepsilon)}$ is called the max-plus-zero class.

3.2.2. The Symmetrized Max-Plus Algebra.

Let us now define the quotient set $\mathbb{S} = \mathcal{P}_\varepsilon / \mathcal{B}$. The algebraic structure $\mathbb{S}_{\max} = (\mathbb{S}, \oplus, \otimes)$ is called the *symmetrized max-plus algebra*. By associating $\overline{(w, -\infty)}$ with $w \in \mathbb{R}_\varepsilon$, we can identify $\mathbb{R}_\varepsilon$ with the set of max-plus-positive or max-plus-zero classes denoted by $\mathbb{S}^\oplus$. The set of max-plus-negative or max-plus-zero classes will be denoted by $\mathbb{S}^\ominus$, and the set of

Table 3.2 *Some analogies between conventional algebra and the symmetrized max-plus algebra.*

Conventional algebra		Symmetrized max-plus algebra
$+$	$\leftrightarrow$	$\oplus$
$\times$	$\leftrightarrow$	$\otimes$
$-$	$\leftrightarrow$	$\ominus$
$=$	$\leftrightarrow$	∇
0	$\leftrightarrow$	$a^\bullet$ $(a \in \mathbb{R}_\varepsilon)$
$\mathbb{R}^+$	$\leftrightarrow$	$\mathbb{S}^\oplus$
$\mathbb{R}^-$	$\leftrightarrow$	$\mathbb{S}^\ominus$

balanced classes will be represented by $\mathbb{S}^\bullet$. This results in the following decomposition: $\mathbb{S} = \mathbb{S}^\oplus \cup \mathbb{S}^\ominus \cup \mathbb{S}^\bullet$. Note that the max-plus-zero class $\overline{(\varepsilon, \varepsilon)}$ corresponds to ε. The max-plus-positive elements, the max-plus-negative elements, and ε are called signed. Define $\mathbb{S}^\vee = \mathbb{S}^\oplus \cup \mathbb{S}^\ominus$. Note that $\mathbb{S}^\oplus \cap \mathbb{S}^\ominus \cap \mathbb{S}^\bullet = \left\{ \overline{(\varepsilon, \varepsilon)} \right\}$ and $\varepsilon = \ominus\varepsilon = \varepsilon^\bullet$. Some analogies between conventional algebra and the symmetrized max-plus algebra are represented in Table 3.2.

Example 3.7. *We have* $(3, 0) \in \overline{(3, -\infty)}$ *and* $(2, 5) \in \overline{(-\infty, 5)}$. *In Example* 3.3 *we have shown that* $(3, 0) \oplus (2, 5) = (3, 5) \in \overline{(-\infty, 5)}$. *Furthermore, it is easy to verify that for any* $(x, y) \in \overline{(3, -\infty)}$ *and any* $(w, z) \in \overline{(-\infty, 5)}$ *we have* $(x, y) \oplus (w, z) \in \overline{(-\infty, 5)}$. *Hence, we can write* $\overline{(3, -\infty)} \oplus \overline{(-\infty, 5)} = \overline{(-\infty, 5)}$, *or* $3 \oplus (\ominus 5) = \ominus 5$ *for short, since the classes* $\overline{(3, -\infty)}$ *and* $\overline{(-\infty, 5)}$ *can be associated with* 3 *and* $\ominus 5$, *respectively. Similarly, we can write* $3 \otimes (\ominus 5) = \ominus 8$ *since* $(3, 0) \otimes (2, 5) = (5, 8) \in \overline{(-\infty, 8)}$.

In general, if $x, y \in \mathbb{R}_\varepsilon$, then we have

$$x \oplus (\ominus y) = x \qquad \text{if } x > y\,, \tag{11}$$

$$x \oplus (\ominus y) = \ominus y \qquad \text{if } x < y\,, \tag{12}$$

$$x \oplus (\ominus x) = x^\bullet\,. \tag{13}$$

In addition, (6)–(10) also hold for $u, v \in \mathbb{R}_\varepsilon$.

Now we give some extra properties of balances that will be used in the next sections. An element with a $\ominus$-sign can be transferred to the other side of a balance as follows.

Proposition 3.8. *For all* $a, b, c \in \mathbb{S} : a \ominus c \,\nabla\, b$ *if and only if* $a \,\nabla\, b \oplus c$.

If both sides of a balance are signed, we may replace the balance by an equality.

Proposition 3.9. *For all* $a, b \in \mathbb{S}^\vee : a \,\nabla\, b \;\Rightarrow\; a = b$.

Let $a \in \mathbb{S}$. The max-plus-positive part $a^\oplus$ and the max-plus-negative part $a^\ominus$ of a are defined as follows.

- if $a \in \mathbb{S}^\oplus$, then $a^\oplus = a$ and $a^\ominus = \varepsilon$,
- if $a \in \mathbb{S}^\ominus$, then $a^\oplus = \varepsilon$ and $a^\ominus = \ominus a$,
- if $a \in \mathbb{S}^\bullet$, then there exists a number $x \in \mathbb{R}_\varepsilon$ such that $a = x^\bullet$ and then $a^\oplus = a^\ominus = x$.

So $a = a^\oplus \ominus a^\ominus$ and $a^\oplus, a^\ominus \in \mathbb{R}_\varepsilon$. Note that a decomposition of the form $a = x \ominus y$ with $x, y \in \mathbb{R}_\varepsilon$ is unique if it is required that either $x \neq \varepsilon$ and $y = \varepsilon$, $x = \varepsilon$ and $y \neq \varepsilon$, or $x = y$. Hence, the decomposition $a = a^\oplus \ominus a^\ominus$ is unique. Note that $|a|_\oplus = a^\oplus \oplus a^\ominus$ for all $a \in \mathbb{S}$. We say that $a \in \mathbb{S}$ is *finite* if $|a|_\oplus \in \mathbb{R}$. If $|a|_\oplus = \varepsilon$, then we say that a is *infinite*. Definition 3.5 can now be reformulated as follows.

Proposition 3.10. *For all* $a, b \in \mathbb{S}$: $a \nabla b$ *if and only if* $a^{\oplus} \oplus b^{\ominus} = a^{\ominus} \oplus b^{\oplus}$.

Example 3.11. *We have* $3^{\oplus} = 3$, $3^{\ominus} = \varepsilon$, *and* $(3^{\bullet})^{\oplus} = (3^{\bullet})^{\ominus} = 3$. *Hence,* $3 \nabla 4^{\bullet}$ *since* $3^{\oplus} \oplus (4^{\bullet})^{\ominus} = 3 \oplus 4 = 4 = \varepsilon \oplus 4 = 3^{\ominus} \oplus (4^{\bullet})^{\oplus}$. *We have* $3 \not\nabla \ominus 4$ *since* $3^{\oplus} \oplus (\ominus 4)^{\ominus} = 3 \oplus 4 = 4 \neq \varepsilon = \varepsilon \oplus \varepsilon = 3^{\ominus} \oplus (\ominus 4)^{\oplus}$.

Example 3.12. *Consider the balance* $x \oplus 3 \nabla \ominus 4$. *From Proposition* 3.8 *it follows that this balance can be rewritten as* $x \nabla \ominus 4 \ominus 3$ *or* $x \nabla \ominus 4$ *since* $\ominus 4 \ominus 3 = \ominus(4 \oplus 3) = \ominus 4$ *by* (9).

If we want a signed solution, the balance $x \nabla \ominus 4$ *becomes an equality by Proposition* 3.9. *This yields* $x = \ominus 4$.

To determine the balanced solutions of $x \nabla \ominus 4$ *we first rewrite* x *as* $x = t^{\bullet}$ *with* $t \in \mathbb{R}_{\varepsilon}$. *We have* $t^{\bullet} \nabla \ominus 4$ *or equivalently* $t \oplus 4 = t$ *if and only if* $t \geq 4$.

So the solution set of $x \oplus 3 \nabla \ominus 4$ *is given by* $\{ \ominus 4 \} \cup \{ t^{\bullet} \mid t \in \mathbb{R}_{\varepsilon}, t \geq 4 \}$.

The balance relation is extended to matrices in the usual way: if $A, B \in \mathbb{S}^{m \times n}$, then $A \nabla B$ if $a_{ij} \nabla b_{ij}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Propositions 3.8 and 3.9 can be extended to the matrix case as follows.

Proposition 3.13. *For all* $A, B, C \in \mathbb{S}^{m \times n}$: $A \ominus C \nabla B$ *if and only if* $A \nabla B \oplus C$.

Proposition 3.14. *For all* $A, B \in (\mathbb{S}^{\vee})^{m \times n}$: $A \nabla B \Rightarrow A = B$.

Finally, we define the norm of a vector and a matrix in the symmetrized max-plus-algebra.

Definition 3.15 (max-plus-algebraic norm). *Let* $a \in \mathbb{S}^n$. *The max-plus-algebraic norm of* a *is defined by*

$$\|a\|_{\oplus} = \bigoplus_{i=1}^{n} |a_i|_{\oplus} .$$

The max-plus-algebraic norm of the matrix $A \in \mathbb{S}^{m \times n}$ *is defined by*

$$\|A\|_{\oplus} = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} |a_{ij}|_{\oplus} .$$

The max-plus-algebraic vector norm corresponds to the p-norms from linear algebra since

$$\|a\|_{\oplus} = \left(\bigoplus_{i=1}^{n} |a_i|_{\oplus}^{\otimes^{p}} \right)^{\otimes^{\frac{1}{p}}} \qquad \text{for every } a \in \mathbb{S}^n \text{ and every } p \in \mathbb{N}_0 .$$

Indeed, we have

$$\begin{aligned}
\left(\bigoplus_{i=1}^{n} |a_i|_{\oplus}^{\otimes^{p}} \right)^{\otimes^{\frac{1}{p}}} &= \frac{1}{p} \cdot \left(\bigoplus_{i=1}^{n} |a_i|_{\oplus}^{\otimes^{p}} \right) = \frac{1}{p} \cdot \left(\bigoplus_{i=1}^{n} p \cdot |a_i|_{\oplus} \right) \\
&= \frac{p}{p} \cdot \left(\bigoplus_{i=1}^{n} |a_i|_{\oplus} \right) \qquad (\text{since}^{1}\ p \geq 0) \\
&= \bigoplus_{i=1}^{m} |a_i|_{\oplus} = \|a\|_{\oplus} .
\end{aligned}$$

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[1] If $\alpha, \beta \in \mathbb{R}_{\varepsilon}$ and $p \in \mathbb{R}^{+}$, then $p \cdot \alpha \oplus p \cdot \beta = \max(p\alpha, p\beta) = p \max(\alpha, \beta) = p \cdot (\alpha \oplus \beta)$.

Similarly, we can show that the max-plus-algebraic matrix norm corresponds to both the Frobenius norm and the p-norms from linear algebra since

$$\|A\|_\oplus = \left(\bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} |a_{ij}|_\oplus^{\otimes^2} \right)^{\otimes^{\frac{1}{2}}} \qquad \text{for every } A \in \mathbb{S}^{m \times n} ,$$

and also $\|A\|_\oplus = \max_{\|x\|_\oplus = 0} \|A \otimes x\|_\oplus$ (the maximum is reached for $x = O_{n \times 1}$).

EXAMPLE 3.16. *Let*

$$a = \begin{bmatrix} 3 \\ \ominus 5 \\ 4^\bullet \end{bmatrix} .$$

We have $\|a\|_\oplus = |3|_\oplus \oplus |\ominus 5|_\oplus \oplus |4^\bullet|_\oplus = 3 \oplus 5 \oplus 4 = 5.$

4. A Link between Conventional Algebra and the Symmetrized Max-Plus Algebra. In [55] Olsder and Roos used a kind of link between conventional algebra and the max-plus algebra based on asymptotic equivalences to show that every matrix has at least one max-plus-algebraic eigenvalue and to prove max-plus-algebraic versions of Cramer's rule and of the Cayley–Hamilton theorem. In [17] we extended and formalized this link. Now we recapitulate the reasoning of [17] but in a slightly different form that is mathematically more rigorous.

In the next section we shall encounter functions that are asymptotically equivalent to an exponential of the form νe^{xs} for $s \to \infty$. Since we want to allow exponents that are equal to ε, we set $e^{\varepsilon s}$ equal to 0 for all positive real values of s by definition. We also define the following classes of functions:

$$\mathcal{R}_{\mathrm{e}}^{+} = \left\{ f : \mathbb{R}_0^+ \to \mathbb{R}^+ \,\middle|\, f(s) = \sum_{i=0}^{n} \mu_i e^{x_i s} \text{ with } n \in \mathbb{N}, \right.$$
$$\left. \mu_i \in \mathbb{R}_0^+, \text{ and } x_i \in \mathbb{R}_\varepsilon \text{ for all } i \right\} ,$$

$$\mathcal{R}_{\mathrm{e}} = \left\{ f : \mathbb{R}_0^+ \to \mathbb{R} \,\middle|\, f(s) = \sum_{i=0}^{n} \nu_i e^{x_i s} \text{ with } n \in \mathbb{N}, \right.$$
$$\left. \nu_i \in \mathbb{R}_0, \text{ and } x_i \in \mathbb{R}_\varepsilon \text{ for all } i \right\} .$$

It is easy to verify that $(\mathcal{R}_{\mathrm{e}}, +, \times)$ is a ring.

For all $x, y, z \in \mathbb{R}_\varepsilon$ we have

$$x \oplus y = z \quad \Leftrightarrow \quad e^{xs} + e^{ys} \sim (1 + \delta_{xy}) e^{zs} , \ s \to \infty, \tag{14}$$

$$x \otimes y = z \quad \Leftrightarrow \quad e^{xs} \cdot e^{ys} = e^{zs} \qquad \text{for all } s \in \mathbb{R}_0^+, \tag{15}$$

where $\delta_{xy} = 0$ if $x \neq y$ and $\delta_{xy} = 1$ if $x = y$. The relations (14) and (15) show that there exists a connection between the operations $\oplus$ and $\otimes$ performed on the real numbers and $-\infty$, and the operations $+$ and $\times$ performed on exponentials. We shall extend this link between $(\mathcal{R}_{\mathrm{e}}^{+}, +, \times)$ and $\mathbb{R}_{\max}$ that was used in [51, 52, 53, 54, 55]—and under a slightly different form in [11]—to $\mathbb{S}_{\max}$.

We define a mapping $\mathcal{F}$ with domain of definition $\mathbb{S} \times \mathbb{R}_0 \times \mathbb{R}_0^+$ and with

$$\begin{aligned} \mathcal{F}(a,\mu,s) &= |\mu| e^{as} && \text{if } a \in \mathbb{S}^{\oplus}, \\ \mathcal{F}(a,\mu,s) &= -|\mu| e^{|a|_{\oplus} s} && \text{if } a \in \mathbb{S}^{\ominus}, \\ \mathcal{F}(a,\mu,s) &= \mu e^{|a|_{\oplus} s} && \text{if } a \in \mathbb{S}^{\bullet}, \end{aligned}$$

where $a \in \mathbb{S}$, $\mu \in \mathbb{R}_0$, and $s \in \mathbb{R}_0^+$.

