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Non-conventional matrix calculus in the analysis of rank deficient Hankel matrices of finite dimensions

Bart De MOOR and Joos VANDEWALLE.
ESAT (Electronics, Systems, Automation, Technology), Katholieke Universiteit Leuven, Kardinaal Mercierlaan 94, B-3030 Heverlee Belgium

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Abstract: New algebraic results are obtained that describe the relation between the structure of rank deficient Hankel matrices of finite dimensions and their singular value decomposition. Use is made of the 'non-conventional' matrix calculus of Khatri-Rao and Kronecker products. One of the important results is the parametrization of all rank deficient Hankel matrices of finite dimensions that share (a subset of) the same set of minimal system poles. The results are illustrated with a clarifying example.

Keywords: Hankel matrix, Singular value decomposition, Kronecker product, Khatri-Rao product, Parametrization with same system poles.

1. Introduction

Hankel matrices play an important role in the analysis and realization of linear systems. Although a lot of their algebraic properties are known for quite a long time [8], it is only since the introduction of the now well known realization scheme of Ho and Kalman [7] that their system theoretic importance has been established. An important step was the introduction in the realization context of the singular value decomposition of the Hankel matrix of Markov parameters [13]. Moreover there exists a close connection between the singular value decomposition and the concept of balanced realization, introduced in [11], where it was demonstrated how the singular values and vectors of Hankel matrices of finite and infinite dimensions can be used as quantitative measures of controllability and observability. This connection is exploited in [4] to solve the celebrated model reduction problem with Hankel norm using state space methods only.

In this paper, some new numerical-algebraic results will be established that describe the link between the Hankel structure and the singular value decomposition. The main result is a parametrization of all rank deficient Hankel matrices of finite dimensions that have a prescribed set of minimal system poles. Use is made of the matrix calculus of Kronecker and Khatri-Rao products.

The motivation for this research arises from the important relation between the singular values of the Hankel matrix and the degree of controllability and observability of a system as described in [11]. If for a system of minimal order n, the smallest singular value is small with respect to the largest one, then there exist states that are difficult to control and to observe in terms of 'energy' [11]. Hence, in the design of linear systems, it might be interesting to optimise the Hankel matrix, which is the product of controllability and observability matrices of the system, such that it is as orthonormal as possible. In this way, the energy to control and observe states may be minimized while the desired minimal system poles are specified (e.g. pole placement). This will be the subject of a forthcoming publication. In this paper, we will take more specifically a closer look at the following problem:

Given a system matrix A $[n \times n]$. Parametrize all Hankel matrices H_{pq} of given dimension $p > n$ and $q > n$ that are filled with Markov parameters $H_k = c^t \cdot P^{k-1} \cdot b$ where (P, b, c) are all possible state space models with P similar to A.

The results to be presented are restricted to Hankel matrices with scalar data (single input, single output systems). It is however expected that the generalization to multivariable systems poses no considerable difficulties. This paper is organized as follows: In Section 2, the definitions and the properties of the numerical tools that will be used to analyse the Hankel matrices, will be presented (the singular value decomposition, Kronecker and Khatri–Rao products). In Section 3, the main results of partial realization theory are summarized. In Section 4, these results are restated with emphasis on the singular value decomposition of the Hankel matrices. It is shown that there is a one-to-one correspondence between the vector space generated by the left and right singular vectors and the minimal system poles. Section 5 contains the main contribution of this paper: With the aid of the Kronecker and Khatri–Rao product, some intriguing new properties of Hankel matrices are explored: This leads to a characterization of all possible rank deficient Hankel matrices of prespecified dimensions with a prespecified set of minimal system poles. This is illustrated with a numerical example in Section 6. The conclusions can be found in Section 7.

2. Notations; the singular value decomposition; Khatri–Rao and Kronecker products

2.1. Notations and conventions

The following notations will be used throughout the paper:

$\underset{m\times n}{A}$	$m \times n$ matrix.
A^t	transpose.
$\odot$	Khatri–Rao product.
$\otimes$	Kronecker product.
$\lambda(A)$	eigenvalue set of A
$J(A)$	Jordan form of A.
rank(A)	rank of A.
I_n	$n \times n$ identity matrix.
ker(A)	$= \{x \in R^n \mid A \cdot x = 0\}$.
cor(A)	corank(A) = min(m,n) − rank(A).
$\overline{A}$	matrix A with first row omittted.
$\underline{A}$	matrix A with last row omitted.
vec(A)	store all columns of A in a long column vector.
vecd(A)	store the diagonal elements of A in a column vector.
$\overline{A}^t$	first omit row, then transpose.
$[A \;\; B]$	concatenated matrix. If A is $m \times n$ and B is $m \times p$ then $[A \;\; B]$ is $m \times (p+n)$.

Moreover, the following conventions are made:
- Small letters $a, b, \ldots$ are used for *column* vectors. Row vectors are denoted as the transpose of a column vector.
- Since the Jordan form of a matrix is only unique up to an ordering of the Jordan blocks, the expression $J(S) = J(T)$ where S and T are square matrices, indicates that S and T have the same Jordan structure.
- SVD is the abbreviation for Singular Value Decomposition.

