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A suboptimal solution to nonconvex optimal control problems involving input affine dynamic models.

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Abstract

This papers presents a convex approximation method for the solution of nonconvex optimal control problems involving input-affine dynamic models. The method relies in the availability of full reference state trajectories. By using these states references as real states trajectories, the dynamic model is approximated such that the resulting problem becomes convex. The convexified problem is solved by efficient convex methods delivering a suboptimal solution. This solution is used to linearize the original nonconvex problem such that the minimizer is refined by solving a new convex problem. Consequently, the solution to the original problem is obtained in two steps. An assessment of the errors in the approximation as a function of the mismatch between state reference trajectories and a perfect traceable trajectory is provided. The method is exemplified by formulating the optimal control problem of an isothermal continuous stirred tank reactor with Van den Vusse reactions.

Keywords: optimal control, convex optimization, input-affine models.

1. Introduction

Nonconvex *Optimal Control Problems (*OCPs*)* are of particular interest in the control community since control problems with nonlinear system dynamics are inherently non-convex. Because non-convex OCPs are, in general, difficult to solve, researchers try to find methods and new formulations in order to solve them efficiently and accurately. In order to tackle OCPs involving dynamic models, several approaches have been proposed. On the one hand, the methods based on calculus of variations and Pontryagin's maximum principle [1] are known as *indirect methods* and, on the other hand, the methods based on the finite parametrization of the continuous functions involved in the optimization task, which are known as *direct methods*. Practical and efficient approaches to solve nonconvex optimal control problems in the direct methods class have been proposed. There, the problem is linearized around an initial guess and a sequence of convex problems are solved until a local solution is found [2],[3].

This work proposes a direct-method based approach to find an approximate solution to nonconvex OCPs of the form (1)-(5). Where the cost (1) is a convex function, the inequality constraints (4)-(5) are defined by convex sets and the dynamic model (2) is affine in the controls. Notice that here a quadratic cost, with $Q > 0$, is used for simplicity in the explanation of the method. Nevertheless, the approach is applicable to any convex cost function. Similarly, terminal cost and terminal constraints are not

mention in (1)-(5) but they can be easily included in the problem formulation as long as they do not affect the convexity of the problem.

$$\begin{aligned}
\mathrm{OCP}_{\mathrm{ncvx}}: \min_{x(\cdot),u(\cdot)} \quad & \int_0^T \|x(t)-x^{\mathrm{ref}}(t)\|_Q^2 \, dt && (1)\\
\text{subject to}: \quad & && \\
\dot{x}(t) &= f(x)+g(x)u(t), \quad t\in[0,T], && (2)\\
x(0) &= x_0, && (3)\\
x(t) &\in \mathcal{X}, \quad t\in[0,T], && (4)\\
u(t) &\in \mathcal{U}, \quad t\in[0,T]. && (5)
\end{aligned}$$

2. Convex approximation

Notice that for the given problem (1)-(5), the source of the nonconvexity lies in the nonlinearity of the model. Here, an approximation to the model in (2) is proposed by introducing a pseudo state x_c and reformulating the problem in the form:

$$\begin{aligned}
\mathrm{OCP}_{\mathrm{cvx}}: \min_{x_c(\cdot),u(\cdot)} \quad & \int_0^T \|x_c(t)-x^{\mathrm{ref}}(t)\|_Q^2 \, dt && (6)\\
\text{subject to}: \quad & && \\
\dot{x}_c(t) &= f(x^{\mathrm{ref}})+g(x^{\mathrm{ref}})u(t), \quad t\in[0,T], && (7)\\
x_c(0) &= x_0, && (8)\\
x_c(t) &\in \mathcal{X}, \quad t\in[0,T], && (9)\\
u(t) &\in \mathcal{U}, \quad t\in[0,T]. && (10)
\end{aligned}$$

It has been shown that (6)-(10) corresponds to the convex extreme of a parametric optimization problem which ranges between (6)-(10) and (1)-(5) when a homotopy parameter varies from zero to one. Consequently, (6)-(10) is not obtained arbitrarily, but achieved by a convexification of the original problem through an homotopy approach [4]. Since (6)-(10) is a convex problem in x_c and u , it can be easily solved delivering an approximate solution $(x^*_{\mathrm{cvx}}, u^*_{\mathrm{cvx}})$. The given solution can be refined by formulating the new problem

