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THE GENERALIZED LINEAR COMPLEMENTARITY PROBLEM APPLIED TO THE COMPLETE ANALYSIS OF RESISTIVE PIECEWISE-LINEAR CIRCUITS

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Abstract

The link between the Linear Complementarity Problem (LCP) and piecewise-linear (PWL) resistive circuits is extended by allowing a generalized form of the LCP. Using this generalized LCP (GLCP) a very broad class of circuits can be described and analyzed. We also give an algorithm to find all solutions of this GLCP and hence of a PWL circuit. The same technique can be applied to driving-point and transfer characteristics.

1 Introduction

The geometry of the solution set of PWL resistive circuits can be very complicated. The circuit can have multiple solutions, the solution set can be continuous or unbounded. Except for enumerative techniques, no algorithm is known for finding the complete solution of a PWL circuit.

A related problem is to trace driving point or transfer characteristics. Here the main difficulty lies in the detection of multiple branches of the curve.

In [1] and [2] the PWL circuits were solved by first synthesizing these as diode-resistor networks. The network equations of this diode-resistor network constitute a Linear Complementarity Problem (LCP), a mathematical programming problem well known as a unified description of a large class of problems in applied mathematics [3]. Most of the algorithms for solving LCP's (iterative methods, homotopy methods, optimization-based methods ...), have in common that they can only deal with a restricted class of problems and that they can find only one solution at the same time. This limits the applicability of complementarity theory to certain PWL resistive circuits. In this paper, the connection between PWL circuits and complementarity problems is extended by allowing a generalized form of the LCP. The application of a recent algorithm for solving this GLCP will also be discussed.

2 Description of piecewise-linear circuit elements

The circuits under study may contain the following elements :

1. all possible linear resistive elements,
2. piecewise linear two terminal resistors (we do not require the resistors to be either voltage- or current-controlled),

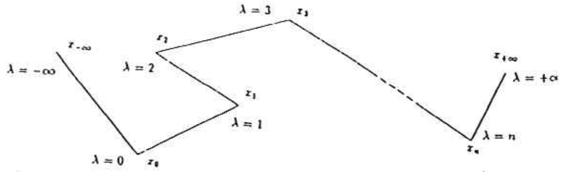

Figure 1: Parametrization of a PWL curve

3. piecewise linear controlled sources (all four types) with one controlling variable (the characteristics may be multi-valued).

The basic technique for the formulation of the GLCP associated to a circuit, is the parametrization given below.

A one-dimensional PWL curve in R^m (fig.1), characterized by a set of $n+1$ breakpoints $x_0, x_1, \ldots, x_n$ and two directions $x_{-\infty}$ and $x_{+\infty}$, can be described as

$$
\begin{aligned}
x &= x_0 + x_{-\infty}.\lambda^- + (x_1 - x_0).\lambda^+ \\
&\quad + \sum_{k=2}^{n} (x_k - 2x_{k-1} + x_{k-2}).(\lambda - k + 1)^+ \\
&\quad + (x_{+\infty} - x_n + x_{n-1}).(\lambda - n)^+ \qquad (1)
\end{aligned}
$$

where λ is a parameter running from $-\infty$ to $+\infty$. We have used the following notation for a real number x :

$$x^+ = \max(x, 0), \quad x^- = \max(-x, 0).$$

The same notation can be used for an n-vector x when all operations are assumed to be performed componentwise. An equivalent definition is:

$$
\begin{aligned}
x &= x^+ - x^- \\
x^+ &\geq 0, \quad x^- \geq 0 \; . \\
(x^+)^t.x^- &= 0
\end{aligned}
$$

Upon introducing the auxiliary variables

$$
\begin{aligned}
\lambda_1 &= \lambda - 1 \\
\lambda_2 &= \lambda - 2 \\
&\ldots \\
\lambda_n &= \lambda - n.
\end{aligned} \qquad (2)
$$

the equations (1) and (2) can be written in matrix form :

$$
\begin{aligned}
x &= x_0 + X^-.\Lambda^- + X^+.\Lambda^+ \\
E.(\Lambda^+ - \Lambda^-) &= e \\
\Lambda^- \geq 0, &\quad \Lambda^+ \geq 0 \\
(\Lambda^-)^t.\Lambda^+ &= 0
\end{aligned} \qquad (3)
$$

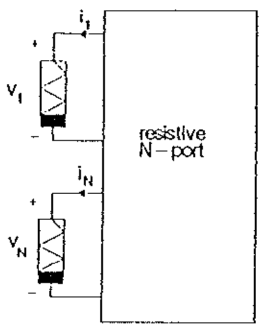

Figure 2: Extraction of the nonlinear resistors

where $\Lambda^- = [\lambda^- \;\; \lambda_1^- \;\; \ldots \lambda_n^-]^t$, $\Lambda^+ = [\lambda^+ \;\; \lambda_1^+ \;\; \ldots \lambda_n^+]^t$.

