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Equivalence of state representations for hidden Markov models

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Abstract

In this paper we consider the following problem for hidden Markov models: given a minimal hidden Markov model, derive conditions for another hidden Markov model to be equivalent and give a description of the complete set of equivalent models. A distinction is made between quasi- and positive hidden Markov models and between Mealy and Moore hidden Markov models. We derive a condition for two positive Mealy models to be equivalent and give a description of the complete set of Mealy models that are equivalent to a given Mealy model. We show that under certain conditions minimal quasi-Moore models are unique up to a permutation of the states. We derive a condition for two positive Moore models to be equivalent and give a description of the complete set of Moore models equivalent to a given Moore model. Finally, we compare the results for hidden Markov models and linear Gaussian systems.

Keywords: Hidden Markov models; Stochastic systems; Realization theory; Equivalent state representations

1. Introduction

Hidden Markov models (HMMs) were introduced in the literature in the late 1950s [2]. HMMs are used extensively in engineering applications, such as speech processing, image processing, and bioinformatics. Despite the success in applications, many questions remain unanswered. An example of an open problem is the *realization problem*: given the string probabilities of finite length strings, find all hidden Markov models that realize these string probabilities. The realization problem can be split up into three subproblems. The first is the realizability problem: derive conditions for string probabilities to be realizable by an HMM. In [13] conditions for the realizability of string probabilities are derived. The second subproblem is the realization problem itself: given realizable string probabilities, find a corresponding hidden Markov model. Partial solutions for this problem are given in [1,6,11,13]. The third subproblem concerns the question of finding all possible realizations that are equivalent to a given realization. For Gauss–Markov systems, where both the states and observations take values in finite-dimensional real vector spaces, this problem is solved in [5]. However, for hidden Markov models not much is known about the equivalence problem. In this paper, we consider the equivalence problem for hidden Markov models.

The structure of the paper is as follows. In Section 2 we introduce Moore and Mealy hidden Markov models and their quasi-forms and also describe a procedure to find a minimal quasi-Mealy model equivalent to a given positive model. In Section 3 we describe the complete set of equivalent Mealy models, both for the quasi- and positive cases. In Section 4 we show that, under certain conditions, the class of equivalent quasi-Moore hidden Markov models consists of only one element up to a permutation of the states and subsequently we give a description of the set of equivalent positive Moore models. In Section 5 we summarize the results concerning the equivalence sets, and in Section 6 we compare the results with the linear Gaussian case. In Section 7, we draw some conclusions.

The following notation is used: if $\mathbb{S}$ is a set, then $|\mathbb{S}|$ denotes its cardinality. $\mathbb{R}_+$ is the set of nonnegative real numbers, $\geq$ and $>$ are "elementwise larger than or equal to" and "elementwise larger than" respectively. If X is a matrix, then $X_{i,j}$ denotes its (i, j)th element, $X_{i,:}$ denotes the ith row and $X_{i:j,:}$ the matrix formed by the ith to the jth row of X. Analogous notations are used for selecting columns instead of rows. With e we denote the column vector with all elements equal to 1, that is $e = [\,1\;1\;\ldots\;1]^\top$. The dimension of e is clear from the context. With

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I we denote the unit matrix of appropriate size. With $*$ we denote an element, a subvector or a submatrix of a matrix of which the exact values are unimportant. Finally $\text{vec}(X)$ is a row vector where the elements are row-wise scanned from the matrix X.

2. Moore and Mealy HMM

Hidden Markov models are used to model finite-valued processes y defined on the time axis $\mathbb{N}$. They assume the existence of an underlying finite-valued Markov process x, called the state process, on which the output process depends in a probabilistic manner. In this section, we introduce two different types of HMMs: Moore HMMs and Mealy HMMs. We also introduce the so-called quasi-forms of these two types of models. We also discuss conversions between Moore and Mealy models. Finally, we describe a procedure to find a minimal quasi-Mealy model equivalent to a positive Mealy model.

2.1. Mealy HMM

A *Mealy HMM* assumes that the state generated at time $t+1$ and the output symbol generated at time t depend probabilistically on the state at time t. A Mealy HMM is specified by the quadruple $(\mathbb{X}, \mathbb{Y}, \Pi, \pi(1))$ where:

- $\mathbb{X}$ with $|\mathbb{X}| < \infty$ is the *state alphabet* and $\mathbb{Y}$ with $|\mathbb{Y}| < \infty$ the *output alphabet*. Without loss of generality, we identify $\mathbb{X} = \{1, 2, \ldots, |\mathbb{X}|\}$.
- Π is a mapping from $\mathbb{Y}$ to $\mathbb{R}_+^{|\mathbb{X}|\times|\mathbb{X}|}$, with $\Pi_{\mathbb{X}} := \sum_{\mathrm{y}\in\mathbb{Y}} \Pi(\mathrm{y})$ is a stochastic matrix, i.e. $\Pi_{\mathbb{X}} e = e$. The element $\Pi_{i,j}(\mathrm{y})$ is $P(x(t+1) = j, y(t) = \mathrm{y} | x(t) = i)$, that is the probability of going from state i to state j while generating the output symbol y. The matrix $\Pi_{\mathbb{X}}$ is called the *transition matrix* of the HMM.
- $\pi(1)$ is a vector in $\mathbb{R}_+^{|\mathbb{X}|}$ for which $\pi(1)e = 1$. It is called the *initial state distribution*. The element $\pi_i(1)$ is $P(x(1) = i)$, that is the probability that the initial state is i.

The number of states $|\mathbb{X}|$ is called the *order* of the HMM. The model is called *stationary* if the state distribution is the same at every time instant, i.e. if the initial state distribution vector is a left eigenvector of the transition matrix corresponding to the eigenvalue 1: $\pi(1)\Pi_{\mathbb{X}} = \pi(1)$.

Denote by $\mathbb{Y}^*$ the set of finite strings with symbols from the set $\mathbb{Y}$ (including the empty string) and by $\mathrm{y} = \mathrm{y}_1\mathrm{y}_2\ldots\mathrm{y}_{|\mathrm{y}|}$ an output sequence from $\mathbb{Y}^*$, where $|\mathrm{y}|$ denotes the length of y. Let $\mathcal{P} : \mathbb{Y}^* \mapsto [0, 1]$ be *string probabilities*, defined as $\mathcal{P}(\mathrm{y}) := P(y(1) = \mathrm{y}_1, y(2) = \mathrm{y}_2, \ldots, y(|\mathrm{y}|) = \mathrm{y}_{|\mathrm{y}|})$. Of course, the string probabilities satisfy $\mathcal{P}(\phi) = 1$, where ϕ denotes the empty string, and $\sum_{\mathrm{y}\in\mathbb{Y}} \mathcal{P}(\mathrm{yy}) = \mathcal{P}(\mathrm{y})$ where yy is the concatenation of the string y with the symbol y. The string probabilities *generated* by a Mealy HMM $(\mathbb{X}, \mathbb{Y}, \Pi, \pi(1))$ are given by

$$\mathcal{P}(\mathrm{y}) = \pi(1)\Pi(\mathrm{y})e,$$

where $\mathrm{y} = \mathrm{y}_1\mathrm{y}_2\ldots\mathrm{y}_{|\mathrm{y}|} \in \mathbb{Y}^*$ and where $\Pi(\mathrm{y}) = \Pi(\mathrm{y}_1)\Pi(\mathrm{y}_2)\ldots\Pi(\mathrm{y}_{|\mathrm{y}|})$.

Two Mealy HMMs $(\mathbb{X}, \mathbb{Y}, \Pi, \pi(1))$ and $(\mathbb{X}', \mathbb{Y}, \Pi', \pi'(1))$ with string probabilities $\mathcal{P}$ and $\mathcal{P}'$, respectively, are said to be *equivalent* if $\mathcal{P} = \mathcal{P}'$. A Mealy HMM $(\mathbb{X}, \mathbb{Y}, \Pi, \pi(1))$ is called *minimal* if for any other equivalent Mealy model $(\mathbb{X}', \mathbb{Y}, \Pi', \pi'(1))$ it holds that $|\mathbb{X}| \leq |\mathbb{X}'|$.

In the *Mealy realization problem*, we are given output string probabilities $\mathcal{P}$ and the problem is to find a Mealy HMM $(\mathbb{X}, \mathbb{Y}, \Pi, \pi(1))$ that generates the string probabilities $\mathcal{P}$. The realization problem is hard because of the positivity constraints on $\pi(1)$ and Π. For that reason, one often (first) solves the quasi-realization problem, which is exactly the same problem as the realization problem but without the positivity constraints. However, the quasi-model which is found from the quasi-realization procedure retains some of the interesting properties of a positive model [12].

