Cluster Synchronization in Oscillatory Networks

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Synchronous behavior in networks of coupled oscillators are commonly observed phenomena attracting a growing interest in physics, biology, communication and other fields of science and technology. Besides global synchronization, one can also observe splitting of the full network into several clusters of mutually synchronized oscillators. In this paper we study conditions for such cluster partitioning into ensembles for the case of identical chaotic systems. Most attention we pay at the existence and the stability of unique *unconditional* clusters which rise does not depend on the origin of the other clusters. Also *conditional* clusters in arrays of globally non-symmetrically coupled identical chaotic oscillators are investigated. The design problem of organizing clusters into a given configuration is discussed.

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Chaotic synchronization has attracted a growing interest in physics with applications to many areas of science. In the context of coupled chaotic elements, many different types of synchronization have been studied in the past two decades. The most important ones are complete or identical synchronization (CS) [1–3], phase synchronization (PS) [4, 5], lag synchronization (LS) [6] and generalized synchronization (GS) [7, 8]. In this paper we focus on CS in ensembles of non-symmetrically coupled chaotic cells. CS was first discovered and is the simplest form of synchronization in chaotic systems. It consists of a perfect hooking of the chaotic trajectories of two or many *identical* systems even though their initial conditions may be different. In arrays of coupled systems CS can take the forms of *global* (full) and *cluster* (partial) synchronization. In global synchronization all the elements of the ensemble are mutually synchronized. The phenomenon of cluster synchronization is observed when an ensemble of oscillators splits into groups of synchronized elements (for a review see [9, 10]). The analytical and numerical study of the conditions for existence and stability of cluster CS are presented.

INTRODUCTION

One of the major collective coherent behaviors in ensembles of identical and non-identical chaotic elements are global and cluster synchronization. Analytical studies of these synchronization phenomena meet some problems, where currently there are only few results in this direction [11–13]. In this paper we study conditions of cluster synchronization of identical chaotic systems. For complete synchronization we focus on the existence and stability of unique *unconditional* clusters which rise does not depend on the origin of the other clusters.

Let us consider the following problem of N *identical* oscillators, which in the absence of interaction exhibit qualitatively *similar* dynamics. For different initial conditions the oscillators show different dynamical trajectories, while for identical initial conditions the trajectories coincide. On the other hand, when considering interaction between the oscillators, one may achieve alignment of a certain state with all other ones. In this way one has *complete synchronization* for the N identical oscillators.

Furthermore, if n from the N identical oscillators, for initial states that are close to each other, possess coinciding states for all time, these n oscillators are said to form one *cluster* $C(n)$. This may be thought then as one oscillator of n sticked together oscillators.

Let us distinguish now the main types of clusters as follows.

Definition 1. If the existence of a cluster $C(n)$ does not depend on the states of the other $N-n$ oscillators, we call $C(n)$ an unconditional cluster. Otherwise, if the cluster $C(n)$ can only coexist with the other clusters or when it is embedded within a larger one vanishing together with $C(n)$, we call $C(n)$ a conditional cluster.

Definition 2. If the cluster $C(n)$ is embedded within a larger cluster and it can be stable only together with the larger one, we call $C(n)$ a hidden cluster. In the opposite case we call the stable cluster $C(n)$ the real cluster.

INVARIANT MANIFOLDS AND CLUSTER SYNCHRONIZATION

We consider the phenomenon of clustering in an ensemble of nonlocally coupled identical oscillators described by the following system

$$\dot{V} = F(V) + (G \otimes P) f(V), \quad V \in R^{dN}, \tag{1}$$

where $V = (v_1, \dots, v_N)^T$ is the set of variables of the N oscillators forming the array, v_i is the d-dimensional vector of i-th oscillator variables, and $F(v) = (F(v_1), \dots, F(v_N))^T$, $f(v_i) = (f(v_i^1), \dots, f(v_i^d))^T$. Elements of the $d \times d$ matrix P that are equal to 1 determine by which variables the oscillators are coupled. The other part of P is zero. G is an arbitrary non-decomposable $N \times N$ matrix. Recall in the case of zero-row sums, that the system has the synchronization manifold