Multiterminal resistors can be described by using more general complementarity conditions $\sum_{i=1}^{l} \prod_{k \in B_i} \lambda_k = 0$, where the λ_k are nonnegative variables and each B_i is an arbitrary index set. This may be incorporated in our analysis, but will be omitted here for the sake of clarity.

3 Determination of the operating points of PWL circuits

The circuit equations can be written in standard form by first extracting all nonlinear resistors (fig.2). Let N be the number of nonlinear resistors. The N-port left behind can always be described by its constraint matrix C :

$$C.[i_1 \;\; v_1 \;\; i_2 \;\; v_2 \cdots i_N \;\; v_N]^t = b \tag{4}$$

where C is a matrix with $2N$ columns and an arbitrary number of rows. The k-th PWL resistor can be parametrized by two complementary vectors Λ_k^+ and Λ_k^- as in (1). Substituting these parametrizations in (4) leads to a set of equations

$$\begin{array}{c} M \begin{bmatrix} \Lambda_1^+ \\ \vdots \\ \Lambda_N^+ \end{bmatrix} + N \begin{bmatrix} \Lambda_1^- \\ \vdots \\ \Lambda_N^- \end{bmatrix} = q \\ \Lambda_i^- \geq 0, \;\; \Lambda_i^+ \geq 0 \;\; \text{for } i = 1 \ldots N \\ (\Lambda_i^+)^t.\Lambda_i^- = 0 \;\; \text{for } i = 1 \ldots N \end{array} \tag{5}$$

Solving this GLCP will yield the complete solution set.

4 Driving-point and transfer characteristics

The analysis given in the previous section can easily be extended to the determination of a driving point plot (i_{IN} vs. v_{IN}) as in fig. 3. In order to obtain the standard form we only have to split the unconstrained variables i_{IN} and v_{IN} as the difference of two complementary variables :

$$\begin{array}{c} i_{IN} = i_{IN}^+ - i_{IN}^- \\ v_{IN} = v_{IN}^+ - v_{IN}^- \\ i_{IN}^-, v_{IN}^-, i_{IN}^+, v_{IN}^+ \geq 0 \\ i_{IN}^-.i_{IN}^+ + v_{IN}^-.v_{IN}^+ = 0 \end{array}$$

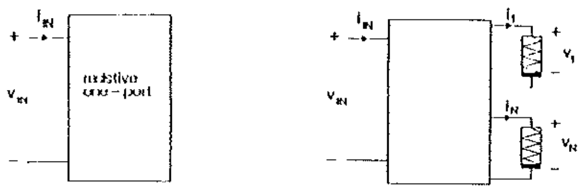

Figure 3: Determination of a driving-point plot

Together with the PWL description of the nonlinear resistors this again yields a GLCP (5). It is important to note that this GLCP will in general be rectangular (i.e. M and N will have more columns than rows).

A similar analysis can be carried out to obtain transfer characteristics.

5 The generalized linear complementarity problem

In the previous sections the circuit equations have been written in the standard form (5). The term Generalized Linear Complementarity Problem for this set of equations is justified by a number of differences with the classical formulation.

The linear complementarity problem :
Given $N \in R^{n \times n}$ and $q \in R^n$, find all solutions $w \in R^n$ and $z \in R^n$ to

$$\begin{array}{c} w + Nz = q \\ w \geq 0, \quad z \geq 0 \\ w^t.z = 0 \end{array} \tag{6}$$

where the shorthand notation $x \geq 0$ for vector inequalities holds componentwise.

A generalization of this problem was given in [4]. For our present purposes, the following form will be sufficient :

The generalized linear complementarity problem (GLCP)
Given $M \in R^{m \times n}$, $N \in R^{m \times n}$, $q \in R^m$, find all $w \in R^n$, $z \in R^n$ and $\alpha \in R$ such that

$$\begin{array}{c} Mw + Nz = q.\alpha, \\ w, z \geq 0, \;\; \alpha \geq 0 \\ w^t.z = 0 \end{array} \tag{7}$$

There are three distinct differences between this formulation and (6) :

- GLCP (7) allows rectangular LCP's ($m < n$). We have seen how these occur in the determination of driving-point and transfer characteristics.
- even if $m = n$, we do not exclude cases where no reordening of the variables w and z provides an invertible matrix M.
- the nonnegative scalar α has been added to allow for solutions at infinity, i.e. directions where the solution set is unbounded. The normalized solution set is the intersection of the solution set of (7) and the hyperplane $\alpha = 1$.