In the remainder of this paper the first two arguments of $\mathcal{F}$ will most of the time be fixed and we shall only consider $\mathcal{F}$ as a function of the third argument; i.e., for a given $a \in \mathbb{S}$ and $\mu \in \mathbb{R}_0$ we consider the function $\mathcal{F}(a,\mu,\cdot)$. Note that if $x \in \mathbb{R}_\varepsilon$ and $\mu \in \mathbb{R}_0$, then we have

$$\begin{aligned} \mathcal{F}(x,\mu,s) &= |\mu| e^{xs}, \\ \mathcal{F}(\ominus x,\mu,s) &= -|\mu| e^{xs}, \\ \mathcal{F}(x^{\bullet},\mu,s) &= \mu e^{xs} \end{aligned}$$

for all $s \in \mathbb{R}_0^+$. Furthermore, $\mathcal{F}(\varepsilon,\mu,\cdot) = 0$ for all $\mu \in \mathbb{R}_0$ since $e^{\varepsilon s} = 0$ for all $s \in \mathbb{R}_0^+$, by definition.

For a given $\mu \in \mathbb{R}_0$ the number $a \in \mathbb{S}$ will be mapped by $\mathcal{F}$ to an exponential function $s \mapsto \nu e^{|a|_{\oplus} s}$, where $\nu = |\mu|$, $\nu = -|\mu|$, or $\nu = \mu$ depending on the max-plus-algebraic sign of a. In order to reverse this process, we define the mapping $\mathcal{R}$, which we shall call the *reverse mapping* and which will map a function that is asymptotically equivalent to an exponential function $s \mapsto \nu e^{|a|_{\oplus} s}$ in the neighborhood of ∞ to the number $|a|_{\oplus}$ or $\ominus |a|_{\oplus}$, depending on the sign of ν. More specifically, if f is a real function, if $x \in \mathbb{R}_\varepsilon$, and if $\mu \in \mathbb{R}_0$, then we have

$$\begin{aligned} f(s) \sim |\mu| e^{xs}, \ s \to \infty \quad &\Rightarrow \quad \mathcal{R}(f) = x, \\ f(s) \sim -|\mu| e^{xs}, \ s \to \infty \quad &\Rightarrow \quad \mathcal{R}(f) = \ominus x. \end{aligned}$$

Note that $\mathcal{R}$ will always map a function that is asymptotically equivalent to an exponential function in the neighborhood of ∞ to a signed number and never to a balanced number that is different from ε. Furthermore, for a fixed $\mu \in \mathbb{R}_0$ the mappings $a \mapsto \mathcal{F}(a,\mu,\cdot)$ and $\mathcal{R}$ are not each other's inverse since these mappings are not bijections, as is shown by the following example.

EXAMPLE 4.1. *Let* $\mu = 2$. *We have* $\mathcal{F}(3,\mu,s) = 2e^{3s}$ *and* $\mathcal{F}(3^{\bullet},\mu,s) = 2e^{3s}$ *for all* $s \in \mathbb{R}_0^+$. *So* $\mathcal{R}(\mathcal{F}(3^{\bullet},\mu,\cdot)) = 3 \neq 3^{\bullet}$.

Consider the real functions f *and* g *defined by* $f(s) = 2e^{3s}$ *and* $g(s) = 2e^{3s} + e^{s}$. *We have* $f(s) \sim g(s) \sim 2e^{3s}$, $s \to \infty$. *Hence,* $\mathcal{R}(f) = \mathcal{R}(g) = 3$. *So* $\mathcal{F}(\mathcal{R}(g),\mu,\cdot) = f \neq g$.

Let $\mu \in \mathbb{R}_0$. It is easy to verify that in general we have $\mathcal{R}(\mathcal{F}(a,\mu,\cdot)) = a$ if $a \in \mathbb{S}^{\oplus} \cup \mathbb{S}^{\ominus}$, $\mathcal{R}(\mathcal{F}(a,\mu,\cdot)) = |a|_{\oplus}$ if $a \in \mathbb{S}^{\bullet}$ and $\mu > 0$, and $\mathcal{R}(\mathcal{F}(a,\mu,\cdot)) = \ominus |a|_{\oplus}$ if $a \in \mathbb{S}^{\bullet}$ and $\mu < 0$. Furthermore, if f is a real function that is asymptotically equivalent to an exponential function in the neighborhood of ∞, then we have $\mathcal{F}(\mathcal{R}(f),\mu,s) \sim f(s)$, $s \to \infty$.

Let us now extend (14)–(15) from $\mathbb{R}_\varepsilon$ to $\mathbb{S}$. For all $a,b,c \in \mathbb{S}$ we have

$$a \oplus b = c \ \Rightarrow \ \left\{ \begin{array}{l} \exists \mu_a, \mu_b, \mu_c \in \mathbb{R}_0 \text{ such that} \\ \mathcal{F}(a,\mu_a,s) + \mathcal{F}(b,\mu_b,s) \sim \mathcal{F}(c,\mu_c,s), \qquad s \to \infty, \end{array} \right. \tag{16}$$

$$\left. \begin{array}{l} \exists \mu_a, \mu_b, \mu_c \in \mathbb{R}_0 \text{ such that} \\ \mathcal{F}(a,\mu_a,s) + \mathcal{F}(b,\mu_b,s) \sim \mathcal{F}(c,\mu_c,s), \ s \to \infty \end{array} \right\} \ \Rightarrow \ a \oplus b \ \nabla \ c, \tag{17}$$

$$a \otimes b = c \;\Rightarrow\; \left\{ \begin{array}{l} \exists \mu_a, \mu_b, \mu_c \in \mathbb{R}_0 \text{ such that} \\ \mathcal{F}(a, \mu_a, s) \cdot \mathcal{F}(b, \mu_b, s) = \mathcal{F}(c, \mu_c, s) \quad \text{for all } s \in \mathbb{R}_0^+, \end{array} \right. \tag{18}$$

$$\left. \begin{array}{l} \exists \mu_a, \mu_b, \mu_c \in \mathbb{R}_0 \text{ such that} \\ \mathcal{F}(a, \mu_a, s) \cdot \mathcal{F}(b, \mu_b, s) = \mathcal{F}(c, \mu_c, s) \quad \text{for all } s \in \mathbb{R}_0^+ \end{array} \right\} \;\Rightarrow\; a \otimes b \;\nabla\; c. \tag{19}$$

As a consequence, we could say that the mapping $\mathcal{F}$ provides a link between the structure $(\mathcal{R}_{\mathrm{e}}^+, +, \times)$ and $\mathbb{R}_{\max} = (\mathbb{R}_\varepsilon, \oplus, \otimes)$ and a link between the structure $(\mathcal{R}_{\mathrm{e}}, +, \times)$ and $\mathbb{S}_{\max} = (\mathbb{S}, \oplus, \otimes)$.

Remark 4.2. *The balance in* (17) *results from the fact that we can have cancellation of equal terms with opposite sign in* $(\mathcal{R}_{\mathrm{e}}^+, +, \times)$*, whereas this is in general not possible in the symmetrized max-plus algebra since for all* $a \in \mathbb{S} \setminus \{\varepsilon\} : a \ominus a \neq \varepsilon$*. We have, e.g.,* $\mathcal{F}(3, 1, s) + \mathcal{F}(\ominus 3, 1, s) = e^{3s} - e^{3s} = 0 = e^{\varepsilon s} = \mathcal{F}(\varepsilon, 1, s)$ *for all* $s \in \mathbb{R}_0^+$*. So* $3 \oplus (\ominus 3) \;\nabla\; \varepsilon$*, but clearly* $3 \oplus (\ominus 3) = 3^\bullet \neq \varepsilon$*.*

The following example shows that the balance on the right-hand side of (19) *is also necessary: we have* $\mathcal{F}(3, 1, s) \cdot \mathcal{F}(3, 1, s) = e^{3s} \cdot e^{3s} = e^{6s} = \mathcal{F}(6^\bullet, 1, s)$ *for all* $s \in \mathbb{R}_0^+$*, but* $3 \otimes 3 = 6 \neq 6^\bullet$*.*

The equality signs in the left-hand sides of (16) *and* (18) *cannot be replaced by a balance sign, as is shown by the following example. We have* $3 \oplus (\ominus 4) = \ominus 4 \,\nabla\, 5^\bullet$*. However, there do not exist real numbers* μ_1*,* μ_2*,* $\mu_3 \in \mathbb{R}_0$ *such that*

$$\mathcal{F}(3, \mu_1, s) + \mathcal{F}(\ominus 4, \mu_2, s) \;\sim\; \mathcal{F}(5^\bullet, \mu_3, s)\,, \qquad s \to \infty,$$

or equivalently

$$|\mu_1|\, e^{3s} \;-\; |\mu_2|\, e^{4s} \;\sim\; \mu_3\, e^{5s}\,, \qquad s \to \infty.$$

This implies that in general (16) *does not hold any more if we replace the equality on the left-hand side by a balance.*

In a similar way we can show that in general $a \otimes b \,\nabla\, c$ *with* $a, b, c, \in \mathbb{S}$ *does not imply that there exist real numbers* μ_a*,* μ_b*,* $\mu_c \in \mathbb{R}_0$ *such that* $\mathcal{F}(a, \mu_a, s) \cdot \mathcal{F}(b, \mu_b, s) = \mathcal{F}(c, \mu_c, s)$ *for all* $s \in \mathbb{R}_0^+$*.*

We extend the mapping $\mathcal{F}$ to matrices as follows. If $A \in \mathbb{S}^{m \times n}$ and if $M \in \mathbb{R}_0^{m \times n}$, then $\tilde{A} = \mathcal{F}(A, M, \cdot)$ is a real m by n matrix-valued function with domain of definition $\mathbb{R}_0^+$ and with $\tilde{a}_{ij}(s) = \mathcal{F}(a_{ij}, m_{ij}, s)$ for all i, j. Note that the mapping is performed entrywise. The reverse mapping $\mathcal{R}$ is extended to matrices in a similar way: if $\tilde{A}$ is a real matrix-valued function with entries that are asymptotically equivalent to an exponential in the neighborhood of ∞, then $(\mathcal{R}(\tilde{A}))_{ij} = \mathcal{R}(\tilde{a}_{ij})$ for all i, j.

If A, B, and C are matrices with entries in $\mathbb{S}$, we have

$$A \oplus B = C \;\Rightarrow\; \left\{ \begin{array}{l} \exists M_A, M_B, M_C \text{ such that} \\ \mathcal{F}(A, M_A, s) + \mathcal{F}(B, M_B, s) \;\sim\; \mathcal{F}(C, M_C, s)\,, \qquad s \to \infty, \end{array} \right. \tag{20}$$

$$\left. \begin{array}{l} \exists M_A, M_B, M_C \text{ such that} \\ \mathcal{F}(A, M_A, s) + \mathcal{F}(B, M_B, s) \;\sim\; \mathcal{F}(C, M_C, s)\,,\; s \to \infty \end{array} \right\} \;\Rightarrow\; A \oplus B \;\nabla\; C, \tag{21}$$

$$A \otimes B = C \;\Rightarrow\; \left\{ \begin{array}{l} \exists M_A, M_B, M_C \text{ such that} \\ \mathcal{F}(A, M_A, s) \cdot \mathcal{F}(B, M_B, s) \;\sim\; \mathcal{F}(C, M_C, s)\,,\; s \to \infty, \end{array} \right. \tag{22}$$

$$\left. \begin{array}{l} \exists M_A, M_B, M_C \text{ such that} \\ \mathcal{F}(A, M_A, s) \cdot \mathcal{F}(B, M_B, s) \;\sim\; \mathcal{F}(C, M_C, s)\,,\; s \to \infty \end{array} \right\} \;\Rightarrow\; A \otimes B \;\nabla\; C. \tag{23}$$

EXAMPLE 4.3. *Let*

$$A = \begin{bmatrix} 3 & 2 \\ \ominus 0 & \varepsilon \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 0 & \ominus(-3) \\ \ominus 1 & 2^{\bullet} \end{bmatrix} .$$

Hence,

$$\begin{aligned} A \otimes B &= \begin{bmatrix} 3 \otimes 0 \oplus 2 \otimes (\ominus 1) & 3 \otimes (\ominus(-3)) \oplus 2 \otimes 2^{\bullet} \\ \ominus 0 \otimes 0 \oplus \varepsilon \otimes (\ominus 1) & \ominus 0 \otimes (\ominus(-3)) \oplus \varepsilon \otimes 2^{\bullet} \end{bmatrix} \\ &= \begin{bmatrix} 3 \oplus (\ominus 3) & \ominus 0 \oplus 4^{\bullet} \\ \ominus 0 \oplus \varepsilon & -3 \oplus \varepsilon \end{bmatrix} = \begin{bmatrix} 3^{\bullet} & 4^{\bullet} \\ \ominus 0 & -3 \end{bmatrix} . \end{aligned}$$

Let M, N, *and* $P \in \mathbb{R}_0^{2 \times 2}$. *In general, we have*

$$\begin{aligned} \mathcal{F}(A, M, s) &= \begin{bmatrix} |m_{11}|\, e^{3s} & |m_{12}|\, e^{2s} \\ -|m_{21}| & 0 \end{bmatrix} , \\ \mathcal{F}(B, N, s) &= \begin{bmatrix} |n_{11}| & -|n_{12}|\, e^{-3s} \\ -|n_{21}| e^{s} & n_{22}\, e^{2s} \end{bmatrix} , \\ \mathcal{F}(A \otimes B, P, s) &= \begin{bmatrix} p_{11}\, e^{3s} & p_{12}\, e^{4s} \\ -|p_{21}| & |p_{22}|\, e^{-3s} \end{bmatrix} \end{aligned}$$

for all $s \in \mathbb{R}_0^{+}$. *Furthermore,*

$$\begin{aligned} &\mathcal{F}(A, M, s) \cdot \mathcal{F}(B, N, s) \\ &\quad = \begin{bmatrix} (|m_{11}|\, |n_{11}| - |m_{12}|\, |n_{21}|)\, e^{3s} & -|m_{11}|\, |n_{12}| + |m_{12}|\, n_{22} e^{4s} \\ -|m_{21}|\, |n_{11}| & |m_{21}|\, |n_{12}|\, e^{-3s} \end{bmatrix} \end{aligned}$$

for all $s \in \mathbb{R}_0^{+}$.

If $|m_{11}|\,|n_{11}| - |m_{12}||n_{21}| \neq 0$ *and if we take*

$$\begin{aligned} p_{11} &= |m_{11}|\, |n_{11}| - |m_{12}|\, |n_{21}| \, , & p_{12} &= |m_{12}|\, n_{22} \, , \\ p_{21} &= |m_{21}|\, |n_{11}| \, , & p_{22} &= |m_{21}|\, |n_{12}| , \end{aligned}$$

then we have $p_{ij} \neq 0$ *for all* $i, j \in \{1, 2\}$ *and*

$$\mathcal{F}(A, M, s) \cdot \mathcal{F}(B, N, s) \ \sim \ \mathcal{F}(A \otimes B, P, s) \ , \qquad s \to \infty \, .$$

If we take $m_{ij} = n_{ij} = 1$ *for all* i, j, *we get*

$$\mathcal{F}(A, s) \cdot \mathcal{F}(B, s) \ \sim \ \begin{bmatrix} 0 & e^{4s} \\ -1 & e^{-3s} \end{bmatrix} \overset{\text{def}}{=} \tilde{C}(s) \ , \qquad s \to \infty.$$

The reverse mapping results in $C = \mathcal{R}(\tilde{C}) = \left[\begin{smallmatrix} \varepsilon & 4 \\ \ominus 0 & -3 \end{smallmatrix}\right]$. *Note that* $A \otimes B \ \nabla \ C$.