$$M(N, 1) = \{v =: v_1 = \ldots = v_N\} \tag{2}$$

corresponding to one cluster of N oscillators, whose dynamics is defined by the single system for one node:

$$\dot{v} = F(v), \quad v \in R^d \tag{3}$$

Suppose the set of nodes is subdivided into m subsets:

$$\{1, \dots, N\} = C_1^0 \cup \ldots \cup C_m^0, \tag{4}$$

Here C_j^0 are clusters in the whole system, where the zero upper index refers to non-ordering. To order these subsets we introduce a transformation of variables in (1):

$$V = \Pi X, \tag{5}$$

where Π is a $N \times N$ permutation matrix, such that each subset in (4) obtains the natural numeration. Under (5) the system (1) gets the form:

$$\dot{X} = F(X) + (E \otimes P) f(X), \quad X \in R^{dN}, \tag{6}$$

where $E = \Pi^{-1} G \Pi$. The matrix E may be introduced in the block form:

$$E = \{\varepsilon^{kl}\}, \quad k, l = 1, \dots, m. \tag{7}$$

An arbitrary partition (4) under the permutation (5) obtains the form:

$$\{1, \dots, N\} = C_1 \cup \ldots \cup C_m = \{1, \dots, n_1\} \cup \{1, \dots, n_2\} \cup ... \cup \{1, \dots, n_m\}, \tag{8}$$

where $N = \sum_{k=1}^{m} n_k$. Each block ε^{kl} of the size $n_k \times n_l$ has the entries:

$$\varepsilon^{kl} = \{\varepsilon_{ij}^{kl}\}, \quad i = 1, \dots, n_k, \quad j = 1, \dots, n_l. \tag{9}$$

Introducing the vector X triple subdivision:

$$\begin{aligned} X &= (X^1, \dots, X^m)^T, \\ X^k &= (x_1^k, \dots, x_{n_k}^k)^T, \\ x_i^k &= (x_{i1}^k, \dots, x_{id}^k)^T \end{aligned} \tag{10}$$

we rewrite the system (6) in the block form:

$$\dot{X}^k = F(X^k) + \sum_{l=1}^{m} (\varepsilon^{kl} \otimes P) f(X^k), \quad k = 1, \dots, m. \tag{11}$$

Now let us discuss the invariance conditions of the linear subspace $M(N, m)$ corresponding to the partition (8) defined as a linear manifold in R^{dN}:

$$M(N, m) = \{y^k =: x_1^k = \ldots = x_{n_k}^k, k = 1, \dots, m\}. \tag{12}$$

Definition 3. If the subspace $M(N,m)$ is the invariant manifold of the system (11), $M(N,m)$ is called the cluster partition.

The subspace $M(N,m)$ becomes an invariant manifold of the system (11) if this system is compatible for $X^k|_M = \widetilde{V}^k = (\widetilde{v}^k, \ldots, \widetilde{v}^k)^T$. The latter is true under the condition

$$\sum_{j=1}^{n_l} \varepsilon_{ij}^{kl} = \widetilde{g}_{kl}, \quad k,l = 1,\ldots,m \tag{13}$$

where $\widetilde{g}_{kl}$ are scalar values being the entries of $m \times m$ matrix $\widetilde{G}$. The system (11) on the manifold $M(N,m)$ takes the form

$$\dot{\widetilde{V}} = F(\widetilde{V}) + (\widetilde{G} \otimes P) f(\widetilde{V}) \tag{14}$$

similar to (1) with $\widetilde{V} = (\widetilde{v}^1, \ldots, \widetilde{v}^m)$ and $\widetilde{G} = (\widetilde{g}_{kl}), k,l = 1,\ldots,m$. We call the system (11) under the condition (13) the normal form of the cluster partition (12). In this way we proved the following assertion.