$$\begin{aligned}
\mathrm{OCP}_{\mathrm{cvx-ref}}: \min_{x(\cdot),u(\cdot)} \quad & \int_0^T \|x(t)-x^{\mathrm{ref}}(t)\|_Q^2 \, dt && (11)\\
\text{subject to}: \quad & && \\
\dot{x}(t) &= Ax(t)+Bu(t)+c, \quad t\in[0,T], && (12)\\
x(0) &= x_0, && (13)\\
x(t) &\in \mathcal{X}, \quad t\in[0,T], && (14)\\
u(t) &\in \mathcal{U}, \quad t\in[0,T], && (15)
\end{aligned}$$

where the equality (12) corresponds to the linearization of (2) around $(x^*_{\mathrm{c}}, u^*_{\mathrm{cxv}})$. By solving $\mathrm{OCP}_{\mathrm{cvx-ref}}$, a refined solution $(x^*_{\mathrm{cvx-ref}}, u^*_{\mathrm{cvx-ref}})$ is obtained. This solution is considered as an approximation to the minimum obtained by solving (1)-(5) directly.

Hence, a 2-step procedure is proposed where the solution to (1)-(5) is approximated by solving the convex problems (5)-(10) and (11)-(15) sequentially.

3. Assessment of errors in the approximation

Notice that all the OCPs have been introduced in continuous time for simplicity in its presentation. Nevertheless, for the solution and analysis of the resulting problems, a time discretization approach is employed. Consequently, under a suitable discretization, the OCP (1)-(5) can be represented by a *Nonlinear Programming* (NLP) problem of the form:

$$\mathrm{NLP}(\bar{\mathbf{x}}): \quad \min_{\mathbf{u},\mathbf{x}} \frac{1}{2} \| \mathbf{x} - \bar{\mathbf{x}} \|_Q^2 \tag{16}$$

subject to:

$$\mathbf{0} = A(\mathbf{x}) - B(\mathbf{x})\mathbf{u} - W\mathbf{x}, \tag{17}$$

$$\mathbf{x} \in \mathbb{X}, \quad \mathbf{u} \in \mathbb{U}, \tag{18}$$

where $\bar{\mathbf{x}}$ is a vector containing references for the states, $\mathbf{x}$ and $\mathbf{u}$ represent vectors in $R^{(N+1).n_x}$ and $R^{N.n_u}$ respectively, and N is the number of discretization points in the time horizon T. $\mathbb{X} = \mathcal{X}_1 \times \mathcal{X}_2 \times \ldots \mathcal{X}_N$ and $\mathbb{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \mathcal{U}_N$. Similarly, the problem (11)-(15) can be discretized to (19)-(21) whose solution leads to an approximated minimum of (16)-(18).

$$\mathrm{OP}_{\mathrm{cvx-ref}}(\bar{\mathbf{x}}): \quad \min_{\mathbf{u},\mathbf{x}} \frac{1}{2} \| \mathbf{x} - \bar{\mathbf{x}} \|_Q^2 \tag{19}$$

subject to:

$$\mathbf{0} = A_L\mathbf{x} - B_L\mathbf{u} - c_L \tag{20}$$

$$\mathbf{x} \in \mathbb{X}, \quad \mathbf{u} \in \mathbb{U}, \tag{21}$$

In order to analyse how the solution to (16)-(18) and (19)-(21) are related, the following conditions are assumed [5]:

A1: The functions $A(\mathbf{x})$ and $B(\mathbf{x})$ are twice continuously differentiable.

A2: There exist a pair $\bar{\bar{\mathbf{x}}} \in \mathbb{X}$ and $\bar{\bar{\mathbf{u}}} \in \mathbb{U}$ such that $0 = A(\bar{\bar{\mathbf{x}}}) - B(\bar{\bar{\mathbf{x}}})\bar{\bar{\mathbf{u}}} - W\bar{\bar{\mathbf{x}}}$.

A3: Both problems, $\mathrm{NLP}(\bar{\bar{\mathbf{x}}})$ and $\mathrm{OP}_{\mathrm{cvx-ref}}(\bar{\bar{\mathbf{x}}})$ satisfy the strong second order sufficient conditions (SOSC), strict complementarity and constraint regularity [6] at their solution $(\bar{\bar{\mathbf{x}}}, \bar{\bar{\mathbf{u}}})$.

Lemma 1: If assumptions A1 to A3 are satisfied, then

$$\|\mathbf{u}^*(\bar{\mathbf{x}}) - \mathbf{u}^*_{\mathrm{cvx-ref}}(\bar{\mathbf{x}})\| = \mathcal{O}(\|\bar{\mathbf{x}} - \bar{\bar{\mathbf{x}}}\|^2) \tag{22}$$

holds.