A *quasi-Mealy HMM* is defined by $(\mathbb{Q}, \mathbb{Y}, A, c, b)$, where $\mathbb{Q}$ is the *quasi-state alphabet* and $\mathbb{Y}$ is the *output alphabet*. b is a column vector in $\mathbb{R}^{|\mathbb{Q}|}$, A is a mapping from $\mathbb{Y}$ to $\mathbb{R}^{|\mathbb{Q}|\times|\mathbb{Q}|}$, where $A_{\mathbb{Q}} := \sum_{\mathrm{y}\in\mathbb{Y}} A(\mathrm{y})$ is a quasi-stochastic matrix, i.e. $A_{\mathbb{Q}} b = b$. The matrix $A_{\mathbb{Q}}$ is called the *quasi-state transition matrix*. c is a row vector in $\mathbb{R}^{|\mathbb{Q}|}$ called the *quasi-initial state distribution* for which $cb = 1$. Notice that A, c and b of a quasi-Mealy model $(\mathbb{Q}, \mathbb{Y}, A, c, b)$ are the analogues of Π, $\pi(1)$ and e of a positive Mealy model $(\mathbb{X}, \mathbb{Y}, \Pi, \pi(1))$. The string probabilities *generated by* a quasi-Mealy HMM $(\mathbb{Q}, \mathbb{Y}, A, c, b)$ are given by

$$\mathcal{P}(\mathrm{y}) = cA(\mathrm{y})b,$$

where $\mathrm{y} = \mathrm{y}_1\mathrm{y}_2\ldots\mathrm{y}_{|\mathrm{y}|} \in \mathbb{Y}^*$ and where $A(\mathrm{y}) = A(\mathrm{y}_1)A(\mathrm{y}_2)\ldots A(\mathrm{y}_{|\mathrm{y}|})$. Equivalence and minimality of quasi-Mealy HMMs are defined in an analogous way as for positive Mealy HMMs. The order of a minimal quasi-HMM is lower than or equal to the order of a minimal equivalent positive HMM.

Now define the $\mathcal{O}(c, A)$-matrix in $\mathbb{R}^{\infty\times|\mathbb{Q}|}$ and the $\mathcal{C}(A, b)$-matrix in $\mathbb{R}^{|\mathbb{Q}|\times\infty}$ of a quasi-Mealy HMM $(\mathbb{Q}, \mathbb{Y}, A, c, b)$ as

$$\mathcal{O}_{i,:}(c, A) := cA(u_i), \tag{1}$$

$$\mathcal{C}_{:,j}(A, b) := A(v_j)b, \tag{2}$$

where u_i is the ith element of a first ordering $u := (u_k, k = 1, 2, \ldots)$ of the strings of $\mathbb{Y}^*$ and v_j is the jth element of a second ordering $v := (v_k, k = 1, 2, \ldots)$ of the strings of $\mathbb{Y}^*$. Typically, in the first ordering the strings are ordered lexicographically from right to left, which gives $(\phi, 0, 1, 00, 10, 01, 11, 000, 100, \ldots)$ for $\mathbb{Y} = \{0, 1\}$. In the second ordering the strings are ordered lexicographically from left to right, which means $(\phi, 0, 1, 00, 01, 10, 11, 000, 001, \ldots)$ for $\mathbb{Y} = \{0, 1\}$. In the case where $\mathbb{Y} = \{0, 1\}$ the matrices $\mathcal{O}(c, A)$ and $\mathcal{C}(A, b)$ are

$$\mathcal{O}(c, A) = \begin{bmatrix} c \\ cA(0) \\ cA(1) \\ cA(00) \\ cA(10) \\ \vdots \end{bmatrix},$$

$$\mathcal{C}(A, b) = \begin{bmatrix} b & A(0)b & A(1)b & A(00)b & A(01)b & \cdots \end{bmatrix}.$$

It can be proven that a quasi-Mealy HMM is minimal if and only if the matrices $\mathcal{C}(A, b)$ and $\mathcal{O}(c, A)$ have full row and full column rank respectively.

2.2. Moore HMM

In a *Moore HMM*, the generation of the next state and the generation of the output are assumed to be independent. A Moore HMM is specified by $(\mathbb{X}, \mathbb{Y}, \Pi_{\mathbb{X}}, \beta, \pi(1))$. As before, $\mathbb{X}$ and $\mathbb{Y}$ are the state and output alphabets. The matrix $\Pi_{\mathbb{X}} \in \mathbb{R}_+^{|\mathbb{X}|\times|\mathbb{X}|}$ with $\Pi_{\mathbb{X}} e = e$ is the *state transition matrix*, defined as $(\Pi_{\mathbb{X}})_{ij} = P(x(t+1) = j|x(t) = i)$. β is a mapping from $\mathbb{Y}$ to $\mathbb{R}_+^{|\mathbb{X}|}$, such that $\beta_i(\mathrm{y}) = P(y(t) = \mathrm{y}|x(t) = i)$. It is required that $\sum_{\mathrm{y}} \beta(\mathrm{y}) = e$. The vector $\pi(1) \in \mathbb{R}_+^{|\mathbb{X}|}$ is the *initial state distribution* defined as $\pi_i(1) = P(x(1) = i)$. Suppose we have an ordering $(\mathrm{y}_k, k = 1, 2, \ldots, |\mathbb{Y}|)$ of the output symbols of the set $\mathbb{Y}$, then the map β can be represented by a matrix, called the *output matrix* B defined as $B := \begin{bmatrix}\beta(\mathrm{y}_1) & \cdots & \beta(\mathrm{y}_{|\mathbb{Y}|})\end{bmatrix}$, with $Be = e$. An equivalent description of the Moore HMM is therefore given by $(\mathbb{X}, \mathbb{Y}, \Pi_{\mathbb{X}}, B, \pi(1))$. The number of states $|\mathbb{X}|$ is the *order* of the Moore model.

String probabilities *generated by* a Moore HMM $(\mathbb{X}, \mathbb{Y}, \Pi_{\mathbb{X}}, \beta, \pi(1))$ are given by

$$\mathcal{P}(\mathrm{y}) = \pi(1)\mathrm{diag}(\beta(\mathrm{y}_1))\Pi_{\mathbb{X}} \cdots \mathrm{diag}(\beta(\mathrm{y}_{|\mathrm{y}|}))\Pi_{\mathbb{X}} e,$$

where $\mathrm{y} = \mathrm{y}_1\mathrm{y}_2 \cdots \mathrm{y}_{|\mathrm{y}|} \in \mathbb{Y}^*$. Equivalence and minimality of Moore models are defined in an analogous way as for Mealy HMMs.

We define a *quasi Moore HMM* as $(\mathbb{Q}, \mathbb{Y}, A_{\mathbb{Q}}, \lambda, c)$ or as $(\mathbb{Q}, \mathbb{Y}, A_{\mathbb{Q}}, L, c)$, where $\mathbb{Q}$ is the *quasi-state alphabet* and $\mathbb{Y}$ is the *output alphabet*. $A_{\mathbb{Q}} \in \mathbb{R}^{|\mathbb{Q}|\times|\mathbb{Q}|}$ with $A_{\mathbb{Q}} e = e$ is the *quasi-state transition matrix*. λ is a mapping from $\mathbb{Y}$ to $\mathbb{R}^{|\mathbb{Q}|}$ for which $\sum_{\mathrm{y}} \lambda(\mathrm{y}) = e$. The vector $c \in \mathbb{R}^{|\mathbb{X}|}$ for which $ce = 1$ is the *quasi-initial state distribution*. The *quasi-output matrix* L is defined as $L := \begin{bmatrix}\lambda(\mathrm{y}_1) & \cdots & \lambda(\mathrm{y}_{|\mathbb{Y}|})\end{bmatrix}$, with $Le = e$. Notice that $A_{\mathbb{Q}}$, λ, L and c of a quasi-Moore model $(\mathbb{Q}, \mathbb{Y}, A_{\mathbb{Q}}, \lambda, c)$ are the analogues of $\Pi_{\mathbb{X}}$, β, B and $\pi(1)$ of a positive Moore model $(\mathbb{X}, \mathbb{Y}, \Pi_{\mathbb{X}}, \beta, \pi(1))$.

String probabilities *generated by* a quasi-Moore HMM $(\mathbb{Q}, \mathbb{Y}, A_{\mathbb{Q}}, \lambda, c)$ are given by

$$\mathcal{P}(\mathrm{y}) = c\, \mathrm{diag}(\lambda(\mathrm{y}_1))A_{\mathbb{Q}} \ldots \mathrm{diag}(\lambda(\mathrm{y}_{|\mathrm{y}|}))A_{\mathbb{Q}} e,$$

where $\mathrm{y} = \mathrm{y}_1\mathrm{y}_2 \ldots \mathrm{y}_{|\mathrm{y}|} \in \mathbb{Y}^*$. Equivalence and minimality of quasi-Moore HMMs are defined in an analogous way as for Mealy HMMs. Equivalence between (quasi-) Moore and (quasi-) Mealy models is defined analogously.

2.3. Conversions between Moore and Mealy models

It can be shown that the expressive power of Moore HMMs and Mealy HMMs is the same [13], meaning that a finite-valued process is realizable with a Moore HMM if and only if it is realizable with a Mealy HMM. However for a given finite process the order of a minimal Mealy model does not exceed the order of a minimal Moore model.

Converting a Moore model $(\mathbb{X}, \mathbb{Y}, \Pi_{\mathbb{X}}, \beta, \pi(1))$ into a Mealy model $(\mathbb{X}, \mathbb{Y}, \Pi, \pi(1))$ is always possible, using

$$\Pi(\mathrm{y}) = \mathrm{diag}(\beta(\mathrm{y}))\Pi_{\mathbb{X}}.$$

However, the obtained Mealy model can be nonminimal, even if the Moore model is minimal.

Converting a Mealy model in a Moore model can be done by connecting a state of the Moore model to every state transition of the Mealy model and then calculating the state transition probabilities and the output probabilities. Typically, this approach leads to a nonminimal Moore model even if the Mealy model is minimal.