Statement 1. Let the coupling matrix G of the system (1) be transformed to the block matrix $E = \{\varepsilon^{kl}\}$ satisfying the condition (13), implying that each matrix ε^{kl} has equal row sums. Then the system (1) has the cluster partition $M(N,m) = C_1(n_1) \cup \ldots \cup C_m(n_m)$ composed of m conditional clusters. Due to the permutation (5) and the reduction from n_k oscillators to one oscillator per cluster, provided the condition (13) holds, the system (1) at $M(N,m)$ obtains the form (14).

Statement 2. If in addition to (13) for some k, say $k = 1$, the off-diagonal matrices $\varepsilon^{1l}, l = 2,\ldots,m$ have equal rows:

$$\varepsilon_{ij}^{1l} = \widehat{\varepsilon}_j^{1l}, \quad i = 1,\ldots,n_k, j = 1,\ldots,n_l, \tag{15}$$

the cluster C_1 in the cluster partition $M(N,m)$ becomes the unconditional cluster.

The first equation in (11) having the diagonal matrix ε^{11} with equal row sums, $\sum_{j=1}^{n_{k_1}} \varepsilon_{ij}^{11} \equiv \widetilde{g}_{11}$, and the off-diagonal matrices with equal rows (15) may be rewritten in the form

$$\dot{X}_i^1 = F(X_i^1) + \sum_{j=1}^{n_1} \varepsilon_{ij}^{11} P f(X_j^1) + h(X^2, \ldots, X^m), \quad i = 1,\ldots,n_1, \tag{16}$$

where the function $h = \sum_{l=2}^{m} \sum_{j=1}^{n_l} \widehat{\varepsilon}_j^{1l} P f(X_j^l)$ does not depend on the index of equation i and the vector X_i^1. Hence, C_1 is the cluster with the equation in it

$$\dot{y}^1 = F(y) + g_{11} P f(y^1) + h(X^2, \ldots, X^m), \tag{17}$$

and the existence of cluster C_1 does not depend on any other elements not included into this cluster.

Remark. Note that the conditions of the statements 1 and 2 are being similar to the statements from [12–14], but in contrast to these papers we present now an explicit scheme for the clusters finding. This scheme involves verifying the conditions (13), (15) for all possible permutation matrices Π forming the matrix E.

EMBEDDINGS AND SIMPLE CLUSTER PARTITIONS

Considering the system (14) standing for the system (1), and applying recurrently transitions similar to (4)-(14), we obtain a sequence of cluster embeddings such that the next cluster partition is formed from the clusters of the previous cluster partition. The limit of this sequence is the full cluster of complete synchronization if it does exist. The example of such embeddings serves the hierarchy reported in [11] for diffusively coupled identical systems.

Let us now clarify two simple cases concerning the system (1):

- The case of equal row sums of the matrix G is similar to the case of zero-row sums. It gives the one cluster partition $C(N) = \{1,\ldots,N\}$ defined as the linear manifold $M(N,1) = \{y =: x_1 = \ldots = x_N\}$. The single system for $C_1(N)$ is

$$\dot{y} = F(y) + \widetilde{g}_{11} P f(y), \quad y \in R^d. \tag{18}$$

- Let the system (1) have an unconditional cluster $C_1(n)$ which is stable and cannot be embedded into any (conditional or unconditional) cluster. This cluster represents a real cluster and we call it an *isolated* cluster. We present an artificial partition $\{1,\dots,N\} = C_1(n) \cup \widetilde{C}_2(N-n)$ such that the group of $N-n$ remaining oscillators $\widetilde{C}_2(N-n)$ may have an arbitrary internal cluster partition. The system (11) for the singled out cluster $C_1(n)$ can then be rewritten in the form