Lemma 1 implies that if the reference is a perfect traceable trajectory $\bar{\bar{\mathbf{x}}}$, the solution to the convex approximation (19)-(21) perfectly matches the solution of the original NLP (16)-(18). Otherwise, the errors in the approximation are of second order in the size of the distance between the real $\bar{\mathbf{x}}$ and the perfect traceable trajectory $\bar{\bar{\mathbf{x}}}$. Lemma 1 is numerically corroborated in the case study presented in the next section. A mathematical proof is not given here for simplicity, but can be found in [5] for the parameter estimation of parameter-affine models which is analogous to the problem formulated here.

4. Case Study

Consider the benchmark problem presented in [7], where the control of an isothermal CSTR with the Van den Vusse reactions is evaluated. The process is governed by the set of nonlinear ordinary differential equations:

$$\dot{C}_a(t) = \frac{F(t)}{V}(C_{a,0}(t) - C_a(t)) - k_1 C_a(t) - k_3 C_a^2(t), \quad (23)$$

$$\dot{C}_b(t) = k_1 C_a(t) - k_2 C_b(t) - \frac{F(t)}{V} C_b(t), \quad (24)$$

where C_a and C_b are species concentration in the reactor and $C_{a,0}(t)$ and $F(t)$ represent input concentration and volumetric inflow respectively. Nominal parameters are: $k_1 = 50\,l/h$, $k_2 = 100\,l/h$, $k_3 = 10\,l/gmol.h$, $V = 1l$, , $F_{\text{min}} = 0\,l/h$ $F_{\text{max}} = 200\,l/h$, $\bar{F} = 32.61\,l/h$, $\bar{C}_{\text{a}} = 2.91\,gmol/l$, $\bar{C}_{\text{b}} = 1.1gmol/l$, $\bar{C}_{\text{a},0} = 10\,gmol/l$. In order to evaluate the 2-step procedure, an optimal control problem of the form (1)-(5) is proposed over a time horizon $T = 6$ min and discretized with a sampling period $T_s = 0.12$ min, i.e., $N = 50$. The control horizon N_u is assumed equal to the prediction horizon. The state penalization matrix Q is set to $\text{diag}(1, 1)l^2/(gmol^2h)$. However, for $N_u = 1$, the problem can be easily enumerated over the search space of u. Figure 1(a) shows the cost to minimize for the problem when a perfect traceable trajectory $\bar{\bar{\mathrm{x}}}$ is used as a reference while Figure 1(b) shows the cost obtained by performing a step change in the reference plus the addition of Gaussian noise to the second state reference trajectory. Notice that as postulated in Lemma 1, the convex approximation delivers a solution that exactly matches the minimum of the original NLP when the desired trajectory equals a perfect traceable trajectory $\bar{\bar{\mathrm{x}}}$, i.e., the desired trajectory is an open loop response. Figure 2 shows the errors obtained in the approximation when Gaussian noise of different amplitudes is added to a perfect traceable state trajectory $\bar{\bar{\mathrm{x}}}$, where Lemma 1 is corroborated numerically.

In order to evaluate the performance of the 2-step procedure, a tracking and disturbance test is performed in the receding horizon framework. One the one hand, a NLP solver is used to solve the discretized version of (1)-(5) at each sampling instant for set point changes. On the other hand the proposed method is used to provide a suboptimal solution.

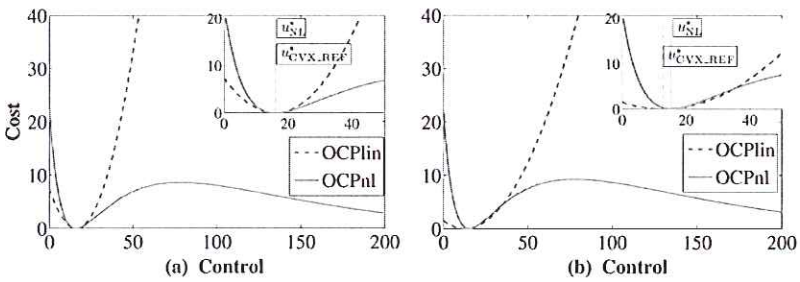

Figure 1. Cost functions for the OCP of a CSTR with Van den Vusse reactions. The reference trajectory exactly matches an open loop response of the system (a) where the approximation delivers the optima solution. On the other hand, a step change is performed in C_b while C_a is contaminated with Gaussian noise of $\sigma = 0.28$. Notice the difference between the solutions.