2.2. The singular value decomposition

The SVD has become a key tool in the analysis and solution of many problems in numerical linear algebra and system theory [5,9]. The reason for this is that powerful algorithms have been developed with very robust and numerically reliable performance. Here only the main result will be stated.

Theorem 1. The Autonne–Eckart–Young theorem restricted to real matrices. *Every real $p \times q$ matrix A can be decomposed in three real matrices U, Σ and V:*

$$\underset{p\times q}{A} = \underset{p\times p}{U} \cdot \underset{p\times q}{\Sigma} \cdot \underset{q\times q}{V^t} \tag{1}$$

with the following properties:
- *U and V are unitary:*

$$U^t \cdot U = I_p = U \cdot U^t, \qquad V^t \cdot V = I_q = V \cdot V^t.$$

- *Σ is $p \times q$ pseudo-diagonal:*

$$\begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}$$

where Σ_1 is a diagonal $r \times r$ matrix with diagonal elements $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ where r is the algebraic rank of the matrix A.

Proof. See [5]. □

The column vectors u_i, $i = 1, \ldots, p$, ($v_j, j = 1, \ldots, q$) of U (V) are the left (right) singular vectors while the diagonal elements of Σ are the singular values of the matrix. If U_1 (V_1) is the $p \times r$ ($q \times r$) matrix consisting of the r first columns of U (V) and Σ_1 is the upper $r \times r$ matrix of Σ, then the singular value decomposition of A

can also be written as

$$\underset{p\times q}{A} = \underset{p\times q}{U} \cdot \underset{p\times q}{\Sigma} \cdot \underset{q\times q}{V^t} = \underset{p\times r}{U_1} \cdot \underset{r\times r}{\Sigma_1} \cdot \underset{r\times q}{V_1^t}$$

$$\text{where } U_1^t \cdot U_1 = V_1^t \cdot V_1 = I_r. \tag{2}$$

For further properties, we refer to [5,9].

2.3. The Kronecker and Khatri–Rao product

There are numerous applications of the Kronecker product in various fields including statistics, economics, optimisation and control. One of the major advantages of the matrix calcules of Kronecker and Khatri–Rao products is that it simplifies considerably complicated calculations, e.g. derivates of a matrix with respect to another matrix. In this paper, the Kronecker and Khatri–Rao product will be introduced for the same purpose.

Definition 1. The Kronecker product of a matrix A $(p\times q)$ and a matrix B $(m\times n)$, denoted by $A\otimes B$, is the $(p\cdot m)\times(q\cdot n)$ matrix defined as

$$A\otimes B = \begin{bmatrix} a_{11}\cdot B & a_{12}\cdot B & \dots & a_{1q}\cdot B \\ a_{21}\cdot B & a_{22}\cdot B & & a_{2q}\cdot B \\ \vdots & & & \vdots \\ a_{p1}\cdot B & a_{p2}\cdot B & \dots & a_{pq}\cdot B \end{bmatrix}.$$

Definition 2. The Kathri–Rao product of two matrices G $(s\times u)$ and F $(t\times u)$ (with the same number of columns !), denoted by $F\odot G$, is the $(s\cdot t)\times u$ matrix defined by

$$F\odot G = [f_1\otimes g_1 \quad f_2\otimes g_2 \quad \cdots \quad f_u\otimes g_u]$$

where f_i and g_i, $i=1,\dots,u$, are the columns of F and G.

The Kathri–Rao product is nothing more than the Kronecker product column wise.

Example. Consider

$$F=\begin{bmatrix}3 & 1\\ 2 & -1\\ 0 & 2\end{bmatrix},\qquad G=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}.$$

Then

$$F\odot G=\left[\begin{bmatrix}3\\2\\0\end{bmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix} \quad \begin{bmatrix}1\\-1\\2\end{bmatrix}\otimes\begin{bmatrix}0\\2\end{bmatrix}\right]$$

$$=\begin{bmatrix}3\cdot\begin{bmatrix}1\\0\end{bmatrix} & 1\cdot\begin{bmatrix}0\\2\end{bmatrix}\\ 2\cdot\begin{bmatrix}1\\0\end{bmatrix} & -1\cdot\begin{bmatrix}0\\2\end{bmatrix}\\ 0\cdot\begin{bmatrix}1\\0\end{bmatrix} & 2\cdot\begin{bmatrix}0\\2\end{bmatrix}\end{bmatrix}=\begin{bmatrix}3 & 0\\ 0 & 2\\ 2 & 0\\ 0 & -2\\ 0 & 0\\ 0 & 4\end{bmatrix}.$$

For a complete survey of the properties and important applications of this matrix calculus in linear system theory, we refer to [1] and [6]. Here, only the properties that will be used are mentioned, without proof. Let A be an $m\times n$ matrix with columns a_i. Then an important vector valued function is

$$\underset{(m\cdot n)\times 1}{\mathrm{vec}(A)} = \begin{bmatrix}a_1\\a_2\\ \vdots\\ a_n\end{bmatrix}$$

obtained by storing the columns a_i of A in a long column vector. For A being square $m\times m$, let vecd(A) be an $m\times 1$ vector containing the diagonal elements of A. A, B, C, D are real matrices of appropriate dimensions.