$$\begin{aligned} \dot{X}^1 &= F(X^1) + (\varepsilon^{11}\otimes P)f(X^1) + (\varepsilon^{12}\otimes P)f(X^2) \\ \dot{X}^2 &= F(X^1) + (\varepsilon^{21}\otimes P)f(X^1) + (\varepsilon^{22}\otimes P)f(X^2) \end{aligned} \tag{19}$$

where the matrix ε^{11} has equal row sums, the matrix ε^{12} has equal rows, and the matrices $\varepsilon^{21}, \varepsilon^{22}$ provide the condition of the cluster $C_1(n)$ to be isolated. Note that for a given matrix G the problem to select the isolated cluster $C_1(n)$ is related to a hard procedure of sorting out permutations (5) and of checking the conditions (13), (15).

STABILITY OF ISOLATED CLUSTER

As the term $\widetilde{g}_{11}f(X^1)$ arising from the equal row sums of the matrix ε^{11} can be combined with the vector $F(X^1)$ in the first equation of the system (19), without loss of generality we consider the matrix ε^{11} in the system (19) with the property of zero row sums.

Assume that the individual system of the system (1) is eventually dissipative and that there exists a compact domain B_0 which attracts all trajectories of the system from the outside. Moreover, we assume that the matrices E and P and the function $f(x)$ are such that the coupled system (19) is eventually dissipative as well, and that there exists a domain B in R^{dN} which attracts all trajectories of the system from outside.

Assume that the scalar function $f: R^1 \to R^1$ satisfies the condition

$$f(0) = 0, Df(x) > K, \quad x \in R^1, \tag{20}$$

where K is a positive real scalar. Note that in the case (20) the property of system (19) to be eventually dissipative can be stated similarly to [14].

For convenience we rewrite the first equation in (19) in the coordinate form

$$\dot{x}_i^1 = F(x_i^1) + \sum_{j=1}^{n} \varepsilon_{ij}^{11} P f(x_j^1) + h(X^2), \quad i = 1,\dots,n, \tag{21}$$

where $h(X^2)$ denotes the equal rows of the matrix $(\varepsilon^{12}\otimes P)f(X^2)$.

Following [14] we introduce the notation for the differences

$$X_{ij} = x_j^1 - x_i^1, \quad i,j = 1,\dots,n. \tag{22}$$

¿From the system (21) we obtain the system of equations for differences

$$\dot{X}_{ij} = F(x_j^1) - F(x_i^1) + \sum_{k=1}^{n} \left[\varepsilon_{jk}^{11} P(f(x_k^1) - f(x_j^1)) - \varepsilon_{jk}^{11} P(f(x_k^1) - f(x_i^1))\right]. \tag{23}$$

Note that the stability of the system (23), being free of the term $h(X^2)$, implies the stability of isolated cluster. Next, following [14] in order to have the explicit presence of X_{ij} in the system (23), we apply the mean value theorem to each difference in (23) such that

$$\begin{aligned} f(x_j^1) - f(x_i^1) &= Df(\theta_1(x_i^1, x_j^1))X_{ij} \\ F(x_j^1) - F(x_i^1) &= DF(\theta_2(x_i^1, x_j^1))X_{ij}, \end{aligned} \tag{24}$$

where $Df,\ DF$ are Jacobian matrices and $\theta_{1,2}(x_i^1, x_j^1) \in [x_i^1, x_j^1]$. Now by denoting the $n\times n$ matrices

$$[\tilde{\varepsilon}_{ij}]_n^n = \left[\varepsilon_{ij}^{11} Df(\theta_1(x_i^1, x_j^1))\right]_n^n, \tag{25}$$

we rewrite the system (23) in the form

$$\dot{X}_{ij} = DF(\theta_2(x_i^1, x_j^1))X_{ij} + \sum_{k=1}^{n} \{\tilde{\varepsilon}_{jk}PX_{jk} - \tilde{\varepsilon}_{ik}PX_{ik}\}, \tag{26}$$

which coincides with the system (1) from [14]. Following this paper we take the next similar assumptions:

(i) P is a diagonal matrix $P = diag(p_1, \ldots, p_d)$ where $p_h = 1$ for $h = 1, 2, \ldots, s$, and $p_h = 0$ for $h = s+1, \ldots, d$.

