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The generalized linear complementarity problem and an algorithm to find all its solutions

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Motivated by a number of typical applications, a generalization of the classical *linear complementarity problem* is presented together with an algorithm to determine the complete solution set. The algorithm is based on the double description method for solving linear inequalities and succeeds in describing continuous as well as unbounded solution sets.

Key words: Linear complementarity, piecewise linear equations, double description method.

1. Introduction

The *generalized linear complementarity problem* (GLCP) can be formulated as follows:

Given a real $m \times n$ matrix M, find all real elementwise nonnegative vectors $x = [\xi_1, \xi_2, \ldots, \xi_n]^t \geqslant 0$ such that

$$Mx = 0, \tag{1}$$

$$C(x) \triangleq \sum_{i=1}^{l} \prod_{k \in \mathscr{B}_i} \xi_k = 0. \tag{2}$$

Equation (2) consists of l complementarity conditions of the form $\prod_{k \in \mathscr{B}_i} \xi_k = 0$ (where each $\mathscr{B}_i$ is a given subset of $\{1, \ldots, n\}$). This formulation is equivalent to the generalization of the LCP stated by Cottle and Dantzig in [3].

The LCP has been recognized as a unifying description of a large class of problems, including linear and quadratic programming [2], fixed point problems and sets of piecewise linear equations [7, 8], bimatrix equilibrium points [9] and variational inequalities [4]. An impressive collection of applications can be found in [12].

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Typical electrical engineering applications lie in the analysis of piecewise linear resistive circuits [14, 16] and in optimization. In [17] we have applied the insights of this paper to the analysis and complete solution of piecewise linear resistive electrical circuits.

An obvious, exponential algorithm for solving the GLCP (1)–(2) consists in enumerating all sign patterns allowed by the complementarity conditions and checking their feasibility. Excluding such drastic methods, no method is known for processing GLCPs of the form (1)–(2) although the recently described algorithm of [1] could possibly be modified to solve our GLCP.

The aim of this paper is to indicate the relevance of the generalized form of the LCP in electrical engineering applications, and to describe a new method for implicitly generating all solutions together with a geometrical description of the solution set. In essence, the proposed method interleaves the construction of vertices of the polytope induced by the equality and inequality constraints with the verification of their complementarity.

The paper is organized as follows. In Section 2, some motivating examples of the GLCP are presented. Section 3 contains a geometrical description of the solution set of the GLCP, which will lead us to the statement of the algorithm in Section 4. Section 5 provides some insight in the computational complexity of the algorithm.

2. Generalizations of the LCP

In this section, we motivate the generalized form of the LCP as in (1)–(2) with a number of typical examples. The interested reader may find additional examples in [6, 17].

2.1. Current-voltage characteristic of a series connection of two piecewise linear resistors

A simple, but typical application of complementarity theory in circuit analysis is depicted in Figure 1. The problem is to trace the current-voltage characteristic of the series connection of the two resistors. We start with the formulation of Kirchhoff's laws:

$$\begin{bmatrix} 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v \\ i \\ v_1 \\ i_1 \\ v_2 \\ i_2 \end{bmatrix} = 0. \tag{3}$$

The resistor characteristics can be parametrized with the parameters λ and μ as shown in Figure 1. Using the notation