Property 1. Mixed product rule for the Kronecker product.

$$[A\otimes B]\cdot[C\otimes D]=[A\cdot C]\otimes[B\cdot D].$$

Property 2. Vector function of a product.

$$\mathrm{vec}[A\cdot D\cdot B]=[B^t\otimes A]\cdot\mathrm{vec}(D).$$

Property 3. Mixed product rule for the Khatri–Rao product.

$$[A\otimes B]\cdot[C\odot D]=[A\cdot C]\odot[B\cdot D].$$

Property 4. Vector function of a product with a diagonal matrix.

$$\mathrm{vec}[A\cdot D\cdot B]=[B^t\odot A]\cdot\ \mathrm{vecd}(D)$$

if D is square diagonal.

Corollary 1. Vector function of the singular value decomposition. *Let the SVD of A be $A=U\cdot\Sigma\cdot V^t$. Then*

$$\mathrm{vec}(A)=[V\odot U]\cdot\mathrm{vecd}(\Sigma).$$

Property 5. Singular value decomposition of the Kronecker product. If $A=U_a\cdot\Sigma_a\cdot V_a^t$ and $B=U_b\cdot\Sigma_b\cdot V_b^t$ are the SVD's of A and B, then the

SVD of $A \otimes B$ is given by

$$A \otimes B = (U_a \otimes U_b) \cdot (\Sigma_a \otimes \Sigma_b) \cdot (V_a \otimes V_b)^t.$$

Corollary 2. *If* $\text{rank}(A) = r_A$ *and* $\text{rank}(B) = r_B$, *then*

$$\text{rank}([A \otimes B]) = r_A \cdot r_B.$$

3. Properties of Hankel matrices of linear systems

In this section, a brief summary is given of the now well known properties of Hankel matrices in the context of (partial) realization theory, but also some new insights are presented that will be exploited in Section 4.

The discrete time invariant single input, single output system with state space description

$$\underset{n\times 1}{x_{k+1}} = \underset{n\times n}{A} \cdot \underset{n\times 1}{x_k} + \underset{n\times 1}{b} \cdot \underset{1\times 1}{u_k}, \tag{3}$$

$$\underset{1\times 1}{y_k} = \underset{1\times n}{c^t} \cdot \underset{n\times 1}{x_k} + \underset{1\times 1}{d} \cdot \underset{1\times 1}{u_k}, \tag{4}$$

will be denoted by (A, b, c, d). It has a set of Markov parameters $h_0 = d$, $h_k = c^t \cdot A^{k-1} \cdot b$, which are the samples of the impulse response of the system. The transfer function $H(z)$ of the system (3) equals

$$H(z) = c^t \cdot (z \cdot I - A)^{-1} \cdot b + d.$$

The extended observability and controllability matrices Γ_p and Δ_q are defined as

$$\Gamma_p = \begin{bmatrix} c^t \\ c^t \cdot A \\ c^t \cdot A^2 \\ \vdots \\ c^t \cdot A^{p-1} \end{bmatrix},$$

$$\Delta_q = [b \quad A \cdot b \quad A^2 \cdot b \quad \cdots \quad A^{q-1} \cdot b]. \tag{5}$$

For $p = q = n$, these are of course the observability and controllability matrices of the system (A, b, c). Throughout the paper, it will be assumed that the triplet (A, b, c) is minimal, i.e. $\text{rank}(\Gamma_n) = \text{rank}(\Delta_n) = n$, which is equivalent with the system being completely observable and controllable. An important structured matrix in linear algebra and its applications is the so called Hankel matrix which can be defined as follows:

Definition 3. Given a vector t with components $[t_1, t_2, \ldots, t_K]$, the rectangular $p \times q$ (with $(p + q - 1) \leq K$) Hankel matrix constructed from the elements of t is defined as

$$H_{pq}(i, j) = t_{i+j-1}.$$

Now let us introduce some conventions: If H is a $p \times q$ matrix, then $\overline{H}$ ($\underline{H}$) is the $(p-1) \times q$ matrix constructed from H by omission of the first (last) row. In the expression $\overline{H}^t$ ($\underline{H}^t$) the first (last) row is omitted before the transpose is taken.

The specific structure of a Hankel matrix also reveals itself in any possible factorization of the matrix:

Theorem 2. Factorization structure of Hankel matrices of finite dimensions. *Let H_{pq} be a rectangular $p \times q$ matrix that can be factored as $H_{pq} = X \cdot Y^t$ where X and Y are arbitrary matrices. Then H_{pq} is a Hankel matrix if and only if*

$$\underline{X} \cdot \overline{Y}^t = \overline{X} \cdot \underline{Y}^t.$$

Proof. Follows immediately from the Hankel structure. □

A fundamental tool in the analysis of linear systems is the Hankel matrix, constructed from the Markov parameters of the linear system (3):

$$H_{pq}(i, j) = h_{i+j-1} = c^t \cdot A^{i+j-2} \cdot b, \quad i = 1, \ldots, p;\ j = 1, \ldots, q. \tag{6}$$

The following properties of this Hankel matrix can be considered as classical in realization theory of linear systems:

Lemma 1. Factorization property of Hankel matrices from Markov parameters. *Let $h_k = c^t \cdot A^{k-1} \cdot b$ be the Markov parameters of a single input, single output linear system and $H_{pq}(i, j) = h_{i+j-1}$ a $p \times q$ Hankel matrix. Then*

$$H_{pq} = \Gamma_p \cdot \Delta_q. \tag{7}$$

Proof. Trivial. □

Lemma 2. Rank property. *Let $h_k = c^t \cdot A^{k-1} \cdot b$ be the Markov parameters of a single input, single output linear system and $H_{pq}(i, j) = h_{i+j-1}$ a $p \times q$ Hankel matrix with $p \geq n$ and $q \geq n$. Then* $\text{rank}(H_{pq}) = n$.