(ii) The matrix $\{\tilde{\varepsilon}_{ij}\}$ is a $n \times n$ symmetric matrix with vanishing row-sums and non-negative off-diagonal elements, i.e. $\tilde{\varepsilon}_{ij} = \tilde{\varepsilon}_{ji}$, $\tilde{\varepsilon}_{ij} \geq 0$ for $i \neq j$, and $\tilde{\varepsilon}_{ii} = -\sum_{j=0, j\neq i}^{n} \varepsilon_{ij}$, $i = 1, \ldots, n$.

(iii) The matrix $\{\tilde{\varepsilon}_{ij}\}$ defines a connected graph with n vertices and m edges where m equals the number of non-zero above diagonal elements of the matrix $\{\varepsilon_{ij}^{11}\}$. For these elements we use an internal numbering k such that $\varepsilon_k^{11} > 0, \ k = 1, \ldots, m$.

(iv) The auxiliary system

$$\dot{X}_{ij} = \left[DF(\theta_2(x_i^1, x_j^1)) - A\right] X_{ij}, \quad i, j = 1, \ldots, n \tag{27}$$

where the matrix $A = diag(a_1, \ldots, a_d)$, $a_h \geq 0$, $h = 1, \ldots, s$ and $a_h = 0$ for $h = s+1, \ldots, d$ is globally stable along any trajectory from the attracting domain B of the system (21). We use parameter a to denote the parameters $a_1, \ldots, a_s$.

Hence, by applying the connection graph stability method we obtain the following condition for the isolated cluster stability.

Statement 3. Under the above assumptions the isolated cluster $C_1(n)$ of the system (1) is globally asymptotically stable if

$$\varepsilon_k^{11} > \frac{a}{nK} b_k(n, m), \quad k = 1, \ldots, m \tag{28}$$

where $b_k(n, m)$ is the sum of the lengths of all chosen paths which pass through a given edge k that belongs to the coupling configuration.

Indeed, we rewrite the inequality (24) from [14] for $\tilde{\varepsilon}_{ij} = \varepsilon_k^{11} Df > \varepsilon_k^{11} \cdot K$, and thus obtain (28).

Corollary. Assume that the condition (20) is valid within an interval $b^* = \{|x| < x^*\}$. Denote a compact $B^* \subset R^{nd}$ being a direct product of the intervals b^* along the coordinates of the system (21). Assume also that the system (21) for such a function $f(x)$ is eventually dissipative and has an attracting domain B such that $B \subset B^*$. Then statement 3 is valid for this type of functions $f(x)$.

CLUSTER STABILITY DIAGRAM FOR IDENTICAL SYSTEMS

In order to check our theoretical results we study cluster synchronization in an ensemble of five coupled identical Lorenz systems:

$$\begin{array}{rcl} \dot{x}_i &=& \sigma(y_i - x_i) + \sum_{j=1}^{5} g_{ij} x_j \\ \dot{y}_i &=& x_i(\rho - z_i) - y_i \\ \dot{z}_i &=& x_i y_i - \beta z_i \end{array} \tag{29}$$

where $\sigma = 10, \ \rho = 28, \ \beta = 8/3$ and g_{ij} are the elements of the coupling block matrix G:

$$G = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix}$$

We take the matrices ε_{ij} as follows

$$\begin{array}{rclrcl} \varepsilon_{11} &=& \alpha_{11} \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} & \varepsilon_{12} &=& \alpha_{12} \begin{pmatrix} 2 & 3 \\ 2 & 3 \\ 1 & 4 \end{pmatrix} \\ \varepsilon_{21} &=& \alpha_{21} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} & \varepsilon_{12} &=& \alpha_{22} \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}, \end{array} \tag{30}$$

where α_{ij} are changeable control parameters to achieve stability conditions.