In Section 3, we show that, under certain conditions, a (quasi-) Mealy model with two outputs can always be converted into a quasi-Moore model with the same number of states.

2.4. Conversion from positive Mealy to minimal quasi-Mealy model

In this section we describe a method to reduce a nonminimal quasi-Mealy model. Because a positive Mealy model is typically nonminimal as a quasi-Mealy model, the method can also be used to find a minimal quasi-Mealy model which is equivalent to a given (minimal or nonminimal) positive Mealy model.

As mentioned before, a quasi-Mealy model $(\mathbb{Q}, \mathbb{Y}, A, c, b)$ is minimal if and only if the matrices $\mathcal{C}(A, b)$ and $\mathcal{O}(c, A)$ defined by (2) and (1) have full row and full column rank respectively.

Given a nonminimal Mealy model $(\mathbb{Q}, \mathbb{Y}, A, c, b)$ finding an equivalent minimal Mealy model can be done in two steps. In the first step one determines an equivalent quasi-Mealy model $(\mathbb{Q}^{\mathrm{c}}, \mathbb{Y}, A^{\mathrm{c}}, c^{\mathrm{c}}, b^{\mathrm{c}})$ for which $\mathcal{C}(c^{\mathrm{c}}, A^{\mathrm{c}})$ has full row rank and in the second step one determines an equivalent quasi-model $(\mathbb{Q}^{co}, \mathbb{Y}, A^{co}, c^{co}, b^{co})$ for which $\mathcal{C}(A^{co}, b^{co})$ and $\mathcal{O}(c^{co}, A^{co})$ have full row and full column rank respectively. We now describe both steps subsequently.

It is clear that for a quasi-Mealy model $(\mathbb{Q}, \mathbb{Y}, A, c, b)$ every nonsingular matrix $T \in \mathbb{R}^{|\mathbb{Q}|\times|\mathbb{Q}|}$ gives rise to an equivalent quasi-Mealy model $(\mathbb{Q}, \mathbb{Y}, TAT^{-1}, cT^{-1}, Tb)$. Now let $(\mathbb{Q}, \mathbb{Y}, A, c, b)$ be such that

$$\mathrm{rank}\, \mathcal{C}(A, b) = r < |\mathbb{Q}|$$

then it can be shown that there exists a nonsingular matrix R such that the equivalent Mealy model $(\mathbb{Q}, \mathbb{Y}, A' = RAR^{-1}, c' = cR^{-1}, b' = Rb)$ has the form

$$A'(\mathrm{y}) = \overset{\begin{matrix}|\mathbb{Q}|-r & r\end{matrix}}{\left[\begin{array}{c|c} * & 0 \\ \hline * & A^{\mathrm{c}}(\mathrm{y}) \end{array}\right]}\begin{matrix}|\mathbb{Q}|-r \\ r\end{matrix}, \quad \forall \mathrm{y} \in \mathbb{Y},$$

$$b' = \left[\begin{array}{c} 0 \\ \hline b^{\mathrm{c}} \end{array}\right]\begin{matrix}|\mathbb{Q}|-r \\ r\end{matrix}, \qquad c' = \overset{\begin{matrix}|\mathbb{Q}|-r & r\end{matrix}}{\left[\begin{array}{c|c} * & c^{\mathrm{c}} \end{array}\right]}.$$

Any realization of this form has the property that the rth order subsystem $(\mathbb{Q}^{\mathrm{c}}, \mathbb{Y}, A^{\mathrm{c}}, c^{\mathrm{c}}, b^{\mathrm{c}})$ is equivalent with the system $(\mathbb{Q}, \mathbb{Y}, A, c, b)$ and that $\mathcal{C}(A^{\mathrm{c}}, b^{\mathrm{c}})$ has full row rank.

Now notice that

$$\mathcal{C}(A', b') = R\mathcal{C}(A, b) = \left[\begin{array}{c} 0 \\ \hline \mathcal{C}(A^{c}, b^{c}) \end{array}\right] \begin{array}{l} {\scriptstyle |\mathbb{Q}|-r} \\ {\scriptstyle r} \end{array},$$

which suggests a procedure to compute the transformation R. Indeed, R is such that $R\mathcal{C}(A, b)$ has its first $|\mathbb{Q}| - r$ rows equal to 0. Such a transformation R can be found using the Singular Value Decomposition (SVD).

We now describe an algorithm, inspired by [8] to find the transformation R directly from the system matrices without computing the $\mathcal{C}$-matrix. Therefore, we first define the matrix P for the model $(\mathbb{Q}, \mathbb{Y}, A, c, b)$ as

$$P(A, b) := \Big[A_{:,1}(\mathrm{y}^{(1)}) \cdots A_{:,1}(\mathrm{y}^{(|\mathbb{Y}|)}) \cdots \\ A_{:,|\mathbb{Q}|}(\mathrm{y}^{(1)}) \cdots A_{:,|\mathbb{Q}|}(\mathrm{y}^{(|\mathbb{Y}|)})|b\Big].$$

Algorithm 1. Given the quasi-model $(\mathbb{Q}, \mathbb{Y}, A, c, b)$ with the corresponding matrix P, run the following steps.

(1) Set $(\mathbb{Q}', \mathbb{Y}, A', c', b') = (\mathbb{Q}, \mathbb{Y}, A, c, b)$, $P' = P$, $i = |\mathbb{X}| \cdot |\mathbb{Y}| + 1$ and $j = |\mathbb{Q}|$.
(2) If every element of $P'_{1:j,i}$ is equal to 0 then goto step 5.
(3) Find a transformation $R_i = \begin{bmatrix} R_i' & 0 \\ & I_{|\mathbb{Q}|-j} \end{bmatrix}$ such that the vector $R_i P_{:,i}$ is of the form $\begin{bmatrix} 0 & \cdots & 0 & \mathrm{x} & \cdots & \mathrm{x} \end{bmatrix}^{\top}$ where the number of 0's is equal to $j-1$ and the number of x's is equal to $|\mathbb{Q}| - j + 1$. Transform $(\mathbb{Q}', \mathbb{Y}, A', c', b')$ into $(\mathbb{Q}', \mathbb{Y}, R_i A' R_i^{-1}, c' R_i^{-1}, R_i b')$ and recalculate the matrix P'.
(4) Decrease i by 1, and decrease j by 1. If $j = 0$ goto step 6. If $j > 0$, go to step 2.
(5) Increase i by 1. Goto step 2.
(6) Calculate R as $R = R_i R_{i-1} \ldots R_2 R_1$.

So far we described a method to find for a given nonminimal Mealy model $(\mathbb{Q}, \mathbb{Y}, A, c, b)$ an equivalent quasi-Mealy model $(\mathbb{Q}^{c}, \mathbb{Y}, A^{c}, c^{c}, b^{c})$ for which $\mathcal{C}(c^{c}, A^{c})$ has full row rank. We now give a procedure to determine an equivalent quasi-model $(\mathbb{Q}^{co}, \mathbb{Y}, A^{co}, c^{co}, b^{co})$ for which $\mathcal{C}(A^{co}, b^{co})$ and $\mathcal{O}(c^{co}, A^{co})$ have full row and full column rank respectively. For this second step, suppose that it holds for $(\mathbb{Q}^{c}, \mathbb{Y}, A^{c}, c^{c}, b^{c})$ that

$$\operatorname{rank} \mathcal{O}(c^{c}, A^{c}) = s < |\mathbb{Q}^{c}| = r,$$

then it can be shown that there exists a nonsingular matrix S such that the equivalent Mealy model $(\mathbb{Q}^{c}, \mathbb{Y}, A^{c\prime} = SA^{c}S^{-1}, c^{c\prime} = c^{c}S^{-1}, b^{c\prime} = Sb)$ has the form

$$A^{c\prime}(\mathrm{y}) = \begin{array}{c} {\scriptstyle r-s \quad\; s} \\ \left[\begin{array}{c|c} * & * \\ \hline 0 & A^{co}(\mathrm{y}) \end{array}\right] \end{array} \begin{array}{l} {\scriptstyle r-s} \\ {\scriptstyle s} \end{array}, \quad \forall \mathrm{y} \in \mathbb{Y},$$

$$b^{c\prime} = \left[\begin{array}{c} * \\ \hline b^{co} \end{array}\right] \begin{array}{l} {\scriptstyle r-s} \\ {\scriptstyle s} \end{array}, \qquad c^{c\prime} = \begin{array}{c} {\scriptstyle r-s \quad s} \\ \left[\begin{array}{c|c} 0 & c^{co} \end{array}\right] \end{array}.$$

This realization has the property that the sth order subsystem $(\mathbb{Q}^{co}, \mathbb{Y}, A^{co}, c^{co}, b^{co})$ is equivalent with the system $(\mathbb{Q}, \mathbb{Y}, A, c, b)$ and that $\mathcal{C}(A^{co}, b^{co})$ has full row rank and that $\mathcal{O}(c^{co}, A^{co})$ has full column rank, i.e. the subsystem $(\mathbb{Q}^{co}, \mathbb{Y}, A^{co}, c^{co}, b^{co})$ is minimal. The procedure to find the transformation S is dual to the procedure to find the transformation R as described before.