Proof. Follows from Lemma 1 and the minimality of (A, b, c). □

The second lemma indicates that the rank of a Hankel matrix of Markov parameters is always equal to the dimension of the observable and controllable part of the state space, if the dimensions p and q are chosen large enough ($p,q \geq n$). A unique correspondence between a linear system defined by a transfer function $H(z)$ and a Hankel matrix only exists for Hankel matrices with infinite dimension. Indeed, the transfer function $H(z)$ is uniquely represented by the corresponding infinite sequence of Markov parameters. When dealing with the correspondence of a finite dimensional Hankel matrix of a linear system $H(z)$, uniqueness of the realization can only be achieved by posing extra conditions on the extension of the sequence of Markov parameters to infinity, e.g. the extra condition of minimality of the partial realization of the finite sequence. As a result, the problem of minimal partial realization is involved [2,12].

Let H_{pq} be a $p \times q$ Hankel matrix of rank n where $n < p$ and $n < q$. Define H_1 to be the $(p-1) \times q$ submatrix of H_{pq} consisting of its first $p-1$ rows and H_2 the $p \times (q-1)$ submatrix of H_{pq} consisting of its first $q-1$ columns.

Lemma 3. *If*

$$\operatorname{rank}(H_{pq}) = \operatorname{rank}(H_1) = \operatorname{rank}(H_2) = n,$$

then the elements of H_{pq} are Markov parameters of a linear system of minimal state space dimension n.

Proof. See e.g. [2,12]. □

Note that the condition of rank deficiency of a finite Hankel matrix is not enough to guarantee a unique solution to the corresponding minimal realization problem. The necessary and sufficient condition hereto is the rank condition on the submatrices H_1 and H_2 in Lemma 3. This criterion is well known in the analysis of the minimal partial realization property as the '(partial) realizability criterion'.

$$\operatorname{rank}(H_{pq}) = \operatorname{rank}(H_1) = \operatorname{rank}(H_2) = n \quad \text{with } p, q > n. \tag{8}$$

It will be assumed throughout the paper that this partial realizability criterion is satisfied.

4. Singular value decomposition and the Hankel structure

In this section, the structural relations between the SVD and the Hankel structure of rank deficient Hankel matrices of finite dimensions are explored. They will provide the necessary insights that will lead to the new results to be presented in Section 5. The matrix $(\underline{H} \ \overline{H})$ is a $(p-1) \times (2q)$ matrix constructed by concatenation of $\underline{H}$ and $\overline{H}$.

The singular value decomposition provides a factorization of the Hankel matrix so that the following property is easy to prove.

Corollary 3. *Let H_{pq} be a rectangular $p \times q$ matrix. Let the SVD of H_{pq} be $H_{pq} = U \cdot \Sigma \cdot V^t$. Then H_{pq} is a Hankel matrix if and only if*

$$\underline{U} \cdot \Sigma \cdot \overline{V}^t = \overline{U} \cdot \Sigma \cdot \underline{V}^t. \tag{9}$$

Proof. Choose for X and Y in Theorem 2: $X = U$ and $Y = V \cdot \Sigma^t$. □

Hence the structure of the Hankel matrix reveals itself in the SVD. From now on, the symbol H_{pq} will be used exclusively to denote a $p \times q$ rectangular Hankel matrix. In Theorem 3, the specific structure of rank deficient matrices will be translated into a condition on the rank of some concatenated matrices that are constructed from the singular vectors.

Theorem 3. *Let H_{pq} be a $p \times q$ Hankel matrix of rank n with $p > n$ and $q > n$ and SVD*

$$H_{pq} = \underset{p\times n}{U} \cdot \underset{n\times n}{\Sigma} \cdot \underset{n\times q}{V^t}$$

and submatrices H_1 (first $p-1$ rows) H_2 (first $q-1$ columns) satisfying

$$\operatorname{rank}(H_{pq}) = \operatorname{rank}(H_1) = \operatorname{rank}(H_2) = n.$$

Then

$$\operatorname{rank}(\underline{U} \quad \overline{U}) = \operatorname{rank}(\underline{V} \quad \overline{V}) = n. \tag{10}$$

Proof. From Corollary 3,

$$(\underline{U} \quad \overline{U}) \cdot \begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix} \cdot \begin{bmatrix} \overline{V}^t \\ \underline{V}^t \end{bmatrix} = 0.$$

Now define $s = \text{rank}(\underline{U}\ \overline{U})$. From the SVD of H_{pq} it follows that $H_1 = \underline{U} \cdot \Sigma \cdot V^t$ and from the rank condition $\text{rank}(H_1) = n$, it follows that $\text{rank}(\underline{U}) = n$. Hence, $s \geq n$. Then

$$\dim \ker\left[\begin{pmatrix} \underline{U} & \overline{U} \end{pmatrix} \cdot \begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix}\right] = 2n - s.$$

Now define $\text{rank}(\overline{V}\ \underline{V}) = t$. From the SVD of H_{pq} it follows that $H_2 = U \cdot \Sigma \cdot \underline{V}^t$ and from the condition $\text{rank}(H_2) = n$, it follows that $\text{rank}(\underline{V}) = n$. Hence, $t \geq n$.