The matrices ε_{ij} are chosen in such a way that the following clusters can exist:

(i) one unconditional cluster: C_{12} (elements 1 and 2 can be synchronized)

(ii) four conditional clusters: $C_{13}, C_{23}, C_{45}, C_{123}$.

Note that cluster C_{23} is completely similar to C_{13}. Therefore we only discuss the dynamics of the latter cluster. Indices after the letter C correspond to the number of oscillators that form each cluster. The conditions (13), (15) for each cluster may be easily checked.

Unfortunately, an analytical study of cluster stability is impossible. Therefore our aim is to obtain stability diagrams for all clusters depending on specified parameters. We set $\alpha_{12} = \alpha_{21} = 0.5$ and varied two other parameters α_{11} and α_{22} from 0 to 3.5. We integrated the whole system and tested the difference of x-variables of each element. As a synchronization criterion between elements i^{th} and j^{th} we consider

$$\frac{1}{S}\sum_{k=1}^{S} |x_i(T^* + 100k\Delta t) - x_j(T^* + 100k\Delta t)| < \delta, \tag{31}$$

where $T^* = 25000$ is the calculation time assigned for transient process, $\delta = 0.01$, $S = 1000$ and $\Delta t = 0.001$ is the integration step. The calculations were also performed for other values of T^* and δ (for example, for $T^* = 35000, \delta = 0.005, S = 2000$) and qualitatively similar results were obtained, though some slight quantitative changes took place. Hence, these particular values are considered to be quite satisfactory for our experiments. The multiplier 100 in (31) is introduced in order to compare the trajectories of the i^{th} and j^{th} elements on a larger time scale. In our case the time when the trajectories are compared equals $100 * S * \Delta t = 200$ time units of the model. We built stability diagrams for all clusters (Fig. 1(a) - cluster C_{12}, Fig. 1(b) - cluster C_{123}, Fig. 1(c) - cluster C_{13}, Fig. 1(d) - cluster C_{45}).

Cluster C_{12} is unconditional and it becomes stable starting from a certain value of $\alpha_{11} \approx 1$ (see Fig. 1(a)), meanwhile its stability almost does not depend on α_{22} and hence on other elements. On the other hand the clusters C_{13} and C_{45} are mutually conditional, i.e. they can become stable only simultaneously as shown in Fig. 1(c) and 1(d). Note, that there is a certain inaccuracy of the algorithm appearing due to the finiteness of the calculation and the long transient process in the border zone of the diagram where stability of clusters settles in. Here the intermittent synchronization may take place, in which long epochs of synchrony can be divided by intervals of asynchronous behavior. That is why one can see individual points outside the large domains in the diagrams. They are "spurious" clusters detected by the algorithm. We can say nothing about these states with certainty in contrast to large domains where the cluster stability appears undoubtedly. Finally, there is the stability diagram of the last cluster C_{123} presented in Fig. 1(b). It is easy to see that when C_{123} becomes stable cluster C_{13} can not be detected, i.e. it becomes a hidden cluster.