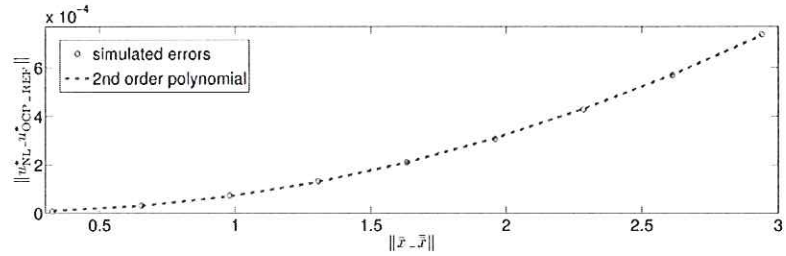

Figure 2. Errors in the approximation as a function of the distance between a perfect traceable trajectory and a reference trajectory. The reference trajectory corresponds to $\bar{\bar{\mathbf{x}}}$ plus Gaussian noise of different amplitudes.

A third scheme, namely linearization around trajectory (LAT), is included here for further discussion. The receding horizon schemes are implemented along with a nonlinear open loop observer plus an output disturbance model [8].

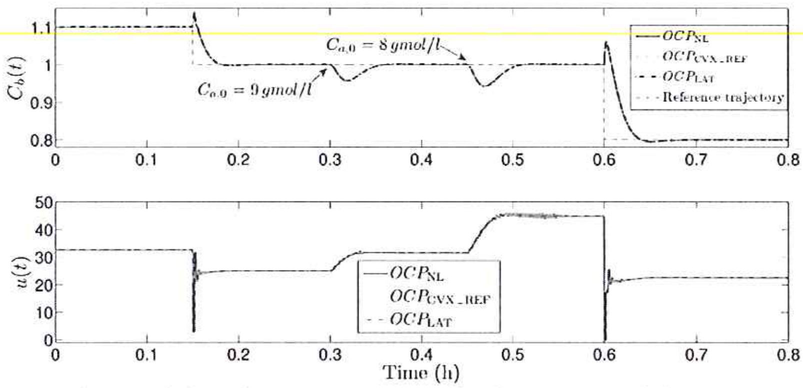

Figure 3. Tracking and disturbance rejection test for the NMPC and the 2-steps approach. Difference in the trajectories are almost negligible.

5. Discussion

For the presented study case, the method performs slightly different from the NMPC scheme. In terms of control performance, the difference is almost negligible even for the disturbance applied to the input concentration. Moreover, the computational demand of the presented approach is less than the one required for solving the original nonconvex optimization problem. The scheme presented can be directly compared with linearization around trajectory where the model inside the MPC is linearized around every point in the reference trajectory, i.e., a linear time-variant model is used inside the MPC. However, the proposed approach presents a computational advantage since the first step provides already a good approximation of the solution, the second step takes less time to achieve the optimal control profile. On the other hand in linearizion around trajectory, the model is solved using a flat control profile which corresponds to the steady state value for the given reference trajectory at the current time. Notice that this is not the case for the 2-step approach since, u^*_{cvx} has already the shape of the optimal control profile. Figure 4 shows the computational time for the test in Figure 3 evaluated with the 3 control approaches as a function of the control horizon N_u.

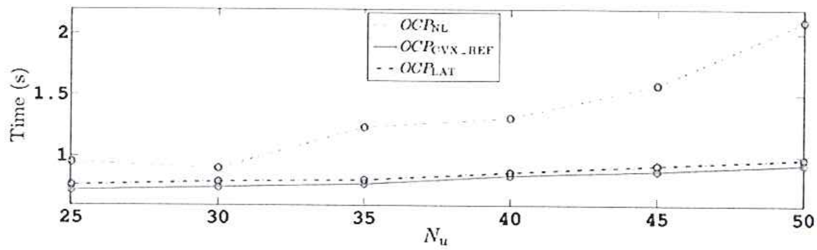

Figure 4. Average computation time for the test performed in Figure 3 as a function of control horizon changes. Notice that the presented 2-step approach presents slightly better computation time than the other control strategies.

6. Conclusions

A 2-step convex optimization method has been presented in the context of optimal control for input affine dynamic models. Although the solution is not optimal, it has been shown that the errors in the approximation are quadratic in terms of the difference between a perfect traceable trajectory and the given reference. A simulated study has shown that proposed approach requires less computational power than the one required by solving the original nonconvex problem. The method can be compared with linearization around trajectory, however, the first step in the approach already provides a better control trajectory that the one in the LAT approach. This property leads to a smaller computation time needed for converging to a local solution.

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