Since the column space of the matrix $(\overline{V}\ \underline{V})^t$ belongs to the kernel of the matrix

$$\left[\begin{pmatrix} \underline{U} & \overline{U} \end{pmatrix} \cdot \begin{bmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{bmatrix}\right],$$

we have that $2n - s \geq t \geq n$ so that both $2n - s \geq n$ (or $s \leq n$) and $s \geq n$ have to be satisfied. Hence $s = n$ or $\text{rank}(\underline{U}\ \overline{U}) = n$.

The proof for $\text{rank}(\underline{V}\ \overline{V}) = n$ is of course similar. □

An immediate consequence of Theorem 3 is the special shift structure of the left and right singular vectors of the Hankel matrices, which is well known and explored in realization theory to obtain a state space model from the Markov parameters [7,10,13]:

Corollary 4. *Let H_{pq} be a $p \times q$ Hankel matrix of rank n with $p > n$ and $q > n$ and SVD*

$$H_{pq} = \underset{p \times n}{U} \cdot \underset{n \times n}{\Sigma} \cdot \underset{n \times q}{V^t}.$$

Then U and V have the following structure (which will be called shift structure):

$$\underset{p \times n}{U} = \begin{bmatrix} u^t \\ u^t \cdot S \\ u^t \cdot S^2 \\ \vdots \\ u^t \cdot S^{p-1} \end{bmatrix}, \qquad \underset{q \times n}{V} = \begin{bmatrix} v^t \\ v^t \cdot T \\ v^t \cdot T^2 \\ \vdots \\ v^t \cdot T^{q-1} \end{bmatrix}$$

where u and v are both $n \times 1$ vectors and where S and T are the unique solution of $\underline{U} \cdot S = \overline{U}$, $\underline{V} \cdot T = \overline{V}$. Moreover, $\Sigma \cdot T = S \cdot \Sigma$.

The matrices S and T will be called shift matrices.

Proof. From Theorem 3 and the partial realization condition (8) it follows that there must exist unique square $n \times n$ matrices S and T such that

$$\underline{U} \cdot S = \overline{U} \quad \text{and} \quad \underline{V} \cdot T = \overline{V}. \tag{11}$$

S and T are given by

$$S = \left(\underline{U}^t \cdot \underline{U}\right)^{-1} \cdot \underline{U}^t \overline{U}$$

and

$$T = \left(\underline{V}^t \cdot \underline{V}\right)^{-1} \cdot \underline{V}^t \cdot \overline{V}.$$

If u^t and v^t are the first rows of U and V, the specific shift structure follows immediately from (11).

From Corollary 3,

$$\underline{U} \cdot \Sigma \cdot \overline{V}^t = \overline{U} \cdot \Sigma \cdot \underline{V}^t.$$

Hence

$$\Sigma \cdot \overline{V}^t \cdot \underline{V} \cdot \left(\underline{V}^t \cdot \underline{V}\right)^{-1} = \left(\underline{U}^t \cdot \underline{U}\right)^{-1} \cdot \underline{U}^t \cdot \overline{U} \cdot \Sigma$$

so that $\Sigma \cdot T = S \cdot \Sigma$. □

Corollary 4 specifies completely how to obtain a state space realization of a set of Markov parameters of a discrete system if the partial realization condition is satisfied. The eigenvalues of the shift matrices S and T are of course the controllable and observable poles of the system. More details can be found in [7,10,13]. An important remark is the fact that the shift structure is a property of the column space of the matrices U and V and not of the specific choice of basis in that space:

Lemma 4. *Let U be a $p \times n$ matrix $(p > n)$ or rank n with shift structure:*

$$\underset{p \times n}{U} = \begin{bmatrix} u^t \\ u^t \cdot S \\ u^t \cdot S^2 \\ \vdots \\ u^t \cdot S^{p-1} \end{bmatrix}$$

satisfying $\text{rank}(\underline{U}) = n$ *and let P be an $n \times n$ nonsingular matrix. Then the matrix $(U \cdot P)$ is also a matrix with shift structure with a shift matrix that is similar to S.*

Proof. Since $\underline{U} \cdot S = \overline{U}$, it follows that $\underline{U} \cdot S \cdot P = \overline{U} \cdot P$ and because P is nonsingular

$$(\underline{U} \cdot P) \cdot (P^{-1} \cdot S \cdot P) = (\overline{U} \cdot P).$$

Hence the matrix $(U \cdot P)$ has shift structure with shift matrix $P^{-1} \cdot S \cdot P$ which is similar to S and hence has the same Jordan structure. □

The interpretation of Lemma 4 is straightforward: With the column space of a matrix U with shift structure, one can associate a set of eigenvalues that only depend on that vector space and not on the choice of basis in that space.