By combining both the steps, we have that for every nonminimal Mealy model $(\mathbb{Q}, \mathbb{Y}, A, c, b)$ there exists a transformation T such that

$$TA(\mathrm{y})T^{-1} = \begin{array}{c} {\scriptstyle |\mathbb{Q}|-r \quad r-s \qquad s} \\ \left[\begin{array}{c|c|c} * & 0 & 0 \\ \hline * & * & * \\ \hline * & 0 & A^{co}(\mathrm{y}) \end{array}\right] \end{array} \begin{array}{l} {\scriptstyle |\mathbb{Q}|-r} \\ {\scriptstyle r-s} \\ {\scriptstyle s} \end{array}, \quad \forall \mathrm{y} \in \mathbb{Y},$$
$$Tb = \left[\begin{array}{c} 0 \\ \hline * \\ \hline b^{co} \end{array}\right] \begin{array}{l} {\scriptstyle |\mathbb{Q}|-r} \\ {\scriptstyle r-s} \\ {\scriptstyle s} \end{array}, \qquad cT^{-1} = \begin{array}{c} {\scriptstyle |\mathbb{Q}|-r \quad r-s \quad s} \\ \left[\begin{array}{c|c|c} * & 0 & c^{co} \end{array}\right] \end{array}. \tag{3}$$

The transformation T can be computed from R and S by

$$T = \begin{bmatrix} I & 0 \\ 0 & S \end{bmatrix} R.$$

3. Equivalence for Mealy HMMs

In this section we investigate the set of equivalent Mealy HMMs. Suppose we are given a minimal Mealy HMM, how is the set of all equivalent Mealy HMMs characterized? Of course, given a certain quasi- or positive Mealy model, one can always obtain an equivalent model by permuting the states. However, there are many more equivalent models than the ones obtained by permuting states. Below we give a description of the complete set of equivalent models both for the quasi-Mealy case as for the positive Mealy case.

3.1. Equivalence for quasi-Mealy HMMs

For quasi-Mealy HMMs the equivalence of realizations is described by the following proposition [13].

Proposition 1. *Given a minimal quasi-Mealy model* $(\mathbb{Q}, \mathbb{Y}, A, c, b)$*. The quasi-Mealy model* $(\mathbb{Q}, \mathbb{Y}, A', c', b')$ *is an equivalent model, if and only if there exists a nonsingular matrix* T*, such that*

$$\forall \mathrm{y} \in \mathbb{Y} : A'(\mathrm{y}) = TA(\mathrm{y})T^{-1},$$
$$c' = cT^{-1}, \tag{4}$$
$$b' = Tb.$$

Define a *vectorized quasi-Mealy HMM* as a vector $\sigma \in \mathbb{R}^n$, where $n = |\mathbb{Y}| \cdot |\mathbb{X}|^2 + 2 \cdot |\mathbb{X}|$

$$\sigma := [\mathrm{vec}(A(\mathrm{y}^{(1)})), \ldots, \mathrm{vec}(A(\mathrm{y}^{(|\mathbb{Y}|)})), c, b^{\top}]^{\top}.$$

Now, the following theorem can be proven as a consequence of Proposition 1.

Theorem 1. *The set of (vectorized) quasi-Mealy models equivalent to a given quasi-Mealy model* $(\mathbb{Q}, \mathbb{Y}, A, c, b)$*, is a semialgebraic set in* $\mathbb{R}^n$*, where* $n = |\mathbb{Y}| \cdot |\mathbb{X}|^2 + 2 \cdot |\mathbb{X}|$.

Proof. The nonsingularity constraint on the matrix T in Proposition 1, can be written as

$$\det(T) \neq 0.$$

From Proposition 1, the set of (vectorized) quasi-Mealy models is a semialgebraic set as it is a projection of a semialgebraic set (see Appendix). ■

It also follows from the Appendix, that the semialgebraic set of (vectorized) quasi-Mealy models equivalent to a given quasi-Mealy model can be constructed, i.e. the quantifier in (4) can be eliminated.

It also follows from Proposition 1 that a quasi-Mealy model $(\mathbb{Q}, \{y^{(1)}, y^{(2)}\}, A, c, b)$ with two outputs, can be converted into an equivalent quasi-Moore model with the same number of states $(\mathbb{Q}, \{y^{(1)}, y^{(2)}\}, A_{\mathbb{Q}}, \lambda, c)$, under the condition that $A(y^{(1)})$ and $A(y^{(2)})$ have full rank and $A(y^{(1)})A(y^{(2)})^{-1}$ has real eigenvalues. Indeed, for the quasi-Moore model to exist, there need to exist a nonsingular matrix $T \in \mathbb{R}^{|\mathbb{Q}|\times|\mathbb{Q}|}$, a matrix $A_{\mathbb{Q}} \in \mathbb{R}^{|\mathbb{Q}|\times|\mathbb{Q}|}$ and two vectors $\lambda(y^{(1)})$ and $\lambda(y^{(2)})$ in $\mathbb{R}^{|\mathbb{Q}|}$ such that

$$\begin{aligned} &Tb = e,\\ &TA(y^{(1)})T^{-1} = \operatorname{diag}(\lambda(y^{(1)}))A_{\mathbb{Q}},\\ &TA(y^{(2)})T^{-1} = \operatorname{diag}(\lambda(y^{(2)}))A_{\mathbb{Q}}. \end{aligned}$$

Such matrices exist if a real diagonal matrix D exists such that $TA(y^{(1)})T^{-1} = DTA(y^{(2)})T^{-1}$ or that $A(y^{(1)})A(y^{(2)})^{-1} = T^{-1}DT$. The matrix D follows from an eigenvalue decomposition of $A(y^{(1)})A(y^{(2)})^{-1}$, because $A(y^{(1)})A(y^{(2)})^{-1}$ has real eigenvalues. Moreover, because eigenvectors are determined up to a constant, T can always be chosen such that $Tb = e$. For Mealy HMMs with more than two outputs the above result does not hold. In general, a minimal quasi-Moore model equivalent to a minimal quasi-Mealy model, has more states than the Mealy model.

3.2. Equivalence for positive Mealy HMMs

We now describe the equivalence sets for positive Mealy models. We first deal with a special situation where the Mealy model is minimal as a quasi-Mealy model, where we say that a model is *minimal as a quasi-Mealy model*, if there does not exist any equivalent quasi-Mealy model of lower order. Next we consider the most general case, where the order of the minimal Mealy model is larger than or equal to the minimal quasi-Mealy order.

For the situation where the Mealy model is minimal as a quasi-Mealy model, we prove the following proposition.

Proposition 2. *Given a minimal Mealy model* $(\mathbb{X}, \mathbb{Y}, \Pi, \pi(1))$ *which is minimal as a quasi-Mealy model. The Mealy model* $(\mathbb{X}, \mathbb{Y}, \Pi', \pi'(1))$ *is equivalent if and only if there exists a nonsingular matrix* $T \in \mathbb{R}^{|\mathbb{X}|\times|\mathbb{X}|}$ *such that*

$$\begin{aligned} &\forall y \in \mathbb{Y} : \Pi'(y) = T\Pi(y)T^{-1},\\ &\pi'(1) = \pi(1)T^{-1},\\ &Te = e,\\ &\forall y \in \mathbb{Y} : \Pi'(y) \geq 0,\\ &\pi'(1) \geq 0. \end{aligned} \qquad (5)$$

Proof. The proposition basically follows from Proposition 1. For the 'if'-case, it remains to be proven that $(\mathbb{X}, \mathbb{Y}, \Pi', \pi'(1))$ fullfilling (5) satisfies the consistency conditions of Mealy models. To see this, first note that $\pi'(1)e = \pi(1)T^{-1}e = \pi(1)e = 1$. Next, from

$$\sum_{y\in\mathbb{Y}} \Pi'(y)e = \sum_{y\in\mathbb{Y}} T\Pi(y)T^{-1}e = T\left(\sum_{y\in\mathbb{Y}} \Pi(y)\right)e = Te = e$$

it follows that $\sum_{y\in\mathbb{Y}} \Pi'(y)$ is a stochastic matrix. ■

In the same way as in the proof of the proposition, it can be proven that if $(\mathbb{X}, \mathbb{Y}, \Pi, \pi(1))$ is stationary (i.e. $\pi(1)\sum_{y\in\mathbb{Y}} \Pi(y) = \pi(1)$), then $(\mathbb{X}, \mathbb{Y}, \Pi', \pi'(1))$ is also stationary. By defining a *vectorized Mealy HMM* as a vector $\sigma \in \mathbb{R}^n$, where $n = |\mathbb{Y}| \cdot |\mathbb{X}|^2 + |\mathbb{X}|$

$$\sigma := [\operatorname{vec}(\Pi(y^{(1)})), \ldots, \operatorname{vec}(\Pi(y^{(|\mathbb{Y}|)})), \pi(1)]^{\top},$$

we can prove the following theorem which is a consequence of Proposition 2.

Theorem 2. *The set of (vectorized) Mealy models equivalent to a given Mealy model* $(\mathbb{X}, \mathbb{Y}, \Pi, \pi(1))$ *which is minimal as a quasi-Mealy model, is a semialgebraic set in* $\mathbb{R}^n$*, where* $n = |\mathbb{Y}| \cdot |\mathbb{X}|^2 + |\mathbb{X}|$.

Proof. The proof is analogous to the proof of Theorem 1. ■

For the most general situation, where the order of the minimal Mealy model is larger than or equal to the minimal quasi-Mealy order, we prove the following proposition.