$$x^+ = \max(0, x), \qquad x^- = \max(0, -x),$$

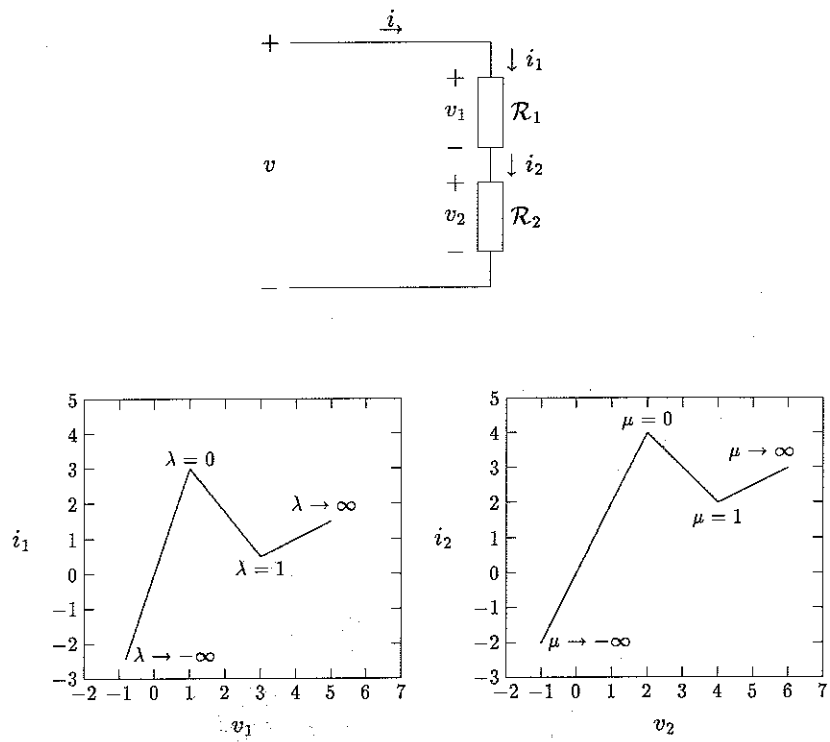

Fig. 1. Series connection of two piecewise linear resistors and their respective current-voltage characteristics.

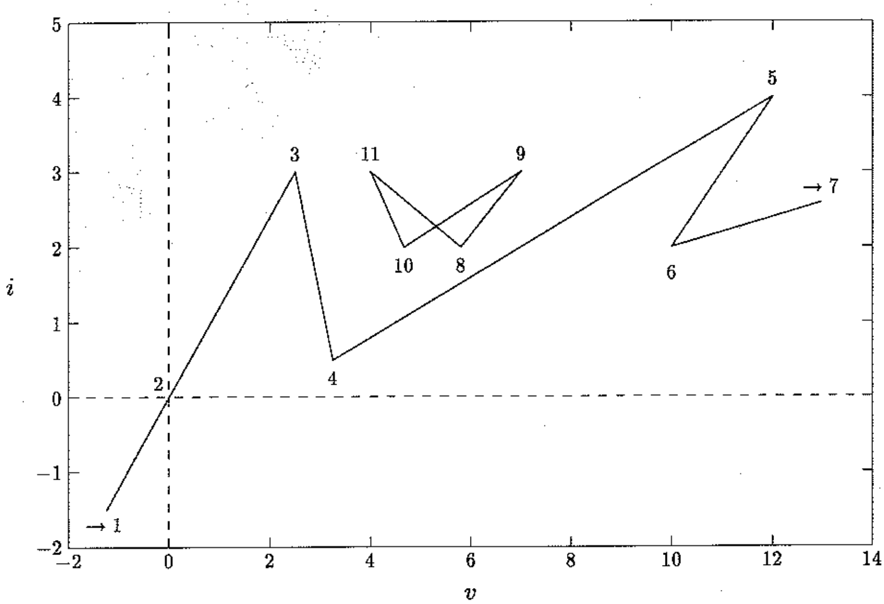

Fig. 2. Resulting current-voltage characteristic of the series connection circuit in Figure 1.

this leads to the equations:

- Resistor $\mathscr{R}_1$:

$$\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ -3 \end{bmatrix} \lambda^- + \begin{bmatrix} 2 \\ -2.5 \end{bmatrix} \lambda^+ + \begin{bmatrix} 0 \\ 3.5 \end{bmatrix} \lambda_1^+, \tag{4}$$

where we used a new variable $\lambda_1 = \lambda - 1$ or

$$\lambda_1^+ - \lambda_1^- = \lambda^+ - \lambda^- - 1. \tag{5}$$

- Resistor $\mathscr{R}_2$:

$$\begin{bmatrix} v_2 \\ i_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \end{bmatrix} \mu^- + \begin{bmatrix} 2 \\ -2 \end{bmatrix} \mu^+ + \begin{bmatrix} 0 \\ 3 \end{bmatrix} \mu_1^+, \tag{6}$$