5. A parametrization of all Hankel matrices of systems with the same set of controllable and observable poles

In this section, a parametrization is obtained of all rank deficient Hankel matrices of finite dimension which have the same set of minimal system poles (or a subset of it). The key observation that will be exploited is Lemma 4 which states that the minimal system poles depend only on the vector space spanned by the left and right singular vectors of the rank deficient Hankel matrix. The parametrization will be obtained by considering all possible bases in those vector spaces and yet still respecting the Hankel structure. In Corollary 4, it was shown that the singular vector matrices U and V of rank deficient Hankel matrices have shift structure. In Theorem 4, it will be shown how the singular values of rank deficient Hankel matrix are uniquely determined by the singular vectors.

Theorem 4. *Let H_{pq} be a rectangular matrix of rank n with $n < p$ and $n < q$ and with SVD*

$$\underset{p\times q}{H_{pq}} = \underset{p\times n}{U} \cdot \underset{n\times n}{\Sigma} \cdot \underset{n\times q}{V^t}.$$

Then H_{pq} is a Hankel matrix if and only if the singular values are the solution to the set of linear equations

$$\left[\overline{V} \odot \underline{U} - \underline{V} \odot \overline{U}\right] \cdot \text{vecd}(\Sigma) = 0.$$

Proof. The theorem follows immediately from Corollary 3 and the properties of the Khatri–Rao product. □

If the singular vectors of a matrix are known and if that matrix is to be of Hankel structure, the singular values are the solution to a set of linear equations, of which the data are determined by the components of the singular vectors. In the next two theorems, it is investigated which Hankel matrices could be constructed from two available matrices U $(p \times n)$ and V $(q \times n)$ with shift structure and with shift matrices that have the same set of minimal system poles. The motivation arises from Lemma 4 and Theorem 4: If U and V contain the left and right singular vectors of a Hankel matrix H_{pq}, then the vector with singular values vecd(Σ) belongs to the kernel of the matrix $[\overline{V} \odot \underline{U} - \underline{V} \odot \overline{U}]$. If now the orthonormal matrices U and V are modified into $U \cdot P$ and $V \cdot Q$ where P and Q are unitary, then from Lemma 4 the minimal system poles remain unchanged. In order to obtain a Hankel matrix having singular vectors $U \cdot P$ and $V \cdot Q$, according to Theorem 4 the 'new' singular values contained in the vector vecd(Σ_{PQ}) will have to satisfy

$$\left[(\overline{V} \cdot Q) \odot (\underline{U} \cdot P) - (\underline{V} \cdot Q) \odot (\overline{U} \cdot P)\right] \cdot \text{vecd}(\Sigma_{PQ}) = 0$$

or

$$\left[\overline{V} \otimes \underline{U} - \underline{V} \otimes \overline{U}\right] \cdot [Q \odot P] \cdot \text{vecd}(\Sigma_{PQ}) = 0$$

(property 3) or

$$\left[\overline{V} \otimes \underline{U} - \underline{V} \otimes \overline{U}\right] \cdot \text{vec}\left(P \cdot \Sigma_{PQ} \cdot Q^t\right) = 0$$

(property 4). Hence, any vector vec(X) of the kernel of the matrix $[\overline{V} \otimes \underline{U} - \underline{V} \otimes \overline{U}]$ where X is an $n \times n$ matrix with SVD $X = P_x \cdot \Sigma_x \cdot Q_x$ will generate a Hankel matrix $U \cdot X \cdot V^t$ with SVD

$$(U \cdot P_x) \cdot \Sigma_x \cdot (V \cdot Q_x)^t.$$

Hence, the kernel of the matrix $[\overline{V} \otimes \underline{U} - \underline{V} \otimes \overline{U}]$ plays a crucial role:

Theorem 5. *Let U $(p \times n)$ and V $(q \times n)$ be orthonormal matrices with shift structure and with shift matrices S and T defined from $\underline{U} \cdot S = \overline{U}$ and $\underline{V} \cdot T = \overline{V}$ such that $J(S) = J(T) = J$. Then*

$$\text{cor}\left[\overline{V} \otimes \underline{U} - \underline{V} \otimes \overline{U}\right] = n.$$

Proof. Let S and T have eigendecompositions

$$S = X_s \cdot J \cdot X_s^{-1} \quad \text{and} \quad T = X_t \cdot J \cdot X_t^{-1}.$$

Then with the properties of the Kronecker product, it follows that

$$\begin{aligned}
&[\bar{V}\otimes\underline{U}-\underline{V}\otimes\bar{U}]\\
&\quad=[(\underline{V}\cdot T)\otimes\underline{U}-\underline{V}\otimes(\underline{U}\cdot S)]\\
&\quad=[\underline{V}\otimes\underline{U}]\cdot[T\otimes I_n-I_n\otimes S]\\
&\quad=[\underline{V}\otimes\underline{U}]\cdot[(X_t\cdot J\cdot X_t^{-1})\otimes I_n\\
&\qquad\qquad -I_n\otimes(X_s\cdot J\cdot X_s^{-1})]\\
&\quad=[\underline{V}\otimes\underline{U}]\cdot[X_t\otimes X_s]\\
&\qquad\cdot[J\otimes I_n-I_n\otimes J]\cdot[X_t\otimes X_s]^{-1}.
\end{aligned}$$

The matrices $[\underline{V}\otimes\underline{U}]$, $[X_t\otimes X_s]$ are nonsingular. It is easily verified that

$$\operatorname{cor}[J\otimes I_n-I_n\otimes J]=n. \qquad \square$$

It will now be proved that every vector $\operatorname{vec}(X)$, where X is an $n\times n$ matrix, in the kernel of the matrix $[\bar{V}\otimes\underline{U}-\underline{V}\otimes\bar{U}]$ can be associated with a Hankel matrix.