Taking $m_{ij} = n_{ij} = -(i + j)$ *for all* i, j *leads to*

$$\begin{aligned} \mathcal{F}(A, s) \cdot \mathcal{F}(B, s) &= \begin{bmatrix} (2 \cdot 2 - 3 \cdot 3)\, e^{3s} & -2 \cdot 3 - 3 \cdot 4 \cdot e^{4s} \\ -3 \cdot 2 & 3 \cdot 3 \cdot e^{-3s} \end{bmatrix} \\ &\sim \begin{bmatrix} -5e^{3s} & -12e^{4s} \\ -6 & 9e^{-3s} \end{bmatrix} \overset{\text{def}}{=} \tilde{D}(s), \qquad s \to \infty. \end{aligned}$$

The reverse mapping results in $D = \mathcal{R}(\tilde{D}) = \left[\begin{smallmatrix} \ominus 3 & \ominus 4 \\ \ominus 0 & -3 \end{smallmatrix}\right]$ *and again we have* $A \otimes B \ \nabla \ D$.

5. The QRD and the SVD in the Symmetrized Max-Plus Algebra.

In [17] we used the mapping from $\mathbb{S}_{\max}$ to $(\mathcal{R}_{\mathrm{e}}, +, \times)$ and the reverse mapping $\mathcal{R}$ to prove the existence of a kind of SVD in $\mathbb{S}_{\max}$. The proof of [17] is based on the *analytic SVD*. In this section we present an alternative proof for the existence theorem of the max-plus-algebraic SVD. The major advantage of the new proof technique that will be developed in this section over that of [17] is that it can be easily extended to prove the existence of many other matrix decompositions in the symmetrized max-plus algebra such as the max-plus-algebraic QRD, the max-plus-algebraic LU decomposition, the max-plus-algebraic eigenvalue decomposition (for symmetric matrices), and so on. This proof technique consists of transforming a matrix with entries in $\mathbb{S}$ to a matrix-valued function with exponential entries (using the mapping $\mathcal{F}$), applying an algorithm from linear algebra, and transforming the result back to the symmetrized max-plus algebra (using the mapping $\mathcal{R}$).

5.1. Sums and Series of Exponentials.

The entries of the matrices that are used in the existence proofs for the max-plus-algebraic QRD and the max-plus-algebraic SVD that will be presented in this section are sums or series of exponentials. Therefore, we first study some properties of this kind of function.

Definition 5.1 (sum or series of exponentials). *Let $\mathcal{S}_{\mathrm{e}}$ be the set of real functions that are analytic and that can be written as a (possibly infinite, but absolutely convergent) sum of exponentials in a neighborhood of ∞:*

$$\mathcal{S}_{\mathrm{e}} = \Big\{ f : A \to \mathbb{R} \;\Big|\; A \subseteq \mathbb{R},\ \exists K \in \mathbb{R}_0^+ \text{ such that } [K,\infty) \subseteq A \text{ and}$$

$$f \text{ is analytic in } [K,\infty) \text{ and either}$$

$$\text{for all } x \geq K : f(x) = \sum_{i=0}^{n} \alpha_i e^{a_i x} \tag{24}$$

$$\text{with } n \in \mathbb{N},\ \alpha_i \in \mathbb{R}_0,\ a_i \in \mathbb{R}_\varepsilon \text{ for all } i \text{ and } a_0 > a_1 > \cdots > a_n \text{ ; or}$$

$$\text{for all } x \geq K : f(x) = \sum_{i=0}^{\infty} \alpha_i e^{a_i x} \tag{25}$$

$$\text{with } \alpha_i \in \mathbb{R}_0,\ a_i \in \mathbb{R},\ a_i > a_{i+1} \text{ for all } i\,,\ \lim_{i\to\infty} a_i = \varepsilon, \text{ and}$$

$$\text{where the series converges absolutely for every } x \geq K \Big\}.$$

If $f \in \mathcal{S}_{\mathrm{e}}$, then the largest exponent in the sum or the series of exponentials that corresponds to f is called the *dominant exponent* of f.

Recall that by definition we have $e^{\varepsilon s} = 0$ for all $s \in \mathbb{R}_0^+$. Since we allow exponents that are equal to $\varepsilon = -\infty$ in the definition of $\mathcal{S}_{\mathrm{e}}$, the zero function also belongs to $\mathcal{S}_{\mathrm{e}}$. Since we require the sequence of the exponents that appear in (24) or (25) to be decreasing and since the coefficients cannot be equal to 0, any sum of exponentials of the form (24) or (25) that corresponds to the zero function consists of exactly one term, e.g., $1 \cdot e^{\varepsilon x}$.

If $f \in \mathcal{S}_{\mathrm{e}}$ is a series of the form (25), then the set $\{a_i \,|\, i = 0, 1, \ldots, \infty\}$ has no finite accumulation point since the sequence $\{a_i\}_{i=0}^{\infty}$ is decreasing and unbounded from below. Note that series of the form (25) are related to (generalized) Dirichlet series [47].

The behavior of the functions in $\mathcal{S}_{\mathrm{e}}$ in the neighborhood of ∞ is given by the following property.

Lemma 5.2. *Every function $f \in \mathcal{S}_\mathrm{e}$ is asymptotically equivalent to an exponential in the neighborhood of ∞:*

$$f \in \mathcal{S}_\mathrm{e} \;\;\Rightarrow\;\; f(x) \,\sim\, \alpha_0 e^{a_0 x},\; x \to \infty,$$

for some $\alpha_0 \in \mathbb{R}_0$ and some $a_0 \in \mathbb{R}_\varepsilon$.

Proof. See Appendix A. □

The set $\mathcal{S}_\mathrm{e}$ is closed under elementary operations such as additions, multiplications, subtractions, divisions, square roots, and absolute values.

Proposition 5.3. *If f and g belong to $\mathcal{S}_\mathrm{e}$, then ρf, $f+g$, $f-g$, fg, f^l, and $|f|$ also belong to $\mathcal{S}_\mathrm{e}$ for any $\rho \in \mathbb{R}$ and any $l \in \mathbb{N}$.*

Furthermore, if there exists a real number P such that $f(x) \neq 0$ for all $x \geq P$, then the functions $\frac{1}{f}$ and $\frac{g}{f}$ restricted to $[P,\infty)$ also belong to $\mathcal{S}_\mathrm{e}$.

If there exists a real number Q such that $f(x) > 0$ for all $x \geq Q$, then the function $\sqrt{f}$ restricted to $[Q,\infty)$ also belongs to $\mathcal{S}_\mathrm{e}$.

Proof. See Appendix B. □

5.2. The Max-Plus-Algebraic QR Decomposition. Let $\tilde{A}$ and $\tilde{R}$ be real m by n matrix-valued functions, and let $\tilde{Q}$ be a real m by m matrix-valued function. Suppose that these matrix-valued functions are defined in $J \subseteq \mathbb{R}$. If $\tilde{Q}(s)\,\tilde{R}(s) = \tilde{A}(s)$, $\tilde{Q}^T(s)\,\tilde{Q}(s) = I_m$, and $\tilde{R}(s)$ is an upper triangular matrix for all $s \in J$, then we say that $\tilde{Q}\tilde{R}$ is a *path of QRDs* of $\tilde{A}$ on J. A path of SVDs is defined in a similar way.

Note that if $\tilde{Q}\tilde{R}$ is a path of QRDs of $\tilde{A}$ on J, then we have $\|\tilde{R}(s)\|_\mathrm{F} = \|\tilde{A}(s)\|_\mathrm{F}$ for all $s \in J$. Now we prove that for a matrix with entries in $\mathcal{S}_\mathrm{e}$, there exists a path of QRDs with entries that also belong to $\mathcal{S}_\mathrm{e}$. Next, we use this result to prove the existence of a max-plus-algebraic analogue of the QRD.

Proposition 5.4. *If $\tilde{A} \in \mathcal{S}_\mathrm{e}^{m \times n}$, then there exists a path of QRDs $\tilde{Q}\tilde{R}$ of $\tilde{A}$ for which the entries of $\tilde{Q}$ and $\tilde{R}$ belong to $\mathcal{S}_\mathrm{e}$.*

Proof. To compute the QRD of a matrix with real entries we can use the Givens QR algorithm (see [31]). The operations used in this algorithm are additions, multiplications, subtractions, divisions, and square roots. Furthermore, the number of operations used in this algorithm is finite.

So if we apply this algorithm to a matrix-valued function $\tilde{A}$ with entries in $\mathcal{S}_\mathrm{e}$, then the entries of the resulting matrix-valued functions $\tilde{Q}$ and $\tilde{R}$ will also belong to $\mathcal{S}_\mathrm{e}$ by Proposition 5.3. □

Theorem 5.5 (max-plus-algebraic QR decomposition). *If $A \in \mathbb{S}^{m \times n}$, then there exist a matrix $Q \in (\mathbb{S}^\vee)^{m \times m}$ and a max-plus-algebraic upper triangular matrix $R \in (\mathbb{S}^\vee)^{m \times n}$ such that*

$$A \;\nabla\; Q \otimes R \tag{26}$$

with $Q^T \otimes Q \;\nabla\; E_m$ and $\|R\|_\oplus = \|A\|_\oplus$.

Every decomposition of the form (26) that satisfies the above conditions is called a max-plus-algebraic QRD of A.

Proof. If $A \in \mathbb{S}^{m \times n}$ has entries that are not signed, we can always define a matrix $\hat{A} \in (\mathbb{S}^\vee)^{m \times n}$ such that

$$\hat{a}_{ij} = \begin{cases} a_{ij} & \text{if } a_{ij} \text{ is signed,} \\ |a_{ij}|_\oplus & \text{if } a_{ij} \text{ is not signed} \end{cases}$$

for all i,j. Since $|\hat{a}_{ij}|_\oplus = |a_{ij}|_\oplus$ for all i,j, we have $\|\hat{A}\|_\oplus = \|A\|_\oplus$. Moreover, we have

$$\text{for all } a,b \in \mathbb{S}:\; a \;\nabla\; b \;\Rightarrow\; a^\bullet \;\nabla\; b,$$

which means that if $\hat{A} \mathrel{\nabla} Q \otimes R$, then also $A \mathrel{\nabla} Q \otimes R$. Therefore, it is sufficient to prove this theorem for signed matrices A.

So from now on we assume that A is signed. We construct $\tilde{A} = \mathcal{F}(A, M, \cdot)$, where $M \in \mathbb{R}^{m \times n}$ with $m_{ij} = 1$ for all i, j. Hence, $\tilde{a}_{ij}(s) = \gamma_{ij} e^{c_{ij} s}$ for all $s \in \mathbb{R}_0^+$ and for all i, j with $\gamma_{ij} \in \{-1, 1\}$ and $c_{ij} = |a_{ij}|_{\oplus} \in \mathbb{R}_\varepsilon$ for all i, j. Note that the entries of $\tilde{A}$ belong to $\mathcal{S}_{\mathrm{e}}$. By Proposition 5.4 there exists a path of QRDs of $\tilde{A}$. So there exist a positive real number L and matrix-valued functions $\tilde{Q}$ and $\tilde{R}$ with entries in $\mathcal{S}_{\mathrm{e}}$ such that

$$\tilde{A}(s) = \tilde{Q}(s)\,\tilde{R}(s) \qquad \text{for all } s \geq L, \tag{27}$$

$$\tilde{Q}^T(s)\,\tilde{Q}(s) = I_m \qquad \text{for all } s \geq L, \tag{28}$$

$$\|\tilde{R}(s)\|_{\mathrm{F}} = \|\tilde{A}(s)\|_{\mathrm{F}} \qquad \text{for all } s \geq L. \tag{29}$$

The entries of $\tilde{Q}$ and $\tilde{R}$ belong to $\mathcal{S}_{\mathrm{e}}$ and are thus asymptotically equivalent to an exponential in the neighborhood of ∞ by Lemma 5.2.

If we define $Q = \mathcal{R}(\tilde{Q})$ and $R = \mathcal{R}(\tilde{R})$, then Q and R have signed entries. If we apply the reverse mapping $\mathcal{R}$ to (27)–(29), we get

$$A \mathrel{\nabla} Q \otimes R\,, \quad Q^T \otimes Q \mathrel{\nabla} E_m, \quad \text{ and } \; \|R\|_{\oplus} = \|A\|_{\oplus}\,. \qquad \Box$$

If QR is a QRD of a matrix $A \in \mathbb{R}^{m \times n}$ in conventional linear algebra, then we always have $\|R\|_{\mathrm{F}} = \|A\|_{\mathrm{F}}$, since Q is an orthogonal matrix. However, the following example shows that this property does not always hold in the symmetrized max-plus algebra; i.e., $A \mathrel{\nabla} Q \otimes R$ and $Q^T \otimes Q \mathrel{\nabla} E_m$ do not always imply that $\|R\|_{\oplus} = \|A\|_{\oplus}$.

EXAMPLE 5.6. *Consider*

$$A = \begin{bmatrix} \ominus 1 & 1 & 1 \\ 1 & \ominus 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} \ominus 0 & 0 & 0 \\ 0 & \ominus 0 & 0 \\ 0 & 0 & \ominus 0 \end{bmatrix}, \quad \textit{and} \quad R(\rho) = \begin{bmatrix} 1 & \varepsilon & \rho \\ \varepsilon & 1 & \rho \\ \varepsilon & \varepsilon & \rho \end{bmatrix}$$

with $\rho \in \mathbb{R}_\varepsilon$. *We have*

$$Q^T \otimes Q = \begin{bmatrix} 0 & 0^{\bullet} & 0^{\bullet} \\ 0^{\bullet} & 0 & 0^{\bullet} \\ 0^{\bullet} & 0^{\bullet} & 0 \end{bmatrix} \mathrel{\nabla} \begin{bmatrix} 0 & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & 0 \end{bmatrix} = E_3$$

and

$$Q \otimes R(\rho) = \begin{bmatrix} \ominus 1 & 1 & \rho^{\bullet} \\ 1 & \ominus 1 & \rho^{\bullet} \\ 1 & 1 & \rho^{\bullet} \end{bmatrix}.$$

So without the condition $\|R\|_{\oplus} = \|A\|_{\oplus}$, $Q \otimes R(\rho)$ *would have been a max-plus-algebraic QRD of* A *for any* $\rho \geq 1$. *However, since* $\|R(\rho)\|_{\oplus} = \rho$ *if* $\rho \geq 1$ *and since* $\|A\|_{\oplus} = 1$, *we do not have* $\|R\|_{\oplus} = \|A\|_{\oplus}$ *if* $\rho > 1$.