with

$$\mu_1^+ - \mu_1^- = \mu^+ - \mu^- - 1.$$

Substituting equations (4, 5, 6) into (3) delivers the LCP:

$$\begin{bmatrix} -1 & 0 & 1 & 0 & 2 & 0 \\ 0 & -1 & 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 4 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} v^- \\ i^- \\ \lambda^- \\ \lambda_1^- \\ \mu^- \\ \mu_1^- \end{bmatrix}$$

$$+ \begin{bmatrix} 1 & 0 & -2 & 0 & -2 & 0 \\ 0 & 1 & 2.5 & -3.5 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & -3 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} v^+ \\ i^+ \\ \lambda^+ \\ \lambda_1^+ \\ \mu^+ \\ \mu_1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 4 \\ 1 \\ 1 \end{bmatrix} \tag{7}$$

with

$$v^-, i^-, \lambda^-, \lambda_1^-, \mu^-, \mu_1^- \geqslant 0,$$
$$v^+, i^+, \lambda^+, \lambda_1^+, \mu^+, \mu_1^+ \geqslant 0,$$
$$v^- \cdot v^+ + i^- \cdot i^+ + \lambda^- \cdot \lambda^+ + \lambda_1^- \cdot \lambda_1^+ + \mu^- \cdot \mu^+ + \mu_1^- \cdot \mu_1^+ = 0.$$

The solution to this problem can be computed by the algorithm of Section 4 and is represented in Figure 2. Equations (7) are of the following type:

$$M_1 w + M_2 z = q, \quad M_1 \in \mathbb{R}^{m \times n}, \quad M_2 \in \mathbb{R}^{m \times n} \tag{8}$$

subject to

$$w, z \geqslant 0, \qquad w^t z = 0.$$

Observe that:

– This LCP is rectangular, i.e., the number of equations is less than the number of complementary pairs ($m = 5 < n = 6$).

– Even if the LCP were square, nothing guarantees the existence of a reordering of the columns of M_1 and M_2 such that the new M_1 becomes invertible and the GLCP reduces to the classical form. Moreover this would require a matrix inversion that might be ill-conditioned.

– In order to trace the complete i–v-characteristic, it is important to find *all* solutions to (8), including the solutions at infinity. This can be achieved by adding to equations (8) an unknown real scalar $\rho \geqslant 0$:

$$M_1 w + M_2 z = q\rho,$$
$$w \geqslant 0, \quad z \geqslant 0, \quad \rho \geqslant 0,$$
$$w^{\mathrm{t}} z = 0.$$

We can now convert this set of equations to the form of the GLCP (1)–(2):

$$[M_1 \ \ M_2 \ \ -q] \begin{pmatrix} w \\ z \\ \rho \end{pmatrix} = 0,$$

with the conditions

$$w \geqslant 0, \quad z \geqslant 0, \quad \rho \geqslant 0, \quad w^{\mathrm{t}} z = 0.$$

Solutions with $\rho \neq 0$ can be normalized to satisfy $\rho = 1$. A solution vector with $\rho = 0$ will be called a *solution at infinity*. These solutions often have physical relevance. For instance, in Figure 2, the solutions at infinity generate the 2 branches left and right (labelled with 1 and 7) for which the breaking point corresponds to a direction.

2.2. A geometrical problem

The resistors in the above examples are two-terminal devices and can be described by a one-dimensional piecewise linear manifold. Many electronic circuits contain *multiterminal* nonlinear devices, which can be modelled by higher-dimensional piecewise linear manifolds. These manifolds can be parametrized in an elegant way by using the more general complementarity conditions in (2) (see [6] for details). The following geometrical problem should give an idea of the usefulness of these generalized complementarity conditions. Consider the object formed by the edges of the tetrahedron in $\mathbb{R}^3$, generated by the origin and the three unit vectors (1, 0, 0), (0, 1, 0) and (0, 0, 1). This object can be described compactly as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \xi_1 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \xi_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \xi_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \xi_4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{9}$$

with $\xi_1, \xi_2, \xi_3, \xi_4 \geqslant 0$, $\xi_1 + \xi_2 + \xi_3 + \xi_4 = 1$ and

$$\xi_1\xi_2\xi_3 + \xi_1\xi_2\xi_4 + \xi_1\xi_3\xi_4 + \xi_2\xi_3\xi_4 = 0. \tag{10}$$

Similar objects could be defined by varying the complementarity conditions, and geometrical problems involving such objects lead to equations of the form (1)–(2).