Theorem 6. *Let U ($p\times n$, $p>n$) and V ($q\times n$, $q>n$) be orthonormal matrices with shift matrices S and T such that $J(S)=J(T)=J$.*

Let $\operatorname{vec}(X)$ be a vector in the kernel of $[\bar{V}\otimes\underline{U}-\underline{V}\otimes\bar{U}]$:

$$[\bar{V}\otimes\underline{U}-\underline{V}\otimes\bar{U}]\cdot\operatorname{vec}(X)=0$$

where X is an $n\times n$ matrix. Suppose $\operatorname{rank}(X)=r$ and that X has SVD

$$\underset{n\times n}{X}=\underset{n\times r}{P}\cdot\underset{r\times r}{\Sigma}\cdot\underset{r\times n}{Q^{\mathrm t}}.$$

Then the matrix $H=U\cdot X\cdot V^{\mathrm t}$ is a Hankel matrix of rank r, of which the r minimal system poles are a subset of the eigenvalues of J, with SVD

$$H=(U\cdot P)\cdot\Sigma\cdot(V\cdot Q)^{\mathrm t}.$$

Proof. Using the properties of the Kronecker and Khatri–Rao product,

$$\begin{aligned}
&[\bar{V}\otimes\underline{U}-\underline{V}\otimes\bar{U}]\cdot\operatorname{vec}(X)=0,\\
&[\bar{V}\otimes\underline{U}-\underline{V}\otimes\bar{U}]\\
&\quad\cdot\operatorname{vec}(P\cdot\Sigma\cdot Q^{\mathrm t})=0\quad(\text{SVD of }X),\\
&[\bar{V}\otimes\underline{U}-\underline{V}\otimes\bar{U}]\cdot(Q\odot P)\cdot\operatorname{vecd}(\Sigma)=0,\\
&[(\bar{V}\cdot Q)\odot(\underline{U}\cdot P)-(\underline{V}\cdot Q)\odot(\bar{U}\cdot P)]\\
&\quad\cdot\operatorname{vecd}(\Sigma)=0,\\
&[(\bar{V}\cdot Q)\odot(\underline{U}\cdot P)]\cdot\operatorname{vecd}(\Sigma)\\
&\quad=[(\underline{V}\cdot Q)\odot(\bar{U}\cdot P)]\cdot\operatorname{vecd}(\Sigma),\\
&(\underline{U}\cdot P)\cdot\Sigma\cdot(Q^{\mathrm t}\cdot\bar{V}^{\mathrm t})=(\bar{U}\cdot P)\cdot\Sigma\cdot(Q^{\mathrm t}\cdot\underline{V}^{\mathrm t}).
\end{aligned}$$

From Theorem 2 it now follows that the matrix

$$H=(U\cdot P)\cdot\Sigma\cdot(V\cdot Q)^{\mathrm t}$$

is a Hankel matrix. Moreover, if $\operatorname{rank}(X)=r$, there exist a S_r and a T_r such that

$$(\underline{U}\cdot P)\cdot S_r=(\bar{U}\cdot P)=(\underline{U}\cdot S)\cdot P$$

and

$$(\underline{V}\cdot Q)\cdot T_r=(\bar{V}\cdot Q)=(\underline{V}\cdot T)\cdot Q.$$

Let S_r and T_r have the eigenvalue decompositions

$$S_r=X_s\cdot J_s\cdot X_s^{-1}\quad\text{and}\quad T_r=X_t\cdot J_t\cdot X_t^{-1}$$

where J_s and J_t are the Jordan form of S_r and T_r. Then

$$(P\cdot X_s)\cdot J_s=S\cdot(P\cdot X_s)$$

and

$$(Q\cdot X_t)\cdot J_t=T\cdot(Q\cdot X_t)$$

so that the diagonal elements of J_s and J_t are also eigenvalues of S and T. It then follows that $J_s=J_t$. $\square$

Theorem 6 provides a parametrization of all Hankel matrices with a given set of minimal system poles. This set of minimal system poles determines the column and row space of all Hankel matrices with those minimal system poles. Once such a basis for this column and row space are known in the form of two matrices U ($p\times n$) and V ($q\times n$), all vectors $\operatorname{vec}(X)$ in the kernel of the matrix $[\bar{V}\otimes\underline{U}-\underline{V}\otimes\bar{U}]$ result in a Hankel matrix

$$H=U\cdot P\cdot\Sigma\cdot Q^{\mathrm t}\cdot V^{\mathrm t}$$

where $P\cdot\Sigma\cdot Q^{\mathrm t}$ is the SVD of X. Clearly, the transformations $U\to U\cdot P$ and $V\to V\cdot Q$ correspond to a change of basis in the column and row space. Theorem 6 can also be interpreted in terms of the zeros of the transfer function

$$G(z)=c^{\mathrm t}\cdot(z\cdot I-A)^{-1}\cdot b+d.$$

For single input, single output, the denumerator of $G(z)$ only depends upon eigenvalues of the matrix A. The choices for other Hankel basis