This example explains why we have included the condition $\|R\|_{\oplus} = \|A\|_{\oplus}$ in the definition of the max-plus-algebraic QRD.

Now we explain why we really need the symmetrized max-plus algebra $\mathbb{S}_{\max}$ to define the max-plus-algebraic QRD: we shall show that the class of matrices with entries in $\mathbb{R}_\varepsilon$ that have max-plus-algebraic QRDs for which the entries of Q and R belong to $\mathbb{R}_\varepsilon$ is rather limited. Let $A \in \mathbb{R}_\varepsilon^{m \times n}$ and let $Q \otimes R$ be a max-plus-algebraic

QRD of A for which the entries of Q and R belong to $\mathbb{R}_\varepsilon$. Since the entries of A, Q, and R are signed, it follows from Proposition 3.14 that the balances $A \nabla Q \otimes R$ and $Q^T \otimes Q \nabla E_m$ result in $A = Q \otimes R$ and $Q^T \otimes Q = E_m$. It is easy to verify that we can only have $Q^T \otimes Q = E_m$ if every column and every row of Q contain exactly one entry that is equal to 0 and if all the other entries of Q are equal to ε. Hence, Q is a max-plus-algebraic permutation matrix. As a consequence, A has to be a row-permuted max-plus-algebraic upper triangular matrix.

So only row-permuted max-plus-algebraic upper triangular matrices with entries in $\mathbb{R}_\varepsilon$ have a max-plus-algebraic QRD with entries in $\mathbb{R}_\varepsilon$. This could be compared with the class of real matrices in linear algebra that have a QRD with only nonnegative entries; using an analogous reasoning one can prove that this class coincides with the set of the real row-permuted upper triangular matrices. Furthermore, it is obvious that every QRD in $\mathbb{R}_{\max}$ is also a QRD in $\mathbb{S}_{\max}$.

5.3. The Max-Plus-Algebraic SVD. In [17] we used the mappings $\mathcal{F}$ and $\mathcal{R}$ to prove the existence of a kind of SVD in the symmetrized max-plus algebra. The proof of [17] was based on the analytic SVD. Now we give an alternative proof for the existence theorem of the max-plus-algebraic SVD that makes use of a linear algebra algorithm. More specifically, we shall use Kogbetliantz's SVD algorithm [43], which can be considered as an extension of Jacobi's method for the computation of the eigenvalue decomposition of a real symmetric matrix. We now state the main properties of this algorithm. The explanation below is mainly based on [7] and [34].

A Givens matrix is a square matrix of the form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \cos(\theta) & \cdots & \sin(\theta) & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & -\sin(\theta) & \cdots & \cos(\theta) & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & 1 \end{bmatrix} .$$

The off-norm of the matrix $A \in \mathbb{R}^{m \times n}$ is defined by

$$\|A\|_{\text{off}} = \sqrt{\sum_{i=1}^{n} \sum_{j=1,\, j \neq i}^{n} a_{ij}^2} \; ,$$

where the empty sum is equal to 0 by definition (so if A is a 1 by 1 matrix, then we have $\|A\|_{\text{off}} = 0$). Let $A \in \mathbb{R}^{m \times n}$. Since USV^T is an SVD of A if and only if VS^TU^T is an SVD of A^T, we may assume without loss of generality that $m \geq n$. Before applying Kogbetliantz's SVD algorithm we compute a QRD of A:

$$A = Q \begin{bmatrix} R \\ O_{(m-n) \times n} \end{bmatrix} ,$$

where R is an n by n upper triangular matrix.

Now we apply Kogbetliantz's SVD algorithm to R. In this algorithm a sequence of matrices is generated as follows:

$$\begin{aligned} &U_0 = I_n\,, \quad V_0 = I_n\,, \quad S_0 = R\,,\\ &U_k = U_{k-1}G_k\,, \quad V_k = V_{k-1}H_k\,, \quad S_k = G_k^T S_{k-1} H_k \qquad \text{for } k = 1,2,3,\dots \end{aligned}$$

such that $\|S_k\|_{\text{off}}$ decreases monotonously as k increases. So S_k tends more and more to a diagonal matrix as the iteration process progresses. The absolute values of the diagonal entries of S_k converge to the singular values of R as k goes to ∞.

The matrices G_k and H_k are Givens matrices that are chosen such that $(S_k)_{i_k j_k} = (S_k)_{j_k i_k} = 0$ for some ordered pair of indices (i_k, j_k). As a result we have

$$\|S_k\|_{\text{off}}^2 = \|S_{k-1}\|_{\text{off}}^2 - (S_{k-1})_{i_k j_k}^2 - (S_{k-1})_{j_k i_k}^2 .$$

Since the matrices G_k and H_k are orthogonal for all $k \in \mathbb{N}_0$, we have

$$\|S_k\|_{\text{F}} = \|R\|_{\text{F}}\,, \quad R = U_k S_k V_k^T, \quad U_k^T U_k = I_n, \quad \text{and} \quad V_k^T V_k = I_n \tag{30}$$

for all $k \in \mathbb{N}$.

We shall use the row-cyclic version of Kogbetliantz's SVD algorithm: in each cycle the indices i_k and j_k are chosen such that the entries in the strictly upper triangular part of the S_k's are selected row by row. This yields the following sequence for the ordered pairs of indices (i_k, j_k):

$$(1,2) \to (1,3) \to \cdots \to (1,n) \to (2,3) \to (2,4) \to \cdots \to (n-1,n).$$

A full cycle $(1,2) \to \cdots \to (n-1,n)$ is called a *sweep*. Note that a sweep corresponds to $N = \frac{(n-1)n}{2}$ iterations. Sweeps are repeated until S_k becomes diagonal. If we have an upper triangular matrix at the beginning of a sweep, then we shall have a lower triangular matrix after the sweep and vice versa.

For triangular matrices the row-cyclic Kogbetliantz algorithm is globally convergent [23, 34]. Furthermore, for triangular matrices the convergence of this algorithm is quadratic if k is large enough [2, 6, 32, 33, 56]:

$$\exists K \in \mathbb{N} \text{ such that for all } k \geq K : \|S_{k+N}\|_{\text{off}} \leq \gamma\, \|S_k\|_{\text{off}}^2 \tag{31}$$

for some constant γ that does not depend on k, under the assumption that diagonal entries that correspond to the same singular value or that are affiliated with the same cluster of singular values occupy successive positions on the diagonal. This assumption is not restrictive since we can always reorder the diagonal entries of S_k by inserting an extra step in which we select a permutation matrix $\hat{P} \in \mathbb{R}^{n\times n}$ such that the diagonal entries of $S_{k+1} = \hat{P}^T S_k \hat{P}$ exhibit the required ordering. Note that $\|S_{k+1}\|_{\text{F}} = \|S_k\|_{\text{F}}$. If we define $U_{k+1} = U_k \hat{P}$ and $V_{k+1} = V_k \hat{P}$, then U_{k+1} and V_{k+1} are orthogonal since $\hat{P}^T \hat{P} = I_n$. We also have

$$U_{k+1} S_{k+1} V_{k+1}^T = \left(U_k \hat{P}\right) \left(\hat{P}^T S_k \hat{P}\right) \left(\hat{P}^T V_k^T\right) = U_k S_k V_k^T = R.$$

Furthermore, once the diagonal entries have the required ordering, they hold it provided that k is sufficiently large [32].

If we define

$$S = \lim_{k\to\infty} S_k, \quad U = \lim_{k\to\infty} U_k, \quad \text{and} \quad V = \lim_{k\to\infty} V_k,$$

then S is a diagonal matrix, U and V are orthogonal matrices, and $USV^T = R$. We make all the diagonal entries of S nonnegative by multiplying S with an appropriate diagonal matrix D. Next we construct a permutation matrix P such that the diagonal entries of P^TSDP are arranged in descending order. If we define $U_R = UP$, $S_R = P^TSDP$, and $V_R = VD^{-1}P$, then U_R and V_R are orthogonal, the diagonal entries of S_R are nonnegative and ordered, and

$$U_R S_R V_R^T = (UP)\left(P^TSDP\right)\left(P^TD^{-1}V^T\right) = USV^T = R.$$

Hence, $U_R S_R V_R^T$ is an SVD of R. If we define

$$U_A = Q\begin{bmatrix} U_R & O_{n\times(m-n)} \\ O_{(m-n)\times n} & I_{m-n} \end{bmatrix},\ S_A = \begin{bmatrix} S_R \\ O_{(m-n)\times n} \end{bmatrix},\ \text{and}\ V_A = V_R,$$

then $U_A S_A V_A^T$ is an SVD of A.

THEOREM 5.7 (max-plus-algebraic SVD). *Let* $A \in \mathbb{S}^{m\times n}$ *and let* $r = \min(m,n)$. *Then there exist a max-plus-algebraic diagonal matrix* $\Sigma \in \mathbb{R}_\varepsilon^{m\times n}$ *and matrices* $U \in (\mathbb{S}^\vee)^{m\times m}$ *and* $V \in (\mathbb{S}^\vee)^{n\times n}$ *such that*

$$A \mathrel{\nabla} U \otimes \Sigma \otimes V^T \tag{32}$$

with $U^T \otimes U \mathrel{\nabla} E_m$, $V^T \otimes V \mathrel{\nabla} E_n$, *and* $\|A\|_\oplus = \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$, *where* $\sigma_i = (\Sigma)_{ii}$ *for* $i = 1, 2, \ldots, r$.

Every decomposition of the form (32) *that satisfies the above conditions is called a max-plus-algebraic SVD of A.*

Proof. Using a reasoning that is similar to the one that was used at the beginning of the proof of Theorem 5.5, we can show that it is sufficient to prove this theorem for signed matrices A. So from now on we assume that A is signed.

Define $c = \|A\|_\oplus$. If $c = \varepsilon$, then $A = \mathcal{E}_{m\times n}$. If we take $U = E_m$, $\Sigma = \mathcal{E}_{m\times n}$, and $V = E_n$, we have $A = U \otimes \Sigma \otimes V^T$, $U^T \otimes U = E_m$, $V^T \otimes V = E_n$, and $\sigma_1 = \sigma_2 = \cdots = \sigma_r = \varepsilon = \|A\|_\oplus$. So $U \otimes \Sigma \otimes V^T$ is a max-plus-algebraic SVD of A.

From now on we assume that $c \neq \varepsilon$. We may assume without loss of generality that $m \geq n$: if $m < n$, we can apply the subsequent reasoning to A^T since $A \mathrel{\nabla} U \otimes \Sigma \otimes V^T$ if and only if $A^T \mathrel{\nabla} V \otimes \Sigma^T \otimes U^T$. So $U \otimes \Sigma \otimes V^T$ is a max-plus-algebraic SVD of A if and only if $V \otimes \Sigma^T \otimes U^T$ is a max-plus-algebraic SVD of A^T.

Now we distinguish between two different situations depending on whether or not all the a_{ij}'s are finite. In Remark 5.8 we shall explain why this distinction is necessary.

Case 1. *All the* a_{ij}*'s are finite.*

We construct $\tilde{A} = \mathcal{F}(A, M, \cdot)$ where $M \in \mathbb{R}^{m\times n}$ with $m_{ij} = 1$ for all i, j. The entries of $\tilde{A}$ belong to $\mathcal{S}_\mathrm{e}$. In order to determine a path of SVDs of $\tilde{A}$, we first compute a path of QRDs of $\tilde{A}$ on $\mathbb{R}_0^+$:

$$\tilde{A} = \tilde{Q}\begin{bmatrix} \tilde{R} \\ O_{(m-n)\times n} \end{bmatrix}, \tag{33}$$

where $\tilde{R}$ is an n by n upper triangular matrix-valued function. By Proposition 5.4 the entries of $\tilde{Q}$ and $\tilde{R}$ belong to $\mathcal{S}_\mathrm{e}$.

Now we use the row-cyclic Kogbetliantz algorithm to compute a path of SVDs of $\tilde{R}$. The operations used in this algorithm are additions, multiplications, subtractions, divisions, square roots, and absolute values. So if we apply this algorithm to a matrix with entries in $\mathcal{S}_{\rm e}$, the entries of all the matrices generated during the iteration process also belong to $\mathcal{S}_{\rm e}$ by Proposition 5.3.

In theory we should run the row-cyclic Kogbetliantz algorithm forever in order to produce a path of exact SVDs of $\tilde{A}$. However, since we are only interested in the asymptotic behavior of the singular values and the entries of the singular vectors of $\tilde{A}$, we may stop the iteration process after a finite number of sweeps, as will be shown next. Let $\tilde{S}_k$, $\tilde{U}_k$, and $\tilde{V}_k$ be the matrix-valued functions that are computed in the kth iteration step of the algorithm. Let $\tilde{\Delta}_p$ be the diagonal matrix-valued function obtained at the end of the pth sweep by removing the off-diagonal entries of $\tilde{S}_{pN}$ (where $N = \frac{n(n-1)}{2}$ is the number of iterations per sweep), making all diagonal entries nonnegative and arranging them in descending order, and adding $m-n$ zero rows (cf. the transformations used to go from S to S_A in the explanation of Kogbetliantz's algorithm given above). Let $\tilde{X}_p$ and $\tilde{Y}_p$ be the matrix-valued functions obtained by applying the corresponding transformations to $\tilde{U}_{pN}$ and $\tilde{V}_{pN}$, respectively. If we define a matrix-valued function $\tilde{C}_p = \tilde{X}_p \tilde{\Delta}_p \tilde{Y}_p^T$, we have a path of *exact* SVDs of $\tilde{C}_p$ on some interval $[L, \infty)$. This means that we may stop the iteration process as soon as

$$\mathcal{F}(A, N, s) \sim \tilde{C}_p(s), \qquad s \to \infty \tag{34}$$

for some $N \in \mathbb{R}_0^{m \times n}$. Note that eventually this condition will always be satisfied due to the fact that Kogbetliantz's SVD algorithm is globally convergent and, for triangular matrices,[2] also quadratically convergent if p is large enough, and due to the fact that the entries of $\tilde{A}$, to which the entries of $\tilde{C}_p$ should converge, are not identically zero since they have a finite dominant exponent (since in Case 1 we assume that all the entries of A are finite).

Let $\tilde{U}\tilde{S}\tilde{V}^T$ be a path of approximate SVDs of $\tilde{A}$ on some interval $[L, \infty)$ that was obtained by the procedure given above. Since we have performed a *finite* number of elementary operations on the entries of $\tilde{A}$, the entries of $\tilde{U}$, $\tilde{S}$, and $\tilde{V}$ belong to $\mathcal{S}_{\rm e}$. We have

$$\mathcal{F}(A, N, s) \sim \tilde{U}(s)\,\tilde{\Sigma}(s)\,\tilde{V}^T(s)\,, \qquad s \to \infty, \tag{35}$$

for some $N \in \mathbb{R}_0^{m \times n}$. Furthermore,

$$\tilde{U}^T(s)\,\tilde{U}(s) = I_m \qquad \text{for all } s \geq L, \tag{36}$$

$$\tilde{V}^T(s)\,\tilde{V}(s) = I_n \qquad \text{for all } s \geq L. \tag{37}$$

The diagonal entries of $\tilde{\Sigma}$ and the entries of $\tilde{U}$ and $\tilde{V}$ belong to $\mathcal{S}_{\rm e}$ and are thus asymptotically equivalent to an exponential in the neighborhood of ∞ by Lemma 5.2. Define $\tilde{\sigma}_i = \tilde{\Sigma}_{ii}$ for $i = 1, 2, \ldots, r$.