6 The solution set of linear systems of constrained equalities and the GLCP

In this section the geometric and algebraic properties of the solution set, and an algorithm to obtain the complete solution set are shortly described.

Consider first the set of linear equations with nonnegativity constraints :

$$\begin{array}{ll} A.x = 0, & \text{with a given } A \in R^{m\times n} \\ x \geq 0 & \end{array} \qquad (8)$$

The polyhedral cone $\mathcal{L}_A = \{x \geq 0 | Ax = 0\}$ can be defined completely by all positive linear combinations of its the extreme rays $\{v^1, v^2, \ldots, v^q\}$.

Theorem 1 *A necessary and sufficient condition for a solution $v \in \mathcal{L}_A$ to be an extreme ray is that no other solutions possess zeros at the same positions as v.*

Theorem 2 *A necessary and sufficient condition for two extreme solutions v and w to be adjacent is that there exist no other extreme solutions with zeros at the same positions as the common zeros of v and w.*

We are now ready to return to the GLCP (7), which will be treated as a set of linear equations

$$[M \;\; N \;\; -q] \begin{bmatrix} w \\ z \\ \alpha \end{bmatrix} = 0, \;\; w \geq 0, z \geq 0, \alpha \geq 0 \qquad (9)$$

with extra constraints $(w^t.z = 0)$. The solution set of (9) is then normalized, taking the intersection with the hyperplane $\alpha = 1$. This intersection is a generalized polytope determined by the set of vertices $v^i = [(w^i)^t \;\; (z^i)^t \;\; \alpha^i]^t$, with $\alpha^i = 1$ for a finite vertex or with $\alpha^i = 0$ for a vertex at infinity, where the polytope becomes unbounded.

Theorem 3 *The solution set of the GLCP (7) consists of all nonnegative combinations of vertices determined by (9) with the following restrictions :*

- *all vertices that are not complementary $((w^i)^t.z^i \neq 0)$ should be discarded,*
- *only convex combinations of cross-complementary vertices are allowed (vertices v^i and v^k with $(w^i)^t.z^k = (w^k)^t.z^i = 0$).*

In 1953 Motzkin et al. [5] proposed an algorithm for the solution of sets of linear inequalities that can be adapted for the solution of the GLCP [4,6]. We first describe the inductive algorithm for the solution of (8).

Call a_i^t = the ith row of A, then we denote with $S^k \subset R^{n\times q_k}$ the matrix formed by the q_k extreme rays of the solution set of

$$\begin{array}{l} a_i^t x = 0, \quad i = 1 \ldots k \\ x \geq 0 \end{array} \qquad (10)$$

- $S^0 = I_n$, the initial set of extreme rays generates the first orthant in R^n.
- The iteration describes how S^k is updated into S^{k+1}, when a new equality

$$a_{k+1}^t x = 0 \qquad (11)$$

is added. Put $(s^{k+1})^t = a_{k+1}^t.S^k$, a $1 \times q_k$ matrix. For each element in $(s^{k+1})^t$ three possibilities exist :

case 1 $s_j^{k+1} = 0$, indicating that S_j^k (j-th column in S^k) lies in the hyperplane $a_{k+1}^t x = 0$,

case 2 and 3 $s_j^{k+1} > 0$ or $s_j^{k+1} < 0$, indicating that S_j^k lies in either of the two halfspaces defined by $a_{k+1}^t x = 0$.

The construction of the extreme rays S^{k+1}, can then proceed as follows :

case 1 if an extreme ray in S^k lies in the hyperplane (11), it is also an extreme ray of S^{k+1},

case 2 and 3 any two adjacent extreme rays, lying on either side of hyperplane (11), define an extreme face, intersecting the hyperplane. This intersection is an extreme ray of S^{k+1} :
if $s_j^{k+1} < 0$ and $s_l^{k+1} > 0$ and S_j^k and S_l^k are adjacent, then $|s_j^{k+1}|.S_l^k + |s_l^{k+1}|.S_j^k \in S^{k+1}$. The adjacency test is described in theorem 2.

The GLCP algorithm is now obvious from a combination of the inductive algorithm for the solution of (8), and theorem 3.