$U \to U \cdot P$ and $V \to V \cdot Q$ correspond to a change of the vectors b, c while the eigenvalues are fixed. Hence, the basis transformations correspond to assigning a new set of zeros to the transfer function $G(z)$. When X is an $n \times n$ matrix such that

$$[\overline{V} \otimes \underline{U} - \underline{V} \otimes \overline{U}] \cdot \mathrm{vec}(X) = 0,$$

then for $\mathrm{rank}(X) = n$, no pole–zero cancellations have occurred. This is the generic situation. However, if $\mathrm{rank}(X) = r < n$, then the new assignment of zeros caused $n - r$ pole–zero cancellations.

6. An example

Consider the 4×4 system matrix

$$A = \begin{bmatrix} 0.9 & 1 & 0 & 0 \\ 0 & 0.9 & 0 & 0 \\ 0 & 0 & 0.7 & -0.6 \\ 0 & 0 & 0.6 & 0.7 \end{bmatrix}$$

with a double eigenvalue 0.9 and a pair of complex conjugated eigenvalues $0.7 \pm 0.6j$. In order to parametrize all possible 8×6 Hankel matrices of rank 4 (or less), having this set of minimal system poles (or a subset of it), the following procedure can be applied: Generate an 8×4 orthonormal matrix U and a 6×4 orthonormal matrix V, both with shift structure. Hereto, choose a 'generator'

$$w^t = [1 \quad 1 \quad 1 \quad 1]$$

(which has components along all eigenvectors of A) and iterate to construct the 8×4 matrix

$$W = \begin{bmatrix} w^t \\ w^t \cdot A \\ w^t \cdot A^2 \\ w^t \cdot A^3 \\ w^t \cdot A^4 \\ w^t \cdot A^5 \\ w^t \cdot A^6 \\ w^t \cdot A^7 \end{bmatrix} = \begin{bmatrix} 1.0000 & 1.0000 & 1.0000 & 1.0000 \\ 0.9000 & 1.9000 & 1.3000 & 0.1000 \\ 0.8100 & 2.6100 & 0.9700 & -0.7100 \\ 0.7290 & 3.1590 & 0.2530 & -1.0790 \\ 0.6561 & 3.5721 & -0.4703 & -0.9071 \\ 0.5905 & 3.8710 & -0.8735 & -0.3528 \\ 0.5314 & 4.0744 & -0.8231 & 0.2771 \\ 0.4783 & 4.1984 & -0.4099 & 0.6879 \end{bmatrix}.$$

The desired orthonormal matrices U and V can now easily be constructed by orthogonalising the columns of (parts of) the matrix W (e.g. with a QR factorization [5]). For V, take the first six rows of W and do a QR factorization. For U, do a QR factorization on the complete matrix W. U and V are now orthonormal matrices and it follows from Lemma 4 that both U and V have the desired shift structure.

Compute the 35×16 matrix $[\overline{V} \otimes \underline{U} - \underline{V} \otimes \overline{U}]$. It can be verified numerically that the rank of this matrix is 12 (as is predicted by Theorem 5). Let a basis of the kernel of this matrix be generated by the columns of the 16×4 matrix V_2:

$$[\overline{V} \otimes \underline{U} - \underline{V} \otimes \overline{U}] \cdot V_2 = 0.$$

Then from Theorem 6 it follows that any linear combination of the columns of V_2 of the form $V_2 \cdot x = \mathrm{vec}(X)$, where x is an arbitrary 4×1 vector, generates a vector $\mathrm{vec}(X)$, such that X is a 4×4 matrix with the property that $H_X = U \cdot X \cdot V^t$ is a Hankel matrix. The kernel of the matrix $[\overline{V} \otimes \underline{U} - \underline{V} \otimes \overline{U}]$ generates all Hankel matrices with the prespecified set of minimal system poles (or at least a subset of it, if pole–zero cancellations occur). There is still some freedom left to pick those Hankel matrices that besides the prespecified poles have some other desirable properties, e.g. prespecified zeros. This freedom will be used in a subsequent publication to make the Hankel matrix as orthogonal as possible.

7. Conclusions

In this paper, new algebraic results that establish the relation between the structure of rank deficient Hankel matrices of finite dimensions and their singular value decomposition have been derived. There exists a close system theoretic connection between minimal system poles and certain vector spaces generated by Hankel bases. Using the properties of the Kronecker and Khatri–Rao product, a parametrization was obtained of all rank deficient Hankel matrices of finite dimensions given a prespecified set of minimal system poles. In a future paper, the problem of optimalisation of the Hankel matrix will be considered with respect to orthogonality. It will be demonstrated how the results of the present paper can be used in optimal design of systems: Starting from a set of desired minimal system poles, controllability and observability will be optimised using the parametrization of Hankel matrices from Theorem 6.

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