Now we apply the reverse mapping $\mathcal{R}$ in order to obtain a max-plus-algebraic SVD of A. If we define

$$\Sigma = \mathcal{R}(\tilde{\Sigma})\,,\ U = \mathcal{R}(\tilde{U})\,,\ V = \mathcal{R}(\tilde{V}), \ \text{ and } \ \sigma_i = (\Sigma)_{ii} = \mathcal{R}(\tilde{\sigma}_i) \text{ for all } i\,,$$

then Σ is a max-plus-algebraic diagonal matrix and U and V have signed entries. If we apply the reverse mapping $\mathcal{R}$ to (35)–(37), we get

$$A \mathrel{\nabla} U \otimes \Sigma \otimes V^T, \quad U^T \otimes U \mathrel{\nabla} E_m, \ \text{ and } \ V^T \otimes V \mathrel{\nabla} E_n.$$

---

[2] Recall that we are applying Kogbetliantz's SVD algorithm to the upper triangular matrix-valued function $\tilde{R}$ (cf. (33)).

The $\tilde{\sigma}_i$'s are nonnegative in $[L, \infty)$ and therefore we have $\sigma_i \in \mathbb{R}_\varepsilon$ for all i. Since the $\tilde{\sigma}_i$'s are ordered in $[L, \infty)$, their dominant exponents are also ordered. Hence, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$.

We have $\|\tilde{A}(s)\|_{\mathrm{F}} \sim \gamma e^{cs}$, $s \rightarrow \infty$, for some $\gamma > 0$ since $c = \|A\|_\oplus$ is the largest exponent that appears in the entries of $\tilde{A}$. Hence, $\mathcal{R}(\|\tilde{A}\|_{\mathrm{F}}) = c = \|A\|_\oplus$.

If $P \in \mathbb{R}^{m \times n}$ then $\frac{1}{\sqrt{n}} \|P\|_{\mathrm{F}} \leq \|P\|_2 \leq \|P\|_{\mathrm{F}}$. As a consequence, we have

$$\frac{1}{\sqrt{n}} \|\tilde{A}\|_{\mathrm{F}} \leq \|\tilde{A}\|_2 \leq \|\tilde{A}\|_{\mathrm{F}} \qquad \text{for all } s \geq L\,.$$

Since $\tilde{\sigma}_1(s) \sim \|\tilde{A}(s)\|_2$, $s \rightarrow \infty$, and since the mapping $\mathcal{R}$ preserves the order, this leads to $\|A\|_\oplus \leq \sigma_1 \leq \|A\|_\oplus$ and, consequently, $\sigma_1 = \|A\|_\oplus$.

Case 2. *Not all the a_{ij}'s are finite.*

First we construct a sequence $\{A_l\}_{l=1}^{\infty}$ of m by n matrices such that

$$(A_l)_{ij} = \begin{cases} a_{ij} & \text{if } |a_{ij}|_\oplus \neq \varepsilon, \\ \|A\|_\oplus - l & \text{if } |a_{ij}|_\oplus = \varepsilon \end{cases}$$

for all i, j. So the entries of the matrices A_l are finite and $\|A\|_\oplus = \|A_l\|_\oplus$ for all $l \in \mathbb{N}_0$. Furthermore, $\lim_{l \rightarrow \infty} A_l = A$.

Now we construct the sequence $\{\tilde{A}_l\}_{l=1}^{\infty}$ with $\tilde{A}_l = \mathcal{F}(A_l, M, \cdot)$ for $l = 1, 2, 3, \ldots$, where $M \in \mathbb{R}^{m \times n}$ and $m_{ij} = 1$ for all i, j. We compute a path of approximate SVDs $\tilde{U}_l\, \tilde{\Sigma}_l\, \tilde{V}_l^T$ of each $\tilde{A}_l$ using the method of Case 1 of this proof.

In general, it is possible that for some of the entries of the $\tilde{U}_l$'s and the $\tilde{V}_l$'s the sequence of the dominant exponents and the sequence of the corresponding coefficients have more than one accumulation point (since if two or more singular values coincide, the corresponding left and right singular vectors are not uniquely defined). However, since we use a fixed computation scheme (the row-cyclic Kogbetliantz algorithm), all the sequences will have exactly one accumulation point. So some of the dominant exponents will reach a finite limit as l goes to ∞, while the other dominant exponents will tend to $-\infty$. If we take the reverse mapping $\mathcal{R}$, we get a sequence of max-plus-algebraic SVDs $\{U_l \otimes \Sigma_l \otimes V_l^T\}_{l=1}^{\infty}$, where some of the entries, viz. those that correspond to dominant exponents that tend to $-\infty$, tend to ε as l goes to ∞.

If we define

$$U = \lim_{l \rightarrow \infty} U_l, \;\; \Sigma = \lim_{l \rightarrow \infty} \Sigma_l, \;\; \text{and} \;\; V = \lim_{l \rightarrow \infty} V_l,$$

then we have

$$A \mathrel{\nabla} U \otimes \Sigma \otimes V^T, \quad U^T \otimes U \mathrel{\nabla} E_m, \;\; \text{and} \;\; V^T \otimes V \mathrel{\nabla} E_n.$$

Since the diagonal entries of all the Σ_l's belong to $\mathbb{R}_\varepsilon$ and are ordered, the diagonal entries of Σ also belong to $\mathbb{R}_\varepsilon$ and are also ordered. Furthermore, $(\Sigma)_{11} = \|A\|_\oplus$ since $(\Sigma_l)_{11} = \|A\|_\oplus$ for all l. Hence, $U \otimes \Sigma \otimes V^T$ is a max-plus-algebraic SVD of A. $\square$

REMARK 5.8. *Now we explain why we have distinguished between two different cases in the proof of Theorem* 5.7.

If there exist indices i and j such that $a_{ij} = \varepsilon$, then $\tilde{a}_{ij}(s) = 0$ for all $s \in \mathbb{R}_0^+$, which means that we cannot guarantee that condition (34) *will be satisfied after a finite number of sweeps. This is why we make a distinction between the case where all the entries of A are finite and the case where at least one entry of A is equal to ε.*

Let us now show that this problem is not an issue for the singular value functions $\tilde{\sigma}_i$ *that should converge to* 0*; i.e., we show that we do not have to take special precautions if* $\tilde{A}$ *has singular values that are identically zero in the neighborhood of* ∞*. If* $\tilde{\Psi}$ *is a real matrix-valued function that is analytic in some interval* $J \subseteq \mathbb{R}$*, then the rank of* $\tilde{\Psi}$ *is constant in* J *except in some isolated points where the rank drops* [30]*. If the rank of* $\tilde{\Psi}(s)$ *is equal to* ρ *for all* $s \in J$ *except for some isolated points, then we say that the* generic rank *of* $\tilde{\Psi}$ *in* J *is equal to* ρ*. The entries of all the matrix-valued functions created in the row-cyclic Kogbetliantz algorithm when applied to* $\tilde{A}$ *are real and analytic in some interval* $[L^*, \infty)$*. Furthermore, for a fixed value of* s*, the matrices* $\tilde{A}(s)$*,* $\tilde{R}(s)$*,* $\tilde{S}_1(s)$*,* $\tilde{S}_2(s)$*, … all have the same rank since they are related by orthogonal transformations. So if* ρ *is the generic rank of* $\tilde{A}$ *in* $[L^*, \infty)$*, then the generic rank of* $\tilde{R}$*,* $\tilde{S}_1$*,* $\tilde{S}_2$*, … in* $[L^*, \infty)$ *is also equal to* ρ*. If* $\rho < n$*, then the* $n - \rho$ *smallest singular values of* $\tilde{R}$ *will be identically zero in* $[L^*, \infty)$*. Since* $\tilde{R}$*,* $\tilde{S}_N$*,* $\tilde{S}_{2N}$*, … are triangular matrices, they have at least* $n - \rho$ *diagonal entries that are identically zero in* $[L^*, \infty)$ *since otherwise their generic rank would be greater than* ρ*. In fact, this also holds for* $\tilde{S}_1$*,* $\tilde{S}_2$*, … since these matrix-valued functions are hierarchically triangular, i.e., block triangular such that the diagonal blocks are again block triangular, etc.* [34]*. Furthermore, if* k *is large enough, diagonal entries do not change their affiliation any more; i.e., if a diagonal entry corresponds to a specific singular value in the* k*th iteration, then it will also correspond to that singular value in all subsequent iterations. Since the diagonal entries of* $\tilde{S}_k$ *are asymptotically equivalent to an exponential in the neighborhood of* ∞*, this means that at least* $n - \rho$ *diagonal entries (with a fixed position) of* $\tilde{S}_k$*,* $\tilde{S}_{k+1}$*, … will be identically zero in some interval* $[L, \infty) \subseteq [L^*, \infty)$ *if* k *is large enough. Hence, we do not have to take special precautions if* $\tilde{A}$ *has singular values that are identically zero in the neighborhood of* ∞ *since convergence to these singular values in a finite number of iteration steps is guaranteed.*

For inner products of two different columns of $\tilde{U}$*, there are no problems either: these inner products are equal to* 0 *by construction since the matrix-valued function* $\tilde{U}_k$ *is orthogonal on* $[L, \infty)$ *for all* $k \in \mathbb{N}$*. This also holds for inner products of two different columns of* $\tilde{V}$*.*

If $U \Sigma V^T$ is an SVD of a matrix $A \in \mathbb{R}^{m \times n}$ in conventional linear algebra, then we have $\sigma_1 = (\Sigma)_{11} = \|A\|_2$. However, in the symmetrized max-plus algebra the balances $A \nabla U \otimes \Sigma \otimes V^T$, $U^T \otimes U \nabla E_m$, and $V^T \otimes V \nabla E_n$, where Σ is a diagonal matrix with entries in $\mathbb{R}_\varepsilon$ and where the entries of U and V are signed, do not always imply that $(\Sigma)_{11} = \|A\|_\oplus$, as is shown by Example 5.9 below; in general, this may occur when A does not have at least one signed entry that is equal to $\|A\|_\oplus$ in max-plus-absolute value [17]. Therefore, we have included the extra condition $\sigma_1 = \|A\|_\oplus$ in the definition of the max-plus-algebraic SVD.

Example 5.9. *Consider*

$$A = \begin{bmatrix} 1^\bullet & \varepsilon \\ \varepsilon & 1^\bullet \end{bmatrix}, \quad U = V = E_2 = \begin{bmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{bmatrix}, \quad \text{and} \quad \Sigma(\rho) = \begin{bmatrix} \rho & \varepsilon \\ \varepsilon & \rho \end{bmatrix}$$

with $\rho \in \mathbb{R}_\varepsilon$*. Note that* $\|A\|_\oplus = 1$ *but that* A *has no signed entry that is equal to* 1 *in max-plus-absolute value. We have* $U^T \otimes U = V^T \otimes V = E_2$*, and* $U \otimes \Sigma(\rho) \otimes V^T = \Sigma(\rho)$ *and* $\sigma_1(\rho) = (\Sigma_{11}(\rho)) = \rho$ *for all* ρ*. Clearly,* $U \otimes \Sigma(\rho) \otimes V^T \; \nabla \; A$ *for any* $\rho \leq 1$*. So without the condition* $\sigma_1 = \|A\|_\oplus$*,* $U \otimes \Sigma(\rho) \otimes V^T$ *would have been a max-plus-algebraic SVD of* A *for any* $\rho \leq 1$*.*

Using a reasoning that is similar to the one that has been used at the end of section 5.2 we can show that only permuted max-plus-algebraic diagonal matrices with entries in $\mathbb{R}_\varepsilon$ have a max-plus-algebraic SVD with entries in $\mathbb{R}_\varepsilon$ [12, 17].

For properties of the max-plus-algebraic SVD the interested reader is referred to [12, 17]. In [12] we also proposed some possible extensions of the definitions of the max-plus-algebraic QRD and the max-plus-algebraic SVD.

The QRD and the SVD are used in many contemporary algorithms for the identification of conventional linear systems [44, 45, 50, 60, 61, 62, 63]. We conjecture that the max-plus-algebraic QRD and SVD can play a similar role in the identification of max-plus-linear discrete-event systems. We could, e.g., use the max-plus-algebraic QRD and SVD to define a max-plus-algebraic matrix rank; this rank could then be used to get an estimate of the system order (i.e., the number of state variables) of a max-plus-linear discrete event system starting from measured input–output data sequences (see also [12, 17]).

Remark 5.10. *If f, g, and h belong to $\mathcal{S}_{\mathrm{e}}$, then they are asymptotically equivalent to an exponential in the neighborhood of ∞ by Lemma* 5.2. *So if L is large enough, then $f(L) \geq 0$ and $g(L) \geq h(L)$ imply that $f(s) \geq 0$ and $g(s) \geq h(s)$ for all $s \in [L, \infty)$. This fact and the fact that $\mathcal{S}_{\mathrm{e}}$ is closed under some elementary algebraic operations (cf. Proposition* 5.3*) explain why many algorithms from linear algebra, such as the Givens QR algorithm and Kogbetliantz's SVD algorithm, also work for matrices with entries that belong to $\mathcal{S}_{\mathrm{e}}$ instead of $\mathbb{R}$. If we apply an algorithm from linear algebra to a matrix-valued function $\tilde{A}$ with entries in $\mathcal{S}_{\mathrm{e}}$ that is defined on some interval $[L, \infty)$, we are in fact applying this algorithm on the (constant) matrix $\tilde{A}(s)$ for every value of $s \in [L, \infty)$ in parallel.*

5.4. Other Max-Plus-Algebraic Matrix Decompositions.

The proof technique that has been used in this section essentially consists of applying an algorithm from linear algebra to a matrix with entries in $\mathcal{S}_{\mathrm{e}}$, where we make use of the fact that $\mathcal{S}_{\mathrm{e}}$ is closed for (finite (nested) compositions of) elementary operations such as addition, multiplication, subtraction, division, square root, and absolute value. This implies that we can consider conventional linear algebra algorithms for the eigenvalue decomposition, the LU decomposition, the Schur decomposition, etc. (see [31, 40]), to prove the existence of the max-plus-algebraic analogues of these matrix decompositions. So the proof technique of this paper can easily be adapted to prove the existence of a max-plus-algebraic eigenvalue decomposition for symmetric matrices (by using the Jacobi algorithm for the computation of the eigenvalue decomposition of a real symmetric matrix), a max-plus-algebraic LU decomposition, a max-plus-algebraic Cholesky decomposition, a max-plus-algebraic Schur decomposition, a max-plus-algebraic Hessenberg decomposition, and so on.