DESIGN OF CLUSTERS

In order to demonstrate the ability to construct clusters of a desired shape by operating with the coupling matrix, we stated the following problem. Our objective is to obtain a cluster of mutually coupled oscillators with a prescribed configuration, e.g. a cluster in the form of the word *"CHAOS"*. In order to do that, firstly, we take a two dimensional 20×60 lattice of globally coupled identical Rössler oscillators. Then, this lattice was presented as a 1D array of 1200 elements. Therefore, the coupling matrix G is 1200×1200. We intend to construct the required cluster here from 295 elements. This auxiliary cluster we call C. In order to obtain this cluster we made the following. We allocated the 295 elements at the beginning of the whole chain. Then, in the coupling matrix G we signed out four blocks: ε_{11} of the size 295×295, ε_{12} of the size 295×905, ε_{21} of the size 905×295 and ε_{22} of the size 905×905. The elements of these blocks were chosen in order to satisfy the existence condition (13) for the block ε_{11} and (15) for ε_{12}. i.e. in the block ε_{11} all row sums are equal and the block ε_{12} consists of identical rows (elements of each column of the block are the same). Blocks ε_{21} and ε_{22} can be chosen as arbitrary because they have no influence on the existence of the desired unconditional cluster. The block matrix G, has then the form

$$G = \begin{pmatrix} \alpha_{11}\varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix}.$$

In order to provide the stability condition of the initial cluster C we introduce the control parameter α_{11} as in the previous examples. After that we apply the permutation (5), in such a way that the stable cluster C will transform