Proposition 3. *Given a minimal Mealy model* $(\mathbb{X}, \mathbb{Y}, \Pi, \pi(1))$. *The Mealy model* $(\mathbb{X}, \mathbb{Y}, \Pi', \pi'(1))$ *is an equivalent model if and only if there exist positive scalars* r, r' *and* s, *nonsingular matrices* T *and* $T' \in \mathbb{R}^{|\mathbb{X}|\times|\mathbb{X}|}$, *and a minimal quasi-Mealy model* $(\{1, \ldots, s\}, \mathbb{Y}, A^{co}, c^{co}, b^{co})$ *such that*

$$\begin{aligned} &\forall y \in \mathbb{Y} : T\Pi(y)T^{-1} = \begin{array}{c} \begin{array}{ccc} \scriptstyle |\mathbb{X}|-r & \scriptstyle r-s & \scriptstyle s \end{array} \\ \left[\begin{array}{c|c|c} * & 0 & 0 \\ \hline * & * & * \\ \hline * & 0 & A^{co}(y) \end{array}\right] \end{array} \begin{array}{l} \scriptstyle |\mathbb{X}|-r \\ \scriptstyle r-s \\ \scriptstyle s \end{array},\\ &\forall y \in \mathbb{Y} : T'\Pi'(y)T'^{-1}\\ &= \begin{array}{c} \begin{array}{ccc} \scriptstyle |\mathbb{X}|-r' & \scriptstyle r'-s & \scriptstyle s \end{array} \\ \left[\begin{array}{c|c|c} A^{(1,1)}(y) & 0 & 0 \\ \hline A^{(2,1)}(y) & A^{(2,2)}(y) & A^{(2,3)}(y) \\ \hline A^{(3,1)}(y) & 0 & A^{co}(y) \end{array}\right] \end{array} \begin{array}{l} \scriptstyle |\mathbb{X}|-r' \\ \scriptstyle r'-s \\ \scriptstyle s \end{array}, \end{aligned} \qquad (6)$$

$$\pi(1)T^{-1} = \Big[\ \underbrace{*}_{|\mathbb{Q}|-r}\ \Big|\ \underbrace{0}_{r-s}\ \Big|\ \underbrace{c^{co}(1)}_{s}\ \Big], \qquad Te = \begin{bmatrix} 0 \\ \hline * \\ \hline b^{co} \end{bmatrix} \begin{matrix} {\scriptstyle |\mathbb{Q}|-r} \\ {\scriptstyle r-s} \\ {\scriptstyle s} \end{matrix},$$

$$\pi'(1)T'^{-1} = \Big[\ \underbrace{c^{(1)}}_{|\mathbb{Q}|-r'}\ \Big|\ \underbrace{0}_{r'-s}\ \Big|\ \underbrace{c^{co}(1)}_{s}\ \Big], \qquad T'e = \begin{bmatrix} 0 \\ \hline b^{(2)} \\ \hline b^{co} \end{bmatrix} \begin{matrix} {\scriptstyle |\mathbb{Q}|-r'} \\ {\scriptstyle r'-s} \\ {\scriptstyle s} \end{matrix},$$

$$\forall \mathrm{y} \in \mathbb{Y} : \Pi'(\mathrm{y}) \geq 0,$$
$$\pi'(1) \geq 0,$$
$$\sum_{\mathrm{y}} \begin{bmatrix} A^{(2,2)}(\mathrm{y}) & A^{(2,3)}(\mathrm{y}) \end{bmatrix} \begin{bmatrix} b^{(2)} \\ b^{co} \end{bmatrix} = b^{(2)}.$$

Proof. The proof follows basically from the procedure for obtaining a minimal quasi-Mealy model from a positive Mealy model described in Section 2.4. For the 'if'-case, it remains to be proven that $(\mathbb{X}, \mathbb{Y}, \Pi', \pi'(1))$ fullfilling (6) satisfies the consistency conditions of Mealy models. To prove that $\pi'(1)$ has element sum equal to 1, one can see that $\pi'(1)e = c^{co}b^{co} = \pi(1)e = 1$. Next, $\sum_{\mathrm{y}} \Pi'(\mathrm{y})e = e$ if and only if

$$\sum_{\mathrm{y}} \begin{bmatrix} A^{(1,1)}(\mathrm{y}) & 0 & 0 \\ A^{(2,1)}(\mathrm{y}) & A^{(2,2)}(\mathrm{y}) & A^{(2,3)}(\mathrm{y}) \\ A^{(3,1)}(\mathrm{y}) & 0 & A^{co}(\mathrm{y}) \end{bmatrix} \begin{bmatrix} 0 \\ b^{(2)} \\ b^{co} \end{bmatrix} = \begin{bmatrix} 0 \\ b^{(2)} \\ b^{co} \end{bmatrix}. \quad (7)$$

This condition is true if and only if $\sum_{\mathrm{y}} A(\mathrm{y})^{co} b^{co} = b^{co}$, which is true because $\sum_{\mathrm{y}} \Pi(\mathrm{y})e = e$. ■

If the original model is stationary (i.e. $\pi(1)\sum_{\mathrm{y}\in\mathbb{Y}} \Pi(\mathrm{y}) = \pi(1)$), then an equivalent model is not necessarily stationary. By adding the condition

$$\begin{bmatrix} c^{(1)} & c^{co} \end{bmatrix} \sum_{\mathrm{y}} \begin{bmatrix} A^{(1,1)}(\mathrm{y}) \\ A^{(3,1)}(\mathrm{y}) \end{bmatrix} = c^{(1)} \quad (8)$$

to (6), a stationary model will give rise to an equivalent stationary model.

To check in practice whether two Mealy models are equivalent, one can use Proposition 3 as follows. First find for both Mealy models, an equivalent minimal quasi-Mealy model using the procedure of Section 2.4. Next check whether there exists a transformation that transforms the first quasi-model into the second quasi-model. If such a transformation exists, the positive models are equivalent, otherwise, they are not equivalent.

As a consequence of Proposition 3, we now have the following theorem.

Theorem 3. *The set of (vectorized) Mealy models equivalent to a given minimal Mealy model* $(\mathbb{X}, \mathbb{Y}, \Pi, \pi(1))$, *is a semialgebraic set in* $\mathbb{R}^n_+$, *where* $n = |\mathbb{Y}| \cdot |\mathbb{X}|^2 + |\mathbb{X}|$.

Proof. The proof is analogous to the proof of Theorem 1. ■

4. Equivalence for Moore HMMs

In this section we investigate the set of equivalent Moore HMMs. As is the case with Mealy models, one can always obtain a model equivalent to a given quasi- or positive Moore model by permuting the states of the original model. However, much more equivalent models are possible. We investigate subsequently the quasi-Moore and the positive Moore case.

4.1. Equivalence for quasi-Moore HMMs

We first deal with a special, though important, class of quasi-Moore models: quasi-Moore models which are minimal as a quasi-Mealy model.

Given a quasi-Moore model that is minimal as a quasi-Mealy model, then we show that under certain conditions every equivalent quasi-Moore model corresponds to a permutation of the states of the given model.

Theorem 4. *Let* $(\mathbb{Q}, \mathbb{Y}, A_{\mathbb{Q}}, L, c)$ *be a quasi-Moore HMM, which is minimal as a quasi-Mealy model. If all the states of the quasi-Moore model have a different output distribution (i.e. no two rows of* L *are equal to each other) and if the state transition matrix* $A_{\mathbb{Q}}$ *has full rank, then every minimal quasi-Moore model that is equivalent to the given quasi-Moore model is obtained by permuting the states of the original model.*

Proof. Suppose that $(\mathbb{Q}, \mathbb{Y}, A'_{\mathbb{Q}}, L', c')$ is equivalent to and of the same order as $(\mathbb{Q}, \mathbb{Y}, A_{\mathbb{Q}}, L, c)$. Then from Theorem 1, there exists a nonsingular matrix T such that

$$\forall \mathrm{y} \in \mathbb{Y} : \mathrm{diag}(\lambda'(\mathrm{y}))A'_{\mathbb{Q}} = T\,\mathrm{diag}(\lambda(\mathrm{y}))A_{\mathbb{Q}}T^{-1}, \quad (9)$$
$$c' = cT^{-1},$$
$$e = Te. \quad (10)$$

Since $A_{\mathbb{Q}}$ has full rank, it follows that $A'_{\mathbb{Q}}$ has full rank, so it follows from (9) that there exist nonsingular matrices T and S such that

$$\forall \mathrm{y} \in \mathbb{Y} : \mathrm{diag}(\lambda'(\mathrm{y})) = T\,\mathrm{diag}(\lambda(\mathrm{y}))S^{-1},$$
$$A'_{\mathbb{Q}} = SA_{\mathbb{Q}}T^{-1}. \quad (11)$$

For the model $(\mathbb{Q}, \mathbb{Y}, A'_{\mathbb{Q}}, L', c')$, it must hold that $\sum_{\mathrm{y}\in\mathbb{Y}} T\,\mathrm{diag}(\lambda(\mathrm{y}))S^{-1} = I$, which gives $T = S$. It follows that

$$\forall \mathrm{y} \in \mathbb{Y} : \mathrm{diag}(\lambda'(\mathrm{y})) = T\,\mathrm{diag}(\lambda(\mathrm{y}))T^{-1}.$$

Together with $Te = e$ and with the fact that all states of the Moore model have a different output distribution, this allows us to conclude that T can only be equal to a permutation matrix. ■

If in a quasi-Moore HMM, there exist states with the same output distribution, and if all the other conditions of Theorem 4 are fulfilled, then there exists a set of equivalent Moore models (apart from the models obtained by permuting the states). Suppose, for instance, that the output distribution of the first state equals the output distribution of the second state. In that case the transformation T of Eq. (9) is of the form

$$T = P\begin{bmatrix} T' & 0 \\ 0 & I \end{bmatrix}, \quad (12)$$

where P is a permutation matrix and $T' \in \mathbb{R}^{2\times 2}$ a nonsingular matrix with $T'e = e$. This representation gives a complete description of the equivalence set. Notice that all the elements of the set of equivalent Moore models have the same output matrix L (up to a permutation of the states). They only differ in the state transition matrix $A_{\mathbb{Q}}$.