6. Computational Methods.

There are several ways to compute a max-plus-algebraic matrix factorization of a given matrix $A \in \mathbb{S}^{m \times n}$:

1. via symbolic computation using linear algebra algorithms,
2. via numerical computation using linear algebra algorithms,
3. via the extended linear complementarity problem (ELCP).

In the following sections we shall discuss these methods in more detail.

6.1. Symbolic Computation Using Linear Algebra Algorithms.

Let $A \in \mathbb{S}^{m \times n}$. To compute a max-plus-algebraic SVD of A, we first select a matrix $M \in \mathbb{R}_0^{m \times n}$ and construct the matrix-valued function $\mathcal{F}(A, M, \cdot)$. Next, we use the constructive proof technique of section 5 and we apply a linear algebra algorithm, corresponding to the matrix decomposition that we want to compute, to $\mathcal{F}(A, M, \cdot)$. Finally, we transform the result back to the symmetrized max-plus algebra via the mapping $\mathcal{R}$. This approach will be illustrated in the worked examples of section 7.

The main disadvantage of this approach is that it requires symbolic calculation, which may be computationally intensive.

6.2. Numerical Computation Using Linear Algebra Algorithms.

In this section we shall focus on the max-plus-algebraic SVD. Note, however, that the numerical computation method can also be used to compute the other max-plus-algebraic matrix decompositions.

Let $A \in \mathbb{S}^{m\times n}$. Just as for the symbolic computation, we first select a matrix $M \in \mathbb{R}_0^{m\times n}$. Next, we define an increasing sequence of points $s_0, s_1, \ldots, s_K \in \mathbb{R}_0^+$, and we numerically compute the (constant) SVD of $\mathcal{F}(A, M, s_k)$ for $k = 0, 1, \ldots, K$. This yields a sequence of SVDs $\tilde{U}(s_k)\,\tilde{\Sigma}(s_k)\,\tilde{V}^T(s_k)$ of $\mathcal{F}(A, M, s_k)$ for $k = 0, 1, \ldots, K$. By taking the logarithm and dividing by s (see the worked example of section 7) we can now determine the dominant exponents of the entries of the matrix-valued functions $\tilde{U}$, $\tilde{\Sigma}$, and $\tilde{V}$ of the path of SVDs $\tilde{U}\tilde{\Sigma}\tilde{V}^T$ of $\mathcal{F}(A, M, \cdot)$. If we take the signs of the entries of $\tilde{U}$ and $\tilde{V}$ into account and apply the reverse mapping $\mathcal{R}$, we obtain a max-plus-algebraic SVD of A. This method will be illustrated in section 7.

The main disadvantage of this approach is that we can run into numerical problems due to very large or almost zero numerical values of entries of $\mathcal{F}(A, M, \cdot)$ for large values of s.

6.3. ELCP Approach.

In this section we shall focus on the max-plus-algebraic QRD. Note, however, that the ELCP method can also be used to compute the other max-plus-algebraic matrix decompositions.

We shall show that the max-plus-algebraic QRD of a matrix $A \in \mathbb{S}^{m\times n}$ can also be computed by solving an ELCP, which is a kind of mathematical programming problem. Although it would lead us too far off to explain this procedure in detail, we shall now give a brief outline of how the equations that appear in the definition of the max-plus-algebraic QRD can be transformed into a system of multivariate max-plus-algebraic polynomial equalities. For the sake of simplicity we assume that all the entries of A are finite. If this were not the case, we could apply a limit argument[3] similar to the one used in Case 2 of the proof of Theorem 5.7.

Consider the equation $A \nabla Q \otimes R$. If we extract the max-plus-positive and the max-plus-negative parts of each matrix, we obtain

$$A^{\oplus} \ominus A^{\ominus} \;\nabla\; (Q^{\oplus} \ominus Q^{\ominus}) \otimes (R^{\oplus} \ominus R^{\ominus})$$

or

$$A^{\oplus} \ominus A^{\ominus} \;\nabla\; Q^{\oplus} \otimes R^{\oplus} \ominus Q^{\oplus} \otimes R^{\ominus} \ominus Q^{\ominus} \otimes R^{\oplus} \oplus Q^{\ominus} \otimes R^{\ominus}.$$

By Proposition 3.13 this can be rewritten as

$$A^{\oplus} \oplus Q^{\oplus} \otimes R^{\ominus} \oplus Q^{\ominus} \otimes R^{\oplus} \;\nabla\; A^{\ominus} \oplus Q^{\oplus} \otimes R^{\oplus} \oplus Q^{\ominus} \otimes R^{\ominus}.$$

Both sides of this balance are signed. So by Proposition 3.14 we may replace the balance by an equality. If we introduce a matrix T of auxiliary variables, we obtain

$$A^{\oplus} \oplus Q^{\oplus} \otimes R^{\ominus} \oplus Q^{\ominus} \otimes R^{\oplus} = T, \tag{38}$$

$$A^{\ominus} \oplus Q^{\oplus} \otimes R^{\oplus} \oplus Q^{\ominus} \otimes R^{\ominus} = T. \tag{39}$$

---

[3] In practice, since the $\oplus$-operator causes large numbers to mask smaller numbers, a limit argument is usually not required, and a "large-number" argument is already sufficient. Loosely speaking, this means that we replace all infinite entries of A by a large negative real number $-\xi$ with $\xi \gg 1$, we perform the computations, and afterward we replace all entries in the resulting max-plus-algebraic QRD decomposition that have the same order of magnitude as $-\xi$ by ε (see [12] for more information on this topic).

Since we have assumed that all entries of A are finite, the entries of T will also be finite and, as a consequence, they will be max-plus-invertible. So if we write out the max-plus-algebraic matrix multiplications in (38) and if we transfer the entries of T to the opposite side, we get

$$\begin{aligned} a_{ij}^{\oplus} \otimes {t_{ij}}^{\otimes^{-1}} & \oplus \bigoplus_{k=1}^{m} q_{ik}^{\oplus} \otimes r_{kj}^{\ominus} \otimes {t_{ij}}^{\otimes^{-1}} \\ & \oplus \bigoplus_{k=1}^{m} q_{ik}^{\ominus} \otimes r_{kj}^{\oplus} \otimes {t_{ij}}^{\otimes^{-1}} = 0 \qquad \text{for all } i, j \ . \end{aligned} \tag{40}$$

Equation (39) can be rewritten in a similar way. The condition $Q^T \otimes Q \,\nabla\, E_m$ also leads to similar equations.

The condition that the entries of Q and R should be signed can be written as[4]

$$q_{ij}^{\oplus} \otimes q_{ij}^{\ominus} = \varepsilon \qquad \text{for all } i, j, \tag{41}$$

$$r_{ij}^{\oplus} \otimes r_{ij}^{\ominus} = \varepsilon \qquad \text{for all } i, j. \tag{42}$$

The condition $\|R\|_{\oplus} = \|A\|_{\oplus}$ is equivalent to

$$\bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} \left(r_{ij}^{\oplus} \oplus r_{ij}^{\ominus} \right) = \|A\|_{\oplus} \qquad \text{for all } i, j. \tag{43}$$

So if we combine all equations of the form (40)–(43), we obtain a system of multivariate max-plus-algebraic polynomial equalities of the following form:

> Given l integers $m_1, m_2, \ldots, m_l \in \mathbb{N}_0$ and real numbers a_{ki}, b_k, and c_{kij} for $k = 1, 2, \ldots, l$, $i = 1, 2, \ldots, m_l$, and $j = 1, 2, \ldots, r$, find $x \in \mathbb{R}_{\varepsilon}^{r}$ such that
>
> $$\bigoplus_{i=1}^{m_l} a_{ki} \otimes \bigotimes_{j=1}^{r} {x_j}^{\otimes^{c_{kij}}} = b_k \qquad \text{for } k = 1, 2, \ldots, l\,,$$
>
> or show that no such x exists,

where the vector x contains the max-plus-positive and the max-plus-negative parts of the entries of Q and R and the auxiliary variables.

In conventional algebra this problem can be rewritten as follows:

> Given l integers $m_1, m_2, \ldots, m_l \in \mathbb{N}_0$ and real numbers a_{ki}, b_k, and c_{kij} for $k = 1, 2, \ldots, l$, $i = 1, 2, \ldots, m_l$, and $j = 1, 2, \ldots, r$, find $x \in \mathbb{R}_{\varepsilon}^{r}$ such that
>
> $$\max_{i=1,\ldots,m_l} a_{ki} + c_{ki1}x_1 + c_{ki2}x_2 + \cdots + c_{kir}x_r = b_k \qquad \text{for } k = 1, 2, \ldots, l\,,$$
>
> or show that no such x exists.

In [12, 16] we showed that this problem can in its turn be rewritten as a mathematical programming problem of the following form[5]:

---

[4] Since we want the resulting ELCP to have finite data entries, we shall also apply a limit or large-number argument (cf. footnote 3) for the right-hand sides of (41)–(42).

[5] Basically, the proof boils down to the fact that for $\alpha, \beta, \gamma \in \mathbb{R}$ the equation $\max(\alpha, \beta) = \gamma$ is equivalent to the system $\alpha \leq \gamma$, $\beta \leq \gamma$, $(\gamma - \alpha)(\gamma - \beta) = 0$.

Given two matrices $A \in \mathbb{R}^{p \times r}$, $B \in \mathbb{R}^{q \times r}$, two vectors $c \in \mathbb{R}^p$, $d \in \mathbb{R}^q$, and s subsets $\phi_1, \phi_2, \ldots, \phi_s$ of $\{1, 2, \ldots, p\}$, find $x \in \mathbb{R}^r$ such that

$$\sum_{j=1}^{s} \prod_{i \in \phi_j} (Ax - c)_i = 0 \tag{44}$$

subject to $Ax \geq c$ and $Bx = d$, or show that no such x exists.

This problem is called the *extended linear complementarity problem* (ELCP). The ELCP arose from our research on discrete-event systems, hybrid systems, and traffic signal control [13, 15, 16, 18, 22].

Condition (44) represents the complementarity condition of the ELCP and can be interpreted as follows. Since $Ax \geq c$, all the terms in (44) are nonnegative. Hence, (44) is equivalent to $\prod_{i \in \phi_j} (Ax - c)_i = 0$ for $j = 1, 2, \ldots, s$. So for each $j \in \{1, 2, \ldots, s\}$ we should have $(Ax - c)_i = 0$ for some index $i \in \phi_j$. Hence, each set ϕ_j corresponds to a group of inequalities in $Ax \geq c$, and in each group at least one inequality should hold with equality (i.e., the corresponding surplus variable is equal to 0).

In general, the solution set of the ELCP consists of the union of a subset of the faces of the polyhedron defined by the system of linear equalities and inequalities $Ax \geq c$, $Bx = d$. In [14] we developed an algorithm to find a parametric representation of the *entire* solution set of an ELCP. However, the execution time of this algorithm increases exponentially as the number of equations and variables of the ELCP increases. Recently, we developed a new algorithm for the ELCP that is based on mixed integer programming [20, 21]. However, although this new approach allows us to solve a much larger instance of the ELCP (see [20]), the general ELCP is intrinsically hard to solve due to the fact that it is an NP-hard problem [12, 14]. As a consequence, the ELCP approach can only be used to compute max-plus-algebraic QRDs of small-sized matrices. So there certainly is a need for efficient algorithms to compute max-plus-algebraic QRDs and max-plus-algebraic other matrix decompositions. This will be an important topic for further research. Another question is whether we can develop efficient algorithms for special classes of matrices; e.g., is it possible to come up with more efficient algorithms by making use of the structure (sparse, banded, triangular, etc.) of the matrix?

For an illustration of the use of the ELCP approach to compute the max-plus-algebraic SVD of a matrix, we refer to [12].

7. A Worked Example of the Max-Plus-Algebraic QRD and the Max-Plus-Algebraic SVD.

Now we give an example of the computation of a max-plus-algebraic QRD and a max-plus-algebraic SVD of a matrix using the mapping $\mathcal{F}$.

Consider the matrix

$$A = \begin{bmatrix} \ominus 5 & 1 & \ominus 0 \\ -3 & \varepsilon & (-2)^{\bullet} \end{bmatrix}.$$

Let us first compute a max-plus-algebraic QRD of A using the mapping $\mathcal{F}$. Let $M = \left[\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix}\right]$ and define $\tilde{A} = \mathcal{F}(A, M, \cdot)$. So

$$\tilde{A}(s) = \begin{bmatrix} -e^{5s} & e^{s} & -1 \\ e^{-3s} & 0 & e^{-2s} \end{bmatrix} \qquad \text{for all } s \in \mathbb{R}_0^+. \tag{45}$$

If we use the Givens QR algorithm, we get a path of QRDs $\tilde{Q}\tilde{R}$ of $\tilde{A}$ with

$$\tilde{Q}(s) = \begin{bmatrix} \dfrac{-1}{\sqrt{1+e^{-16s}}} & \dfrac{-e^{-8s}}{\sqrt{1+e^{-16s}}} \\ \dfrac{e^{-8s}}{\sqrt{1+e^{-16s}}} & \dfrac{-1}{\sqrt{1+e^{-16s}}} \end{bmatrix},$$

$$\tilde{R}(s) = \begin{bmatrix} e^{5s}\sqrt{1+e^{-16s}} & \dfrac{-e^{s}}{\sqrt{1+e^{-16s}}} & \dfrac{1+e^{-10s}}{\sqrt{1+e^{-16s}}} \\ 0 & \dfrac{-e^{-7s}}{\sqrt{1+e^{-16s}}} & \dfrac{-e^{-2s}+e^{-8s}}{\sqrt{1+e^{-16s}}} \end{bmatrix}$$

for all $s \in \mathbb{R}_0^+$. Hence,

$$\tilde{Q}(s) \sim \begin{bmatrix} -1 & -e^{-8s} \\ e^{-8s} & -1 \end{bmatrix}, \qquad s \to \infty,$$

$$\tilde{R}(s) \sim \begin{bmatrix} e^{5s} & -e^{s} & 1 \\ 0 & -e^{-7s} & -e^{-2s} \end{bmatrix}, \qquad s \to \infty.$$

If we define $Q = \mathcal{R}(\tilde{Q})$ and $R = \mathcal{R}(\tilde{R})$, we obtain

$$Q = \begin{bmatrix} \ominus 0 & \ominus(-8) \\ -8 & \ominus 0 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 5 & \ominus 1 & 0 \\ \varepsilon & \ominus(-7) & \ominus(-2) \end{bmatrix}.$$

We have

$$Q \otimes R = \begin{bmatrix} \ominus 5 & 1 & \ominus 0 \\ -3 & (-7)^{\bullet} & -2 \end{bmatrix} \nabla A,$$

$$Q^T \otimes Q = \begin{bmatrix} 0 & (-8)^{\bullet} \\ (-8)^{\bullet} & 0 \end{bmatrix} \nabla E_2,$$

and $\|R\|_{\oplus} = 5 = \|A\|_{\oplus}$.