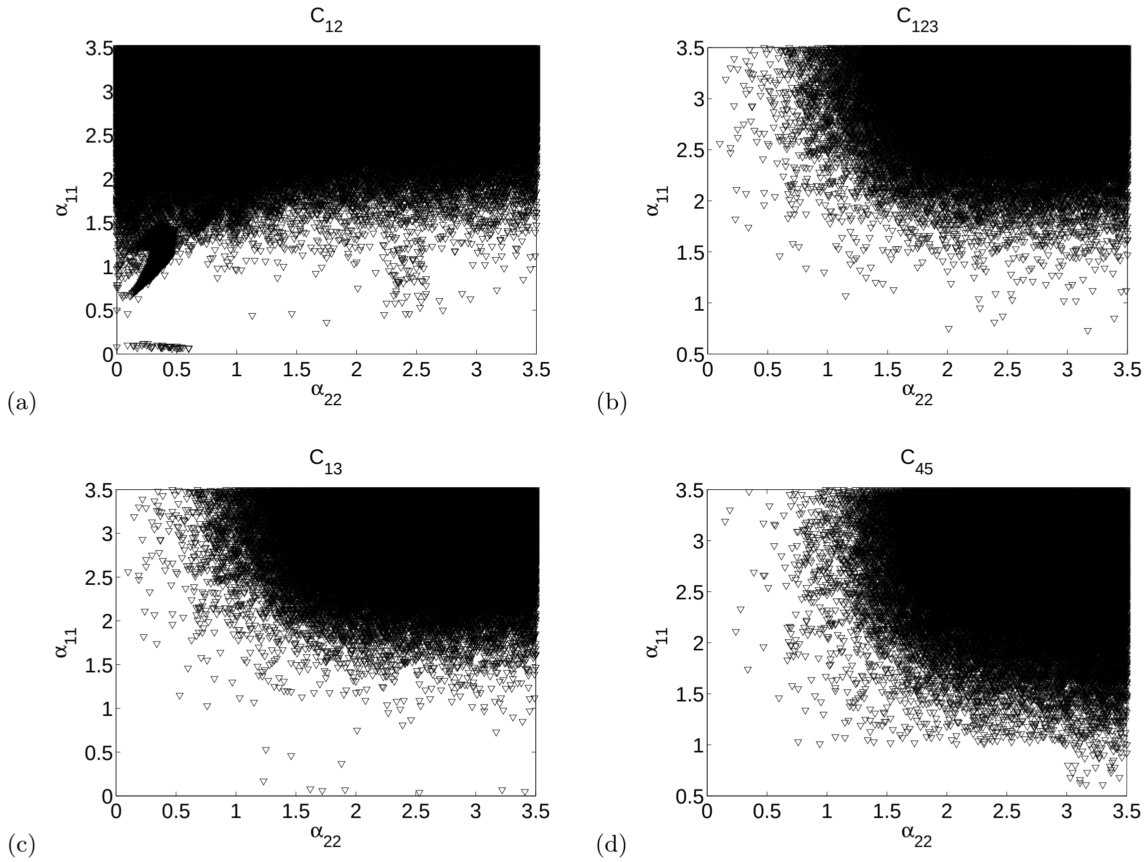

FIG. 1: Cluster stability diagram in the ensemble of five coupled identical Lorenz oscillators for parameters $\alpha_{12} = \alpha_{21} = 0.5$. Diagrams presented for clusters: a) C_{12}, b) C_{123}, c) C_{13}, d) C_{45}. The areas of stability of clusters C_{13}, C_{45}, C_{123} almost coincide, indicating that these clusters are really conditional. Stability of cluster C_{12} does not depend on the stability of any other clusters.

into the desired cluster with the form of the word *"CHAOS"*. This procedure may be performed automatically by the following algorithm:

1. First, we define two elements of the lattice for which we will replace the corresponding coefficients in the coupling matrix. The first one we call - e_1 - belonging to the initial cluster C and the second one - e_2 - belonging to the desired cluster.

2. Second, the coordinates of both elements (row and column numbers in the lattice) should be defined: for element e_1 - (r_1, c_1), and for e_2 - (r_2, c_2).

3. Finally, the elements g_{ij} of the coupling matrix G with the indices $g_{r_1 j}$, g_{ic_1} or $g_{r_1 c_1}$ should be replaced correspondingly by the elements $g_{r_2 j}$, g_{ic_2} or $g_{r_2 c_2}$.

Then this algorithm should be repeated 295 times for all the elements of the cluster C. As a result, we obtain the pattern presented in Fig. 2. Note, that inside this cluster *"CHAOS"* there are other clusters of smaller sizes, that are the parts of the main cluster, e.g. the characters *"C"*, *"H"*, etc.

CONCLUSIONS

We observed the effect of cluster partitioning in networks of coupled oscillators, i.e. splitting of the full network into several clusters of mutually synchronized oscillators. The conditions for existence of unconditional and conditional clusters were given. We also obtained the stability condition for unconditional clusters. A design technique for

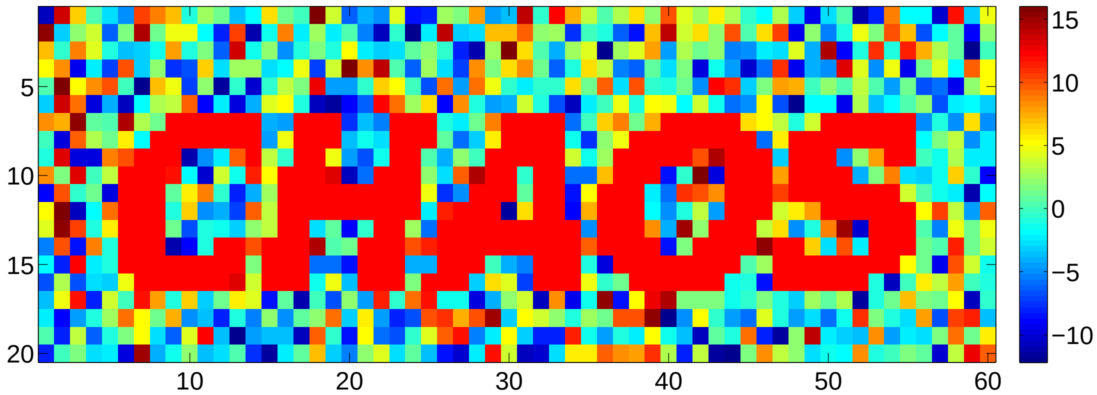

FIG. 2: Example of cluster design in the 2D 20 × 60 lattice of identical Rössler oscillators. The desired cluster in the form of the word "CHAOS" is obtained from randomly distributed initial conditions.

prescribing clusters was also suggested.

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