7 Example

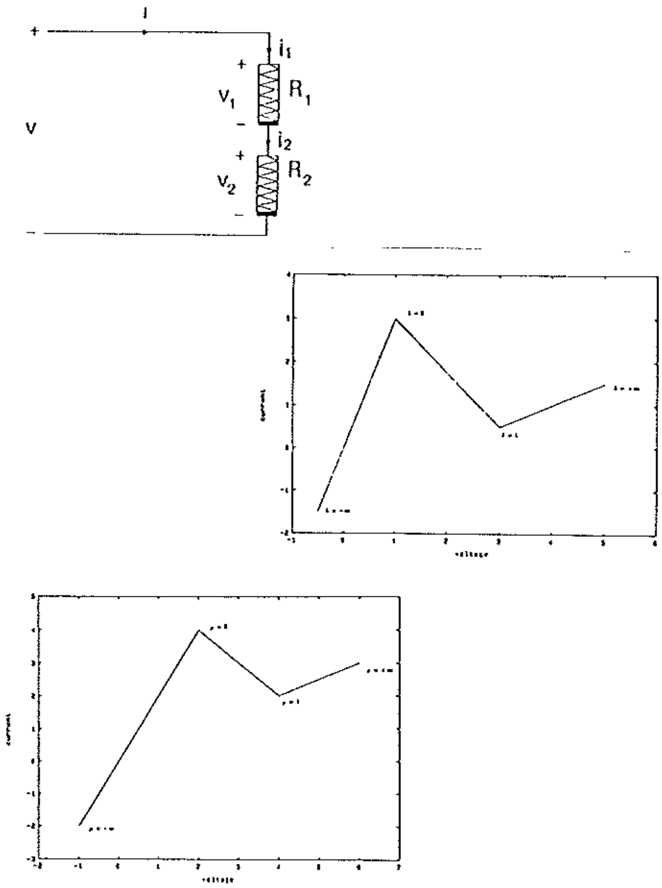

Figure 4: Series connection of two PWL tunnel diodes

We compute the driving-point characteristic for the series connection of two tunnel-diodes (fig.4), whose parametrization is indicated in the figure. The algorithms returns 11 solutions :

	1	2	3	4	5	6	7	8	9	10	11
i^-	0.77	0	0	0	0	0	0	0	0	0	0
v^-	0.64	0	0	0	0	0	0	0	0	0	0
λ^-	0.26	1	0	0	0	0	0	0	0	0.33	0
λ_1^-	0.26	2	1	0	0	0	0	0.6	1	1.33	1
μ^-	0.19	1	0.25	0.88	0	0	0	0	0	0	0
μ_1^-	0.19	2	1.25	1.88	1	0	0	0	0	0	0.5
i^+	0	0	3	0.5	4	2	0.29	2	3	2	3
v^+	0	0	2.5	3.25	12	10	1.48	5.8	7	4.67	4
λ^+	0	0	0	1	4.5	2.5	0.29	0.4	0	0	0
λ_1^+	0	0	0	0	3.5	1.5	0.29	0	0	0	0
μ^+	0	0	0	0	0	1	0.29	1	2	1	0.5
μ_1^+	0	0	0	0	0	0	0.29	0	1	0	0
α	0	1	1	1	1	1	0	1	1	1	1
i	-0.64	0	3	0.50	4	2	0.29	2	3	2	3
v	-0.77	0	2.5	3.25	12	10	1.48	5.8	7	4.67	4

Verifying cross-complementarity conditions allows to trace the complete DP-plot which consists of two pieces : **1 – 2 – 3 – 4 – 5 – 6 – 7** and **8 – 9 – 10 – 11 – 8** (fig. 5).

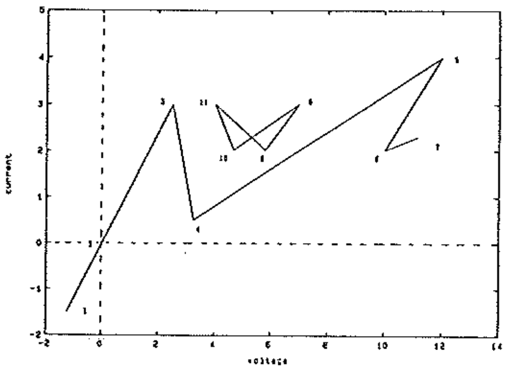

Figure 5: Driving-point plot of the circuit in fig.4

8 Conclusions

The most prominent advantages of this method are as follows :

- a complete description of the solution set is given, even when a continuum of solutions occurs, or when the solution set is unbounded in some direction,
- no restrictions are imposed on the linear part of the circuit, or on the existence of any hybrid representation,
- we allow nonlinear resistors which are neither voltage nor current-controlled.

General statements on computational complexity are difficult because much is dependent on the order in which the equations are processed. Further practical experience is needed before guidelines can be formulated.

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