From Section 3, we know that, under some general conditions, a quasi-Mealy model with two outputs can be converted into an equivalent quasi-Moore model with the same number of states. So, under general conditions, every minimal quasi-Moore model with two outputs is minimal as a quasi-Mealy HMM. From Theorem 4 we conclude that every minimal quasi-Moore model with two outputs, with a full rank transition matrix and with a different output distribution for each state, has no equivalent minimal quasi-Moore representations except trivial ones. However, the theorem is also useful for HMMs with more than two outputs. Consider for example the Moore model $(\mathbb{X}, \mathbb{Y}, \Pi_{\mathbb{X}}, B, \pi(1))$ with

$$\Pi_{\mathbb{X}} = \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.2 & 0.2 & 0.6 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.3 & 0.3 & 0.4 \\ 0.1 & 0.1 & 0.8 \\ 0.2 & 0.6 & 0.2 \end{bmatrix},$$

$$\pi(1) = \begin{bmatrix} 0.5405 & 0.1622 & 0.2973 \end{bmatrix}.$$

It is possible to show that the order of an minimal equivalent quasi-Mealy model equals 3. This allows us to conclude that the Moore model is minimal as a quasi-Mealy model. In addition all the rows of B are different and $\Pi_{\mathbb{X}}$ has full rank, such that we conclude from Theorem 4 that the only way to obtain a minimal Moore equivalent to the given model is by permuting the states.

Now we consider equivalence for general quasi-Moore models of which the order is larger than or equal to the order of an equivalent minimal quasi-Mealy model.

Proposition 4. *Consider the minimal quasi-Moore model* $(\mathbb{Q}, \mathbb{Y}, A_{\mathbb{Q}}, L, c)$. *The quasi-Moore model* $(\mathbb{Q}, \mathbb{Y}, A'_{\mathbb{Q}}, L', c')$ *is an equivalent model if and only if there exist positive scalars* r, r' *and* s, *nonsingular matrices* T *and* $T' \in \mathbb{R}^{|\mathbb{X}|\times|\mathbb{X}|}$, *and a minimal quasi-Mealy model* $(\{1, \ldots, s\}, \mathbb{Y}, A^{co}, c^{co}, b^{co})$ *such that*

$$\forall \mathrm{y} \in \mathbb{Y} : T \operatorname{diag}(\lambda(\mathrm{y})) A_{\mathbb{Q}} T^{-1} = \overset{\quad |\mathbb{X}|-r \;\; r-s \;\; s}{\left[\begin{array}{c|c|c} * & 0 & 0 \\ \hline * & * & * \\ \hline * & 0 & A^{co}(y) \end{array}\right]} \begin{matrix} {\scriptstyle |\mathbb{X}|-r} \\ {\scriptstyle r-s} \\ {\scriptstyle s} \end{matrix},$$

$$\forall \mathrm{y} \in \mathbb{Y} : T' \operatorname{diag}(\lambda'(\mathrm{y})) A'_{\mathbb{Q}} T'^{-1}$$
$$= \overset{\quad |\mathbb{X}|-r' \;\; r'-s \;\; s}{\left[\begin{array}{c|c|c} A^{(1,1)}(y) & 0 & 0 \\ \hline A^{(2,1)}(y) & A^{(2,2)}(y) & A^{(2,3)}(y) \\ \hline A^{(3,1)}(y) & 0 & A^{co}(y) \end{array}\right]} \begin{matrix} {\scriptstyle |\mathbb{X}|-r'} \\ {\scriptstyle r'-s} \\ {\scriptstyle s} \end{matrix},$$

$$cT^{-1} = \overset{|\mathbb{Q}|-r \;\; r-s \;\; s}{\left[\begin{array}{c|c|c} * & 0 & c^{co}(1) \end{array}\right]}, \qquad Te = \left[\begin{array}{c} 0 \\ \hline * \\ \hline b^{co} \end{array}\right] \begin{matrix} {\scriptstyle |\mathbb{Q}|-r} \\ {\scriptstyle r-s} \\ {\scriptstyle s} \end{matrix},$$

$$c'T'^{-1} = \overset{|\mathbb{Q}|-r' \;\; r'-s \;\; s}{\left[\begin{array}{c|c|c} c^{(1)} & 0 & c^{co}(1) \end{array}\right]}, \qquad T'e = \left[\begin{array}{c} 0 \\ \hline b^{(2)} \\ \hline b^{co} \end{array}\right] \begin{matrix} {\scriptstyle |\mathbb{Q}|-r'} \\ {\scriptstyle r'-s} \\ {\scriptstyle s} \end{matrix},$$

$$L'e = e,$$

$$\sum_{\mathrm{y}} \begin{bmatrix} A^{(2,2)}(\mathrm{y}) & A^{(2,3)}(\mathrm{y}) \end{bmatrix} \begin{bmatrix} b^{(2)} \\ b^{co} \end{bmatrix} = b^{(2)}. \tag{13}$$

Proof. The proof is analogous to the proof of Proposition 3. ■

The same remark as in Proposition 3 concerning stationarity holds. An equivalent model to a stationary model is not necessarily stationary. Stationarity can be imposed by adding condition (8) to the set of conditions (13). Define a *vectorized quasi-Moore HMM* as a vector $\sigma \in \mathbb{R}^n$, where $n = |\mathbb{X}|^2 + |\mathbb{X}| \cdot (|\mathbb{Y}| + 1)$

$$\sigma := [\operatorname{vec}(A_{\mathbb{Q}}), \operatorname{vec}(L), c]^\top.$$

The following theorem is a direct consequence of Proposition 4.

Theorem 5. *The set of (vectorized) quasi-Moore models equivalent to a given minimal quasi-Moore model* $(\mathbb{Q}, \mathbb{Y}, A_{\mathbb{Q}}, L, c, b)$ *is a semialgebraic set in* $\mathbb{R}^n$, *where* $n = |\mathbb{X}|^2 + |\mathbb{X}| \cdot (|\mathbb{Y}| + 1)$.

Proof. The proof is analogous to the proof of Theorem 1. ■

4.2. Equivalence for Moore HMMs

We now consider the equivalence of positive Moore models. If the Moore model is minimal as a quasi-Mealy model, then we are in the situation of Theorem 4, and hence there exist only trivial equivalent Moore models. In this section, we consider the general case when the order of the Moore model is larger than or equal to the order of a minimal equivalent quasi-Moore model. We do not consider the situation where the order of the Moore model equals the order of a minimal equivalent quasi-Moore model separately, as this does not give rise to an simplified description of the equivalence in contradiction to the case for Mealy models.

Proposition 5. *Consider a minimal Moore model* $(\mathbb{X}, \mathbb{Y}, \Pi_{\mathbb{X}}, B, \pi(1))$. *The quasi-Moore model* $(\mathbb{X}, \mathbb{Y}, \Pi'_{\mathbb{X}}, B', \pi'(1))$ *is an equivalent model if and only if there exist positive scalars* r, r' *and* s, *nonsingular matrices* T *and* $T' \in \mathbb{R}^{|\mathbb{X}|\times|\mathbb{X}|}$, *and a minimal quasi-Mealy model* $(\{1, \ldots, s\}, \mathbb{Y}, A^{co}, c^{co}, b^{co})$ *such that*

$$\forall \mathrm{y} \in \mathbb{Y} : T \operatorname{diag}(\beta(\mathrm{y})) \Pi_{\mathbb{X}} T^{-1} = \overset{\quad |\mathbb{X}|-r \;\; r-s \;\; s}{\left[\begin{array}{c|c|c} * & 0 & 0 \\ \hline * & * & * \\ \hline * & 0 & A^{co}(y) \end{array}\right]} \begin{matrix} {\scriptstyle |\mathbb{X}|-r} \\ {\scriptstyle r-s} \\ {\scriptstyle s} \end{matrix},$$

$$\forall \mathrm{y} \in \mathbb{Y} : T \operatorname{diag}(\beta'(\mathrm{y})) \Pi'_{\mathbb{X}} T'^{-1}$$
$$= \overset{\quad |\mathbb{X}|-r' \;\; r'-s \;\; s}{\left[\begin{array}{c|c|c} A^{(1,1)}(y) & 0 & 0 \\ \hline A^{(2,1)}(y) & A^{(2,2)}(y) & A^{(2,3)}(y) \\ \hline A^{(3,1)}(y) & 0 & A^{co}(y) \end{array}\right]} \begin{matrix} {\scriptstyle |\mathbb{X}|-r'} \\ {\scriptstyle r'-s} \\ {\scriptstyle s} \end{matrix},$$

$$\pi(1)T^{-1} = \overset{|\mathbb{Q}|-r \;\; r-s \;\; s}{\left[\begin{array}{c|c|c} * & 0 & c^{co}(1) \end{array}\right]}, \qquad Te = \left[\begin{array}{c} 0 \\ \hline * \\ \hline b^{co} \end{array}\right] \begin{matrix} {\scriptstyle |\mathbb{Q}|-r} \\ {\scriptstyle r-s} \\ {\scriptstyle s} \end{matrix},$$