2.3. The regular linear complementarity problem

In conclusion, it can be pointed out that these examples can be captured as special cases of the GLCP (1)–(2) stated in the introduction. This formulation includes the classical LCP

$$P \cdot \begin{bmatrix} w_1 \\ \vdots \\ w_p \end{bmatrix} + \begin{bmatrix} z_1 \\ \vdots \\ z_p \end{bmatrix} = \begin{bmatrix} q_1 \\ \vdots \\ q_p \end{bmatrix}$$

as a special case:

$$M = [P \vdots I_p],$$

$$n = 2p, \quad l = p,$$

$$\alpha_1, \ldots, \alpha_{2p} = w_1, \ldots, w_p, z_1, \ldots, z_p,$$

$$\mathcal{B}_k = \{k, k+p\}, \quad k = 1, \ldots, p.$$

3. Geometric description of the solution set of systems of constrained equalities and of the GLCP

The GLCP can be viewed as a set of linear equations with nonnegativity constraints and complementarity requirements. We drop the latter for a while and discuss the geometric properties of the solution set of

$$\begin{aligned} &Ax = 0, \quad \text{with a given } A \in \mathbb{R}^{m \times n}, \\ &x \geqslant 0. \end{aligned} \tag{11}$$

The geometrical description of the solution set will be summarized in Theorems 1 and 2 below. These theorems follow from basic facts that can be found in most textbooks on linear programming (e.g. [13]). In Theorem 3, we take the complementarity conditions into account and describe the general solution set of the GLCP.

The pointed *polyhedral cone* $\mathcal{L}_A = \{x \geqslant 0 | Ax = 0\}$ (the intersection of the first orthant in $\mathbb{R}^n$ and the subspace ker A) can be defined completely by all positive linear combinations of its q extreme rays $\{v^1, v^2, \ldots, v^q\}$.

Definition 1. A nonzero vector $v^i \in \mathbb{R}^n$ is an *extreme ray* of $\mathcal{L}_A$ if there exists a hyperplane $V = \{x \in \mathbb{R}^n \,|\, h^t x = 0\}$ such that $V \cap \mathcal{L}_A = \{x \,|\, x = \lambda v^i, \lambda \geqslant 0\}$.

The following result provides necessary and sufficient conditions for a vector to be an extreme ray.

Theorem 1. *A* necessary and sufficient *condition for a solution $v \in \mathcal{L}_A$ to be an extreme ray is that no other nonzero solutions possess zeros at the same positions as v: call $\mathcal{I}_v = \{k \,|\, v_k = 0\}$, then v is an extreme ray of $\mathcal{L}_A$ iff $\mathcal{I}_v \subseteq \mathcal{I}_w$, $Aw = 0$, $w \geqslant 0$ implies $w = v$.* □

Corollary 1. *If the rows of A are independent, then a* necessary *condition for extremity is that the number of zeros in v is greater than or equal to $n-m-1$.* □

Definition 2. Two extreme rays v and w are *adjacent* if there exists a supporting hyperplane $V=\{x \mid h^t x=0,\ h^t z \geqslant 0,\ \forall z \in \mathscr{L}_A\}$ such that $V \cap \mathscr{L}_A=\{x=\lambda_1 v+\lambda_2 w, \lambda_1, \lambda_2 \geqslant 0\}$. The set of all convex combinations of two adjacent rays is called a *two-dimensional face* of the cone.