Let us now compute a max-plus-algebraic SVD of A. Since $\tilde{A}$ is a 2 by 3 matrix-valued function, we can compute a path of SVDs $\tilde{U}\tilde{\Sigma}\tilde{V}^T$ of $\tilde{A}$ analytically, e.g., via the eigenvalue decomposition of $\tilde{A}^T\tilde{A}$ (see [31, 59]). This yields[6]

$$\tilde{U}(s) \sim \begin{bmatrix} -1 & -e^{-8s} \\ e^{-8s} & -1 \end{bmatrix}, \qquad s \to \infty,$$

$$\tilde{\Sigma}(s) \sim \begin{bmatrix} e^{5s} & 0 & 0 \\ 0 & e^{-2s} & 0 \end{bmatrix}, \qquad s \to \infty,$$

$$\tilde{V}(s) \sim \begin{bmatrix} 1 & e^{-5s} & -e^{-4s} \\ -e^{-4s} & -e^{-5s} & -1 \\ e^{-5s} & -1 & e^{-5s} \end{bmatrix}, \qquad s \to \infty.$$

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[6] We have used the symbolic computation tool Maple to compute a path of SVDs $\tilde{U}\tilde{\Sigma}\tilde{V}^T$ of $\tilde{A}$. However, since the full expressions for the entries of $\tilde{U}$, $\tilde{S}$, and $\tilde{V}$ are too long and too intricate to display here, we give only the dominant exponentials.

If we apply the reverse mapping $\mathcal{R}$, we get a max-plus-algebraic SVD $U \otimes \Sigma \otimes V^T$ of A with

$$U = \mathcal{R}(\tilde{U}) = \begin{bmatrix} \ominus 0 & \ominus(-8) \\ -8 & \ominus 0 \end{bmatrix},$$

$$\Sigma = \mathcal{R}(\tilde{\Sigma}) = \begin{bmatrix} 5 & \varepsilon & \varepsilon \\ \varepsilon & -2 & \varepsilon \end{bmatrix},$$

$$V = \mathcal{R}(\tilde{V}) = \begin{bmatrix} 0 & -5 & \ominus(-4) \\ \ominus(-4) & \ominus(-5) & \ominus 0 \\ -5 & \ominus 0 & -5 \end{bmatrix}.$$

We have

$$U \otimes \Sigma \otimes V^T = \begin{bmatrix} \ominus 5 & 1 & \ominus 0 \\ -3 & (-7)^{\bullet} & -2 \end{bmatrix} \nabla A,$$

$$U^T \otimes U = \begin{bmatrix} 0 & (-8)^{\bullet} \\ (-8)^{\bullet} & 0 \end{bmatrix} \nabla E_2,$$

$$V^T \otimes V = \begin{bmatrix} 0 & (-5)^{\bullet} & (-4)^{\bullet} \\ (-5)^{\bullet} & 0 & (-5)^{\bullet} \\ (-4)^{\bullet} & (-5)^{\bullet} & 0 \end{bmatrix} \nabla E_3,$$

and $\sigma_1 = 5 = \|A\|_{\oplus} \geq -2 = \sigma_2$.

Although in the example above, the Q matrix of the max-plus-algebraic QRD of A is equal to the U matrix of the max-plus-algebraic SVD of A, this does not hold in general (see, e.g., the example of [19]). Furthermore, even if Q and U are equal, this does not necessarily imply that $R = S \otimes V^T$ or even $R \nabla S \otimes V^T$, since for the example above we have $(S \otimes V^T)_{21} = -7$, whereas $(R)_{21} = \varepsilon$.

Finally, we illustrate the numerical computation approach of section 6.2 for the computation of the SVD of A. Consider $\tilde{A} = \mathcal{F}(A, M, \cdot)$ as defined in (45). We have numerically computed the constant SVD of $\tilde{A}$ in a set $\{s_0, s_1, \ldots, s_{100}\}$ of equidistant, discrete points with $s_k = 0.1 + 0.15k$. This yields a sequence of SVDs $\tilde{U}(s_k)\,\tilde{\Sigma}(s_k)\,\tilde{V}^T(s_k)$ of $\mathcal{F}(A, M, s_k)$ for $k = 0, 1, \ldots, K$. The dominant exponents of the corresponding path of SVDs $\tilde{U}\tilde{\Sigma}\tilde{V}^T$ of $\tilde{A}$ can now be determined as follows. In Figure 7.1 we have plotted the functions $\sigma_{\log,i}$, $u_{\log,ij}$, and $v_{\log,ij}$ defined by

$$(46) \qquad \sigma_{\log,i}(s) = \frac{\log \tilde{\sigma}_i(s)}{s}, \quad u_{\log,ij}(s) = \frac{\log |\tilde{u}_{ij}(s)|}{s}, \quad v_{\log,ij}(s) = \frac{\log |\tilde{v}_{ij}(s)|}{s}$$

for all $s \in \mathbb{R}_0^+$. From these plots we can clearly determine the dominant exponents of the singular value functions $\tilde{\sigma}_i$ and the components of $\tilde{U}$ and $\tilde{V}$. For $s \geq s_1$ the signs of the components of $\tilde{U}$ and $\tilde{V}$ are given by

$$\operatorname{sign}(\tilde{U}(s)) = \begin{bmatrix} - & + \\ + & + \end{bmatrix} \quad \text{and} \quad \operatorname{sign}(\tilde{V}(s)) = \begin{bmatrix} + & - & - \\ - & + & - \\ + & + & + \end{bmatrix}.$$

If we take the limit of the functions $\sigma_{\log,i}$, $u_{\log,ij}$, and $v_{\log,ij}$ for the argument s going to ∞, and if we take the signs of the functions $\tilde{u}_{ij}$ and $\tilde{v}_{ij}$ into account[7]—in other

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[7] These signs determine the max-plus-algebraic sign of the corresponding entries of the matrices U and V of the max-plus-algebraic SVD of A.

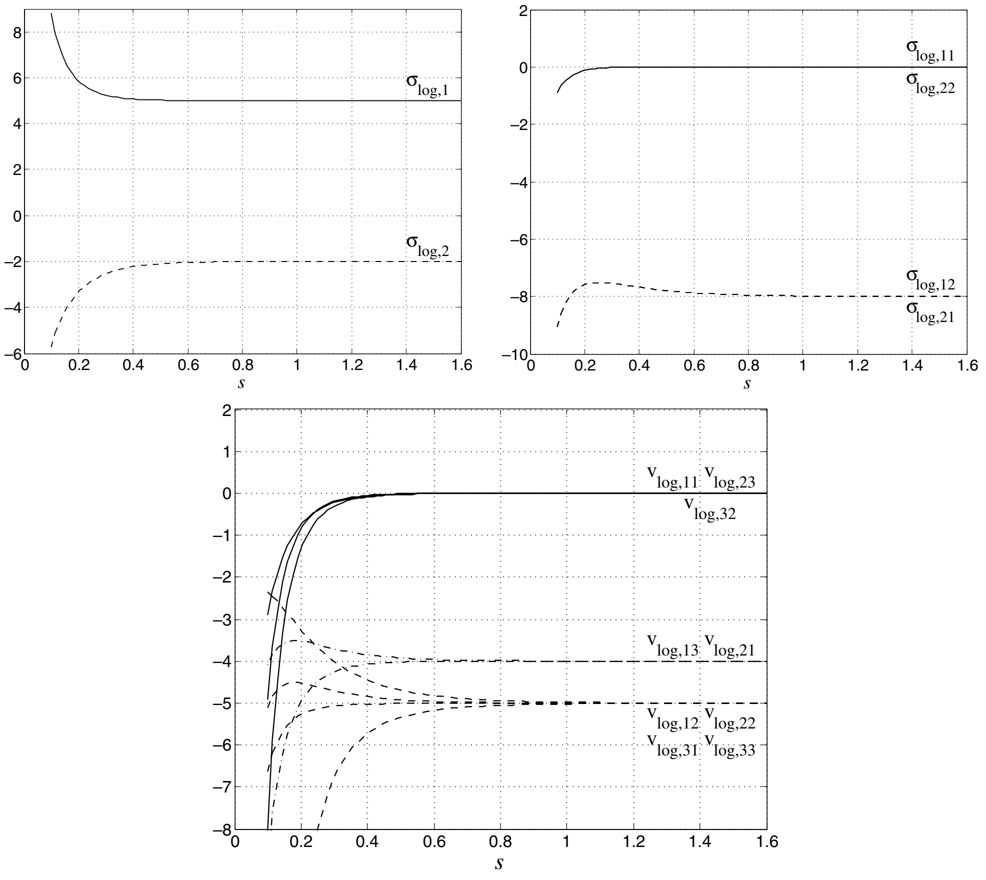

Fig. 7.1 *The functions* $\sigma_{\log,i}$, $u_{\log,ij}$, *and* $v_{\log,ij}$ *defined by* (46) *show the dominant exponents of the entries of* $\tilde{U}$, $\tilde{\Sigma}$, *and* $\tilde{V}$.

words, if we apply the reverse mapping $\mathcal{R}$—we get the following max-plus-algebraic SVD of A:

$$U \otimes \Sigma \otimes V^T = \begin{bmatrix} \ominus 0 & -8 \\ -8 & 0 \end{bmatrix} \otimes \begin{bmatrix} 5 & \varepsilon & \varepsilon \\ \varepsilon & -2 & \varepsilon \end{bmatrix} \otimes \begin{bmatrix} 0 & \ominus(-5) & \ominus(-4) \\ \ominus(-4) & -5 & \ominus 0 \\ -5 & 0 & -5 \end{bmatrix}^T$$

$$= \begin{bmatrix} \ominus 5 & 1 & \ominus 0 \\ -3 & (-7)^{\bullet} & -2 \end{bmatrix} \nabla A.$$

Other examples of the computation of the max-plus-algebraic QRD and SVD can be found in [12, 17, 19].

8. Conclusions and Future Research. In this paper we have tried to fill one of the gaps in the theory of the (symmetrized) max-plus algebra by showing that there exist max-plus-algebraic analogues of many fundamental matrix decompositions from linear algebra.

We have established a link between a ring of real functions (with addition and multiplication as basic operations) and the symmetrized max-plus algebra. Next, we have introduced a class of functions that are analytic and that can be written as a sum or a series of exponentials in a neighborhood of ∞. This class is closed under basic operations such as addition, subtraction, multiplication, division, power, square root, and absolute value. This fact has then been used to prove the existence of a QR decomposition (QRD) and a singular value decomposition (SVD) of a matrix in the symmetrized max-plus algebra. These decompositions are max-plus-algebraic analogues of basic matrix decompositions from linear algebra. The proof technique used to prove the existence of the max-plus-algebraic QRD and SVD consists of applying an exponential mapping to a max-plus-algebraic matrix, using an algorithm from conventional linear algebra for the resulting matrix-valued function, and afterwards transforming the result back to the symmetrized max-plus algebra. In addition to proving the existence of the max-plus-algebraic QRD and the max-plus-algebraic SVD, this approach can also be used to prove the existence of max-plus-algebraic analogues of many other real matrix decompositions from linear algebra such as the LU decomposition, the Hessenberg decomposition, the eigenvalue decomposition (for symmetric matrices), the Schur decomposition, and so on.

We have also discussed three possible methods for computing max-plus-algebraic matrix decompositions: via symbolic computation using linear algebra algorithms, via numerical computation and linear algebra algorithms, and via the extended linear complementarity problems. However, none of these methods can be considered to be efficient, especially for large-size matrices. So the development of efficient algorithms to compute max-plus-algebraic matrix decompositions will be an important issue for further research. One way to address this issue is to make use of the special structure of the matrices appearing in the max-plus-algebraic matrix decompositions and in the corresponding ELCPs. In addition, in view of the fact that the general ELCP is an NP-hard problem, the computational complexity of computing max-plus-algebraic matrix decompositions (in general and for special classes of matrices) should also be investigated.

In [12, 17] we introduced a further extension of the symmetrized max-plus algebra: the max-plus-complex structure $\mathbb{T}_{\max}$, which corresponds to a ring of complex functions (with addition and multiplication as basic operations). We could also define max-plus-algebraic matrix decompositions in $\mathbb{T}_{\max}$. These decompositions would then be analogues of matrix decompositions from linear algebra for complex matrices (such as the eigenvalue decomposition or the Jordan decomposition).

Other important topics for future research are further investigation of the properties of the max-plus-algebraic matrix decompositions that have been introduced in this paper and application of the max-plus-algebraic QRD, the max-plus-algebraic SVD, and other max-plus-algebraic matrix decompositions in the system theory for max-plus-linear discrete-event systems.

Appendix A. Proof of Lemma 5.2. In this appendix we show that functions that belong to the class $\mathcal{S}_{\mathrm{e}}$ are asymptotically equivalent to an exponential in the neighborhood of ∞. We shall use the following lemma.

LEMMA A.1. *If $f \in \mathcal{S}_{\mathrm{e}}$ is a series with a nonpositive dominant exponent, i.e., if there exists a positive real number K such that $f(x) = \sum_{i=0}^{\infty} \alpha_i e^{a_i x}$ for all $x \geq K$ with $\alpha_i \in \mathbb{R}$, $a_0 \leq 0$, $a_{i+1} < a_i$ for all i, $\lim_{i \to \infty} a_i = \varepsilon$, and where the series converges absolutely for every $x \geq K$, then the series $\sum_{i=0}^{\infty} \alpha_i e^{a_i x}$ converges uniformly in $[K, \infty)$.*

Proof. If $x \geq K$, then we have $e^{a_i x} \leq e^{a_i K}$ for all $i \in \mathbb{N}$ since $a_i \leq 0$ for all i. Hence, $|\alpha_i e^{a_i x}| \leq |\alpha_i e^{a_i K}|$ for all $x \geq K$ and for all $i \in \mathbb{N}$. We already know that

$\sum_{i=0}^{\infty} |\alpha_i e^{a_i K}|$ converges. Now we can apply the Weierstrass M-test (see [42, 48]). As a consequence, the series $\sum_{i=0}^{\infty} \alpha_i e^{a_i x}$ converges uniformly in $[K, \infty)$. $\Box$

Proof of Lemma 5.2. If $f \in \mathcal{S}_{\mathrm{e}}$, then there exists a positive real number K such that $f(x) = \sum_{i=0}^{n} \alpha_i e^{a_i x}$ for all $x \geq K$ with $n \in \mathbb{N} \cup \{\infty\}$, $\alpha_i \in \mathbb{R}_0$, and $a_i \in \mathbb{R}_\varepsilon$ for all i. If $n = \infty$, then f is a series that converges absolutely in $[K, \infty)$.

If $a_0 = \varepsilon$, then there exists a real number K such that $f(x) = 0$ for all $x \geq K$ and then we have $f(x) \sim 0 = 1 \cdot e^{\varepsilon x}$, $x \to \infty$, by Definition 2.4.

If $n = 1$, then $f(x) = \alpha_0 e^{a_0 x}$ and thus $f(x) \sim \alpha_0 e^{a_0 x}$, $x \to \infty$, with $\alpha_0 \in \mathbb{R}_0$ and $a_0 \in \mathbb{R}_\varepsilon$.