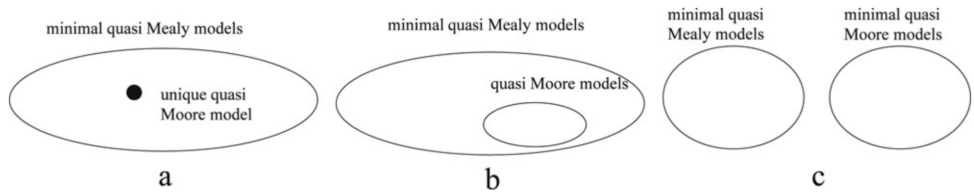

Fig. 1. The three main cases concerning the equivalence classes of quasi-Mealy and Moore models.

$$\pi'(1)T'^{-1} = \left[\ \overset{|\mathbb{Q}|-r'}{c^{(1)}} \ \middle|\ \overset{r'-s}{0} \ \middle|\ \overset{s}{c^{co}(1)}\ \right], \qquad T'e = \begin{bmatrix} 0 \\ \hline b^{(2)} \\ \hline b^{co} \end{bmatrix} \begin{matrix} {\scriptstyle |\mathbb{Q}|-r'} \\ {\scriptstyle r'-s} \\ {\scriptstyle s} \end{matrix},$$

$$\Pi'_{\mathbb{X}},\ B',\ \pi'(1) \geq 0,$$
$$B'e = e,$$
$$\sum_{\mathrm{y}} \left[A^{(2,2)}(\mathrm{y}) \quad A^{(2,3)}(\mathrm{y}) \right] \begin{bmatrix} b^{(2)} \\ b^{co} \end{bmatrix} = b^{(2)}. \qquad (14)$$

Proof. The proof is analogous to the proof of Proposition 2. ■

The same remark concerning stationarity holds as for Proposition 3. Defining a *vectorized Moore HMM* as a vector $\sigma \in \mathbb{R}^n$, where $n = |\mathbb{X}|^2 + |\mathbb{X}| \cdot (|\mathbb{Y}| + 1)$

$$\sigma := [\mathrm{vec}(\Pi_{\mathbb{X}}), \mathrm{vec}(B), \pi(1)]^\top,$$

leads to the following theorem, which is an immediate consequence of Proposition 5.

Theorem 6. *The set of (vectorized) Moore models equivalent to a given minimal Moore model* $(\mathbb{X}, \mathbb{Y}, \Pi_{\mathbb{X}}, B, \pi(1))$*, is a semialgebraic set in* $\mathbb{R}^n$*, where* $n = |\mathbb{X}|^2 + |\mathbb{X}| \cdot (|\mathbb{Y}| + 1)$.

Proof. The proof is analogous to the proof of Theorem 2. ■

5. Summary of the results concerning equivalence sets

We now summarize the results concerning the equivalence sets for quasi- and positive HMMs of Moore and Mealy type.

In Fig. 1, we consider quasi-Markov models. First of all, as described in Theorem 1 there exists a set of equivalent quasi-Mealy models. The order of a minimal equivalent quasi-Moore model is either equal to the order of the minimal quasi-Mealy model (Fig. 1(a),(b)), or larger than the order of the minimal quasi-Mealy model (Fig. 1(c)). If in the first case, every state of the Moore model has a different output distribution and the transition matrix has full rank, then the Moore model is unique by Theorem 4 (Fig. 1(a)). Otherwise there exists a set of equivalent Moore models described by a transformation of the form (12) (Fig. 1(b)). If the order of a minimal equivalent quasi-Moore models is larger than the order of the quasi-Mealy model, then there exists a class of quasi-Moore models described by Proposition 4 and Theorem 5.

In Fig. 2, we consider Mealy models. From Theorem 1 it follows that there exists a set of equivalent quasi-Mealy models. The order of an equivalent minimal positive Mealy model is either equal to the order of the minimal quasi-Mealy model (Fig. 2(a)), or larger than the order of the minimal quasi-Mealy model (Fig. 3(b)). In the first case the set of equivalent positive Mealy models is described by Proposition 2 and Theorem 2, while in the second case the set is described by Proposition 3 and Theorem 3.

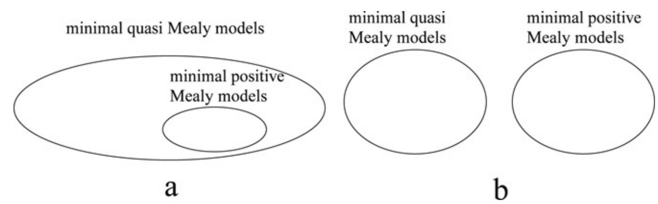

Fig. 2. The two main cases concerning the equivalence classes of quasi- and positive Mealy models.

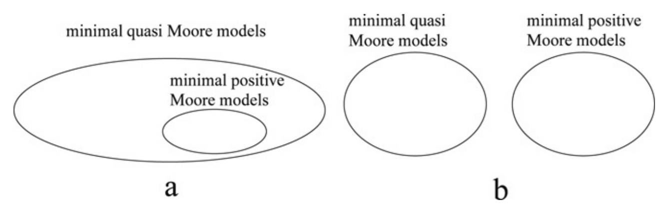

Fig. 3. The two main cases concerning the equivalence classes of quasi- and positive Moore models.

In Fig. 3, we consider Moore models. In general, minimal quasi-Moore models form a set described by a transformation of the form (12), or by Proposition 4 and Theorem 5. Under certain conditions this set has only one element (Theorem 4). The order of an equivalent minimal positive Moore model is either equal to the order of the minimal quasi-Moore model (Fig. 3(a)), or larger than the order of the minimal quasi-Moore model (Fig. 3(b)). In both the cases the set is described by Proposition 5 and Theorem 6.

6. Comparison with linear Gaussian case

We now compare the results for HMMs with stationary Gauss–Markov models. It turns out that the situation for stationary Gauss–Markov systems is very analogous to the situation for quasi-HMMs (Fig. 1).

The state and output process for Gauss–Markov processes take values in a finite-dimensional real vector space, in contrast to HMMs, where the output and state process take values in finite sets. In addition, for Gauss–Markov models the state and output process are typically vector processes, which is not the case for HMMs. Gauss–Markov systems are given by

$$x(t+1) = Ax(t) + w(t),$$

$y(t) = Cx(t) + v(t)$,

where $x(t) \in \mathbb{R}^n$ with n the *order* of the Gauss–Markov model, $y(t) \in \mathbb{R}^p$ and where A is Schur. Next,

$$E\left(\begin{bmatrix} w(p) \\ v(p) \end{bmatrix} \begin{bmatrix} w(q)^\top & v(q)^\top \end{bmatrix}\right) = \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \delta_{pq},$$

where $E(X)$ denotes the expected value of X, and where δ_{pq} is the Kronecker delta, i.e. $\delta_{pq} = 1$ if and only if $p = q$, $\delta_{pq} = 0$ otherwise. As a shorthand notation for a Gauss–Markov system, we use (A, C, Q, R, S).

First notice that this is the analogon of a Mealy model since the generation of the next state is dependent on the output given the present state. A Moore model corresponds to the case $S = 0$. For Gauss–Markov models we do not make a distinction between positive and quasi-models.

Let $\Lambda : \mathbb{Z}_+ \mapsto \mathbb{R}^{p\times p}$ be the *autocovariance* of y, defined as $\Lambda(\tau) = E(y(t+\tau)y(t)^\top)$. By defining $P = E(x(t)x(t)^\top)$ and $G = E(x(t+1)y(t)^\top)$, the autocovariance *generated* by the Gauss–Markov model (A, C, Q, R, S) can be calculated as

$$\begin{aligned} P &= APA^\top + Q, \\ \Lambda(0) &= CPC^\top + R, \\ G &= APC^\top + S, \\ \Lambda(t) &= CA^{t-1}G. \end{aligned}$$

We say that two Gauss–Markov models with autocovariances Λ and Λ' are *equivalent* if $\Lambda = \Lambda'$. A Gauss–Markov model (A, C, Q, R, S) of order n is *minimal* if for any other equivalent model (A', C', Q', R', S') of order n', it holds that $n \leq n'$.

Now one can easily see that for Mealy as well as for Moore models, an equivalent model is obtained by changing the basis in the state space as $x \mapsto Tx$, with T nonsingular. The equivalent model is then given by $(TAT^{-1}, CT^{-1}, TQT^\top, R, TS)$. This state transformation is the analogon of the permutation of the states which is always possible for quasi-HMMs.

However, again analogous to the quasi-hidden-Markov case, for Mealy Gauss–Markov systems, there are more equivalent models than those obtained by permuting the states. Indeed, it can be proven [5] that for a given A, C, G and $\Lambda(0)$, i.e. for a given autocovariance sequence and given state space basis, every $P \succeq 0$ which fullfills

$$\left[\begin{array}{c|c} P - APA^\top & G - APC^\top \\ \hline G^\top - CPA^\top & \Lambda(0) - CPC^\top \end{array}\right] \succeq 0,$$

where $X \succeq 0$ means that X is nonnegative definite, gives rise to an equivalent model $(A, C, P - APA^\top, \Lambda(0) - CPC^\top, G - APC^\top)$. This observation is the analogon of the fact that for quasi-Mealy HMMs one has many equivalent models which are not obtained by permuting states.

For quasi-Moore HMMs on the other hand, under certain conditions, there exists only one equivalent model to a given model (Theorem 4). We prove here the analogous theorem for Gauss–Markov models.

Theorem 7. *For a minimal Moore Gauss–Markov model $(A, C, Q, R, 0)$ the following holds: if C has full column rank and A full rank, then there does not exist any minimal Moore model which is equivalent to the given Moore model and which cannot be formed from the given model by performing a change of basis in the state space.*

Proof. From the fact that $S = 0$ we find that $G - APC^\top = 0$, and from the fact that C has full column rank and A full rank, we find that $P = A^{-1}G(C^\top)^\dagger$, where $X^\dagger$ denotes the Moore–Penrose pseudo-inverse of X. So for a given state space basis there is only one possible choice of P, which proves the theorem. ∎

The condition that C has full column rank (condition of Theorem 7) is equivalent to the condition that a different state at two time instants gives a different output distribution at these time instants. This corresponds to the condition for HMMs of Theorem 4 which requires every state to have a different output distribution. The fact that A has full rank is the analogon of the fact that for HMMs $\Pi_{\mathbb{X}}$ has full rank. We conclude that Theorem 7 is the Gauss–Markov equivalent of Theorem 4.

7. Conclusions

In this paper we considered the following problem for HMMs: given a minimal HMM, describe the set of all equivalent HMMs of the same order. For quasi Mealy HMMs, necessary and sufficient conditions for two models to be equivalent are already proven in literature. We give necessary and sufficient conditions for two positive Mealy models to be equivalent as well as a description of the complete set of all equivalent Mealy models. We also prove that, under certain conditions, the set of minimal quasi Moore models equivalent to a given minimal quasi Moore model consists of only one element (up to a permutation of the states). Next, we derive necessary and sufficient conditions for two positive Moore models to be equivalent as well as a description of the complete set of all equivalent Moore models. Finally, we give a comparison with the situation for Gauss–Markov systems showing that the equivalence sets for hidden Markov models are analogous to the equivalence sets for Gauss–Markov models.

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Appendix. Semialgebraic sets — Tarski–Seidenberg quantifier elimination

In this appendix, we summarize the principle of Tarski–Seidenberg and the relation to semialgebraic sets. This summary is based on [3,4].

A *semialgebraic subset* of $\mathbb{R}^n$ is the subset of $[x_1, \ldots, x_n]^\top$ in $\mathbb{R}^n$ satisfying a boolean combination of polynomial equations and inequalities with real coefficients. In other words, the semialgebraic subsets of $\mathbb{R}^n$ form the smallest class $\mathcal{SA}_n$ of subsets of $\mathbb{R}^n$ such that:

- If g is a polynomial in n variables, then $\{x \in \mathbb{R}^n : g(x) = 0\} \in \mathcal{SA}_n$ and $\{x \in \mathbb{R}^n : g(x) > 0\} \in \mathcal{SA}_n$.
- If $A \in \mathcal{SA}_n$ and $B \in \mathcal{SA}_n$, then $A \cup B$, $A \cap B$ and $A \setminus B$ are in $\mathcal{SA}_n$.

As a consequence of the Tarski–Seidenberg principle [9,10], the class of semialgebraic sets is closed under projection.

Theorem 8. *Let A be a semialgebraic subset of $\mathbb{R}^n$ and $P : \mathbb{R}^n \mapsto \mathbb{R}^p$, the projection on the first p coordinates. Then $P(A)$ is a semialgebraic subset of $\mathbb{R}^p$.*

Now consider a first-order formula over the reals having the form

$$(Q_1 x^{(1)} \in \mathbb{R}^{n_1}) \ldots (Q_l x^{(l)} \in \mathbb{R}^{n_l}) P(y, x^{(1)}, \ldots, x^{(l)}), \qquad (15)$$

where Q_λ, $\lambda = 1, \ldots, l$ is a quantifier: either $\exists$ ("there exists") or $\forall$ ("for all"), where $y = [y_1, \ldots, y_{n_0}]^\top$ are free variables and where $P(y, x^{(1)}, \ldots, x^{(l)})$ is a quantifier-free Boolean formula, i.e. a combination of atomic predicates. The atomic predicates are supposed to be of the form $g_\kappa(y, x^{(1)}, \ldots, x^{(l)}) \Delta_\kappa 0$, $\kappa = 1, \ldots, k$, where $g_\kappa : \prod_{\lambda=0}^{l} \mathbb{R}^{n_\lambda} \mapsto \mathbb{R}$ is a polynomial of degree at most $d \geq 2$ and Δ_i is one of the following relations $\geq, >, =, \neq, \leq$ and $<$. $P(y, x^{(1)}, \ldots, x^{(l)})$ is determined by a Boolean function $\mathbb{P} : \{0, 1\}^k \mapsto \{0, 1\}$ and a function $B : \prod_{\lambda=0}^{l} \mathbb{R}^{n_\lambda} \mapsto \{0, 1\}^k$, where $P := \mathbb{P} \circ B$, and for $\kappa = 1, \ldots, k$

$$B(y, x^{(1)}, \ldots, x^{(l)})_\kappa = \begin{cases} 1 & \text{if } g_\kappa(y, x^{(1)}, \ldots, x^{(l)}) \Delta_\kappa 0, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that the solution set in $\mathbb{R}^{n_0}$ of (15) is a semialgebraic set as it is the projection of a semialgebraic set in $\prod_{\lambda=0}^{l} \mathbb{R}^{n_\lambda}$. It can now be shown [7,9,10], that the semialgebraic set can be *constructed*, i.e. the first-order formula (15) can be written in an equivalent form without quantifiers. The operations that are needed to eliminate the quantifiers are restricted to additions, subtractions, multiplications, divisions, comparisons and the evaluation of Boolean functions. In [7], an algorithm for quantifier elimination is described that requires at most $(kd)^{2^{O(l)} \prod_\lambda n_\lambda}$ multiplications and additions, and at most $(kd)^{O(\sum_\lambda n_\lambda)}$ calls to $\mathbb{P}$. The method requires no divisions. The quantifier elimination algorithm constructs a quantifier-free formula of the following form

$$\bigvee_{\mu=1}^{m} \bigwedge_{\nu=1}^{n_\mu} h_{\mu\nu}(y) \Delta_{\mu\nu} 0,$$

where $m \leq (kd)^{2^{O(l)} \prod_\lambda n_\lambda}$, where $n_\mu \leq (kd)^{2^{O(l)} \prod_\lambda n_\lambda}$, for $\mu = 1, \ldots m$, where the degree of each of the polynomials $h_{\mu\nu}$ is at most $(kd)^{2^{O(l)} \prod_\lambda n_\lambda}$ and where each $\Delta_{\mu\nu}$ is one of the following relations $\geq, >, =, \neq, \leq$ and $<$.

References

[1] B.D.O. Anderson, The realization problem for hidden Markov models, Mathematics of Control, Signals, and Systems 12 (1) (1999) 80–120.
[2] D. Blackwell, L. Koopmans, On the identifiability problem for functions of finite Markov chains, Annals of Mathematical Statistics 28 (1957) 1011–1015.
[3] J.E. Cohen, U.G. Rothblum, Nonnegative ranks, decompositions and factorizations of nonnegative matrices, Linear Algebra and its Applications 190 (1993) 149–168.
[4] J. Bochnak, M. Coste, M.F. Roy, Real algebraic geometry, second ed. in English, in: Ergebnisse der Mat., vol. 36, Springer, Berlin, Heidelberg, New York, 1998.
[5] P. Faurre, Stochastic realization algorithms, in: R. Mehra, D. Lainiotis (Eds.), System Identification, Advances and Case Studies, Academic Press, 1976.
[6] L. Finesso, A. Grassi, P. Spreij, Approximation of stationary processes by hidden Markov models, manuscript, 2006.
[7] J. Renegar, On the computational complexity and geometry of the first-order theory of reals, parts I, II, III, Journal of Symbolic Logic 13 (1992) 255–352.
[8] H. Rosenbrock, State-space and Multivariable Theory, Thomas Nelson and Sons, London, 1970.
[9] A. Seidenberg, A new decision method for elementary algebra, Annals of Mathematics 60 (1954) 365–374.
[10] A. Tarski, A Decision Method for Elementary Algebra and Geometry, second ed., Univ. of California Press, Berkeley, 1951.
[11] B. Vanluyten, J.C. Willems, B. De Moor, Matrix factorization and stochastic state representations, in: Proc. of the 45th IEEE Conference on Decision and Control, San Diego, California, Dec. 2006, pp. 4188–4193.
[12] B. Vanluyten, J.C. Willems, B. De Moor, Recursive Filtering Using Quasi Realizations, in: Lecture Notes in Control and Information Sciences, vol. 341, Springer, 2006, pp. 366–374.
[13] M. Vidyasagar, A realization theory for hidden Markov models: The complete realization problem, manuscript, 2005.