From now on we assume that $n > 1$ and $a_0 \neq \varepsilon$. Then we can rewrite $f(x)$ as

$$f(x) = \alpha_0 e^{a_0 x} \left(1 + \sum_{i=1}^{n} \frac{\alpha_i}{\alpha_0} \, e^{(a_i - a_0)x} \right) = \alpha_0 e^{a_0 x} (\, 1 + p(x)\,)$$

with $p(x) = \sum_{i=1}^{n} \gamma_i e^{c_i x}$, where $\gamma_i = \frac{\alpha_i}{\alpha_0} \in \mathbb{R}_0$ and $c_i = a_i - a_0 < 0$ for all i. Note that $p \in \mathcal{S}_{\mathrm{e}}$ and that p has a negative dominant exponent. Since $c_i < 0$ for all i, we have

$$\lim_{x \to \infty} p(x) = \lim_{x \to \infty} \left(\sum_{i=1}^{n} \gamma_i e^{c_i x} \right) = \sum_{i=1}^{n} \left(\lim_{x \to \infty} \gamma_i e^{c_i x} \right) = 0. \tag{47}$$

If $n = \infty$, then the series $\sum_{i=1}^{\infty} \gamma_i e^{c_i x}$ converges uniformly in $[K, \infty)$ by Lemma A.1. As a consequence, we may also interchange the summation and the limit in (47) if $n = \infty$ (cf. [42]).

So we have

$$\lim_{x \to \infty} \frac{f(x)}{\alpha_0 e^{a_0 x}} = \lim_{x \to \infty} \frac{\alpha_0 e^{a_0 x} (1 + p(x))}{\alpha_0 e^{a_0 x}} = \lim_{x \to \infty} (1 + p(x)) = 1$$

and thus $f(x) \sim \alpha_0 e^{a_0 x}$, $x \to \infty$, where $\alpha_0 \in \mathbb{R}_0$ and $a_0 \in \mathbb{R}$. $\Box$

Appendix B. Proof of Proposition 5.3. In this appendix we show that $\mathcal{S}_{\mathrm{e}}$ is closed under elementary operations such as addition, multiplication, subtraction, division, square root, and absolute value.

Proof of Proposition 5.3. If f and g belong to $\mathcal{S}_{\mathrm{e}}$, then we may assume without loss of generality that the domains of definition of f and g coincide, since we can always restrict the functions f and g to $\operatorname{dom} f \cap \operatorname{dom} g$ and since the restricted functions also belong to $\mathcal{S}_{\mathrm{e}}$.

Since f and g belong to $\mathcal{S}_{\mathrm{e}}$, there exists a positive real number K such that

$$f(x) = \sum_{i=0}^{n} \alpha_i e^{a_i x} \quad \text{and} \quad g(x) = \sum_{j=0}^{m} \beta_j e^{b_j x} \qquad \text{for all } x \geq K$$

with $m, n \in \mathbb{N} \cup \{\infty\}$, $\alpha_i, \beta_j \in \mathbb{R}_0$, and $a_i, b_j \in \mathbb{R}_\varepsilon$ for all i, j. If m or n is equal to ∞, then the corresponding series converges absolutely in $[K, \infty)$.

We may assume without loss of generality that both m and n are equal to ∞. If m or n are finite, then we can always add dummy terms of the form $0 \cdot e^{\varepsilon x}$ to $f(x)$ or $g(x)$. Afterwards we can remove terms of the form $re^{\varepsilon x}$ with $r \in \mathbb{R}$ to obtain an expression with nonzero coefficients and decreasing exponents. So from now on we assume that both f and g are absolute convergent series of exponentials. Now we show that under the conditions stated in Proposition 5.3 the functions $|f|$, ρf, $f + g$, $f - g$, fg, f^l, $\frac{1}{f}$, $\frac{g}{f}$, and $\sqrt{f}$ belong to $\mathcal{S}_{\mathrm{e}}$ for any $\rho \in \mathbb{R}$ and any $l \in \mathbb{N}$.

$|f| \in \mathcal{S}_{\mathrm{e}}$. If $a_0 = \varepsilon$, then we have $f(x) = 0$ for all $x \geq K$, which means that $|f(x)| = 0$ for all $x \geq K$. So if $a_0 = \varepsilon$, then $|f|$ belongs to $\mathcal{S}_{\mathrm{e}}$.

If $a_0 \neq \varepsilon$, then there exists a real number $L \geq K$ such that either $f(x) > 0$ or $f(x) < 0$ for all $x \geq L$ since $f(x) \sim \alpha_0 e^{a_0 x}$, $x \to \infty$, with $\alpha_0 \neq 0$ by Lemma 5.2. Hence, either $|f(x)| = f(x)$ or $|f(x)| = -f(x)$ for all $x \geq L$. So in this case $|f|$ also belongs to $\mathcal{S}_{\mathrm{e}}$.

Since f and g are analytic in $[K, \infty)$, the functions ρf, $f + g$, $f - g$, $f \cdot g$, and f^l are also analytic in $[K, \infty)$ for any $\rho \in \mathbb{R}$ and any $l \in \mathbb{N}$. So in order to prove that these functions belong to $\mathcal{S}_{\mathrm{e}}$, now we only have to prove that they can be written as a sum of exponentials or as an absolutely convergent series of exponentials.

$\rho f \in \mathcal{S}_{\mathrm{e}}$. Consider an arbitrary $\rho \in \mathbb{R}$. If $\rho = 0$, then $\rho f(x) = 0$ for all $x \geq K$ and thus $\rho f \in \mathcal{S}_{\mathrm{e}}$. If $\rho \neq 0$, then we have $\rho f(x) = \sum_{i=0}^{\infty} (\rho \alpha_i) e^{a_i x}$. The series $\sum_{i=0}^{\infty} (\rho \alpha_i) e^{a_i x}$ also converges absolutely in $[K, \infty)$ and has the same exponents as $f(x)$. Hence, $\rho f \in \mathcal{S}_{\mathrm{e}}$.

$f + g, f - g \in \mathcal{S}_{\mathrm{e}}$. The sum function $f + g$ is a series of exponentials since

$$f(x) + g(x) = \sum_{i=0}^{\infty} \alpha_i e^{a_i x} + \sum_{j=0}^{\infty} \beta_j e^{b_j x}.$$

Furthermore, this series converges absolutely for every $x \geq K$. Therefore, the sum of the series does not change if we rearrange the terms [42]. So $f(x) + g(x)$ can be written in the form of Definition 5.1 by reordering the terms, adding up terms with equal exponents, and removing terms of the form $r e^{\varepsilon x}$ with $r \in \mathbb{R}$, if there are any. If the result is a series, then the sequence of exponents is decreasing and unbounded from below. So $f + g \in \mathcal{S}_{\mathrm{e}}$.

Since $f - g = f + (-1)g$, the function $f - g$ also belongs to $\mathcal{S}_{\mathrm{e}}$.

$fg \in \mathcal{S}_{\mathrm{e}}$. The series corresponding to f and g converge absolutely for every $x \geq K$. Therefore, their Cauchy product will also converge absolutely for every $x \geq K$ and it will be equal to fg [42]:

$$f(x)g(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \alpha_j \beta_{i-j} e^{(a_j + b_{i-j})x} \qquad \text{for all } x \geq K\,.$$

Using the same procedure as for $f + g$, we can also write this product in the form (24) or (25). So $fg \in \mathcal{S}_{\mathrm{e}}$.

$f^l \in \mathcal{S}_{\mathrm{e}}$. Let $l \in \mathbb{N}$. If $l = 0$, then $f^l = 0 \in \mathcal{S}_{\mathrm{e}}$, and if $l = 1$, then $f^l = f \in \mathcal{S}_{\mathrm{e}}$. If $l > 1$, we can make repeated use of the fact that $fg \in \mathcal{S}_{\mathrm{e}}$ if $f, g \in \mathcal{S}_{\mathrm{e}}$ to prove that f^l also belongs to $\mathcal{S}_{\mathrm{e}}$.

$\frac{1}{f}, \frac{g}{f} \in \mathcal{S}_{\mathrm{e}}$. If there exists a real number P such that $f(x) \neq 0$ for all $x \geq P$, then $\frac{1}{f}$ and $\frac{g}{f}$ are defined and analytic in $[P, \infty)$. Note that we may assume without loss of generality that $P \geq K$. Furthermore, since the function f restricted to the interval $[P, \infty)$ also belongs to $\mathcal{S}_{\mathrm{e}}$, we may assume without loss of generality that the domain of definition of f is $[P, \infty)$.

If $f(x) \neq 0$ for all $x \geq P$, then we have $a_0 \neq \varepsilon$. As a consequence, we can rewrite $f(x)$ as

$$f(x) = \sum_{i=0}^{\infty} \alpha_i e^{a_i x} = \alpha_0 e^{a_0 x} \left(1 + \sum_{i=1}^{\infty} \frac{\alpha_i}{\alpha_0} \; e^{(a_i - a_0)x} \right) = \alpha_0 e^{a_0 x} (\, 1 + p(x)\,)$$

with $p(x) = \sum_{i=1}^{\infty} \gamma_i e^{c_i x}$, where $\gamma_i = \frac{\alpha_i}{\alpha_0} \in \mathbb{R}_0$ and $c_i = a_i - a_0 < 0$ for all i. Note that p is defined in $[P, \infty)$, that $p \in \mathcal{S}_{\mathrm{e}}$, and that p has a negative dominant exponent.

If $c_1 = \varepsilon$, then $p(x) = 0$ and $\frac{1}{f(x)} = \frac{1}{\alpha_0} e^{-a_0 x}$ for all $x \geq P$. Hence, $\frac{1}{f} \in \mathcal{S}_{\mathrm{e}}$.

Now assume that $c_1 \neq \varepsilon$. Since $\{c_i\}_{i=1}^{\infty}$ is a nonincreasing sequence of negative numbers with $\lim_{i \to \infty} c_i = \varepsilon = -\infty$ and since p converges uniformly in $[P, \infty)$ by Lemma A.1, we have $\lim_{x \to \infty} p(x) = 0$ (cf. (47)). So $|\, p(x)\, |$ will be less than 1 if x is large enough, say if $x \geq M$. If we use the Taylor series expansion of $\frac{1}{1+x}$, we obtain

$$\frac{1}{1 + p(x)} = \sum_{k=0}^{\infty} (-1)^k p^k(x) \qquad \text{if } |p(x)| < 1\,. \tag{48}$$

We already know that $p \in \mathcal{S}_{\mathrm{e}}$. Hence, $p^k \in \mathcal{S}_{\mathrm{e}}$ for all $k \in \mathbb{N}$. We have $|p(x)| < 1$ for all $x \geq M$. Moreover, for any $k \in \mathbb{N}$ the highest exponent in p^k is equal to kc_1, which implies that the dominant exponent of p^k tends to $-\infty$ as k tends to ∞. As a consequence, the coefficients and the exponents of increasingly successive terms of the partial sum function s_n, which is defined by $s_n(x) = \sum_{k=0}^{n} (-1)^k p^k(x)$ for $x \in [M, \infty)$, will no longer change as n becomes larger and larger. Therefore, the series on the right-hand side of (48) also is a sum of exponentials:

$$\frac{1}{1 + p(x)} = \sum_{k=0}^{\infty} (-1)^k \left(\sum_{i=1}^{\infty} \gamma_i e^{c_i x} \right)^k = \sum_{k=0}^{\infty} d_i e^{\delta_i x} \qquad \text{for all } x \geq M\,.$$

Note that the set of exponents of this series will have no finite accumulation point since the highest exponent in p^k is equal to kc_1. Let us now prove that this series also converges absolutely. Define $p^*(x) = \sum_{i=1}^{\infty} |\gamma_i| e^{c_i x}$ for all $x \geq P$. Since the terms of the series p^* are the absolute values of the terms of the series p and since p converges absolutely in $[P, \infty)$, p^* also converges absolutely in $[P, \infty)$. By Lemma A.1 the series p^* also converges uniformly in $[P, \infty)$. Furthermore, $\{c_i\}_{i=1}^{\infty}$ is a nonincreasing and unbounded sequence of negative numbers. As a consequence, we have $\lim_{x \to \infty} p^*(x) = 0$ (cf. (47)). So $|\, p^*(x)\, |$ will be less than 1 if x is large enough, say if $x \geq N$. Therefore, we have

$$\frac{1}{1 + p^*(x)} = \sum_{k=0}^{\infty} (-1)^k \left(p^*(x) \right)^k \qquad \text{for all } x \geq N\,.$$

This series converges absolutely in $[N, \infty)$. Since

$$\sum_{k=0}^{\infty} |d_i| e^{\delta_i x} \leq \sum_{k=0}^{\infty} \left(\sum_{i=1}^{\infty} |\gamma_i| e^{c_i x} \right)^k = \sum_{k=0}^{\infty} \left| (p^*(x))^k \right| ,$$

the series $\sum_{k=0}^{\infty} d_i e^{\delta_i x}$ also converges absolutely for any $x \in [N, \infty)$. Since the series $\sum_{k=0}^{\infty} d_i e^{\delta_i x}$ converges absolutely, we can reorder its terms. After reordering the terms, adding up terms with the same exponents, and removing terms of the form $re^{\varepsilon x}$ with $r \in \mathbb{R}$ if necessary, the sequence of exponents will be decreasing and unbounded from below.

This implies that $\frac{1}{1+p} \in \mathcal{S}_{\mathrm{e}}$ and thus also $\frac{1}{f} \in \mathcal{S}_{\mathrm{e}}$. As a consequence, it follows from the above results that $\frac{g}{f} = g\frac{1}{f}$ also belongs to $\mathcal{S}_{\mathrm{e}}$.

$\boldsymbol{\sqrt{f} \in \mathcal{S}_{\mathrm{e}}}$**.** If there exists a real number Q such that $f(x) > 0$ for all $x \geq Q$, then the function $\sqrt{f}$ is defined and analytic in $[Q, \infty)$. We may assume without loss of generality that $Q \geq K$ and that the domain of definition of f is $[Q, \infty)$.

If $a_0 = \varepsilon$, then we have $\sqrt{f(x)} = 0$ for all $x \geq Q$ and thus $\sqrt{f} \in \mathcal{S}_{\mathrm{e}}$.

If $a_0 \neq \varepsilon$, then $\alpha_0 > 0$ and then we can rewrite $\sqrt{f(x)}$ as

$$\sqrt{f(x)} = \sqrt{\alpha_0}\; e^{\frac{1}{2}a_0 x} \sqrt{1 + p(x)}\,.$$

Now we can use the Taylor series expansion of $\sqrt{1+x}$. This leads to

$$\sqrt{1+p(x)} = \sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-k\right)\, k!}\; p^k(x) \qquad \text{if } |p(x)| < 1\,,$$

where Γ is the gamma function. If we apply the same reasoning as for $\frac{1}{1+p}$, we find that $\sqrt{1+p} \in \mathcal{S}_{\mathrm{e}}$ and thus also $\sqrt{f} \in \mathcal{S}_{\mathrm{e}}$. ☐

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