Theorem 2. *A* necessary and sufficient *condition for two extreme solutions v and w to be adjacent is that there exist no other extreme solutions with zeros at the same positions as the common zeros of v and w. Call $\mathscr{I}_{vw}=\{k \mid v_k=0$ and $w_k=0\}$, then v and w are adjacent iff $\mathscr{I}_{vw} \subseteq \mathscr{I}_z$, $Az=0$, $z \geqslant 0$ implies $z=\lambda_1 v+\lambda_2 w$ for some $\lambda_1, \lambda_2 \geqslant 0$.* □

Corollary 2. *If the rows of A are linearly independent, then a* necessary *condition for extreme solutions to be adjacent is that the number of common zeros in v and w is greater than or equal to $n-m-2$.* □

We are now ready to return to the GLCP (1)-(2), which will be treated as a set of linear equations

$$\begin{aligned} &Mx=0, \quad x^t=(\xi_1, \xi_2, \ldots, \xi_n), \\ &\xi_i \geqslant 0, \end{aligned} \tag{12}$$

with extra constraints

$$C(x)=\sum_{i=1}^{l} \prod_{k \in \mathscr{B}_i} \xi_k=0. \tag{13}$$

The solution set of (12) *generalized polytope* determined by a set of vertices. For instance, if $x^t=[\xi_1\ \xi_2 \ldots \xi_n\ \rho]$ the solution set is a generalized polytope defined by a set of vertices $(v^i)^t=[\xi_1^i\ \xi_2^i \ldots \xi_n^i\ \rho^i]$, with $\rho^i=1$ for a finite vertex or with $\rho^i=0$ for a vertex at infinity, where the polytope becomes unbounded.

The complementarity conditions (13) change this picture quite drastically. In order to provide a geometrical description of the solution set to (12)-(13), we first need to introduce the notion of *cross-complementarity*:

Definition 3. A set of nonnegative complementary vectors v^l, $l \in \mathscr{B}$, $C(v^l)=0$, where $\mathscr{B}$ is a given index set are called *cross-complementary* if their sum is complementary: $C(\sum_{l \in \mathscr{B}} v^l)=0$. This implies that each convex combination of these vertices satisfies the complementarity condition.

We are now in a position to describe the solution set of the GLCP (12)-(13):

Theorem 3. *The solution set of the GLCP*

$$Mx=0, \quad x \geqslant 0, \quad C(x)=0 \tag{14}$$

consists of all convex combinations of vertices of

$$Mx = 0, \quad x \geqslant 0, \tag{15}$$

with the following restrictions:

- *all vertices of* (15) *that are not complementary* $(C(x) > 0)$ *should be discarded,*
- *only convex combinations of vertices that are cross-complementary are allowed.*

Hence, the solution set is a collection of generalized polytopes. □

These generalized polytopes correspond to the cliques in a graph where each node represents a complementary vertex of (15), and where two nodes are joined by an edge if the corresponding vertices are cross-complementary.

4. The GLCP-algorithm

In 1953 Motzkin et al. ([11]) proposed an algorithm for the solution of sets of linear inequalities that can easily be adapted for the GLCP [5, 6, 17]. We first describe an inductive algorithm for the solution of (11) and then investigate the modifications necessary to include the complementarity conditions.

Call $(a^i)^t =$ the ith row of A. We denote by $S^k \in \mathbb{R}^{n \times q_k}$ the matrix formed by the q_k extreme rays of the solution set of

$$\begin{aligned} &(a^i)^t x = 0, \quad i = 1, \ldots, k, \\ &x \geqslant 0. \end{aligned} \tag{16}$$

Starting from $S^0 = I_n$, the algorithm moves inductively from S^k to S^{k+1} by computing the intersection between S^k and the hyperplane $\{x \mid (a^{k+1})^t x = 0\}$. This can be done as follows:

Let $(s^{k+1})^t = (a^{k+1})^t S^k$, a $1 \times q_k$ matrix. For each element in $(s^{k+1})^t$ three possibilities exist:

Case 1. $s_j^{k+1} = 0$, indicating that S_j^k (jth column in S^k) lies in the hyperplane $(a^{k+1})^t x = 0$.

Case 2 and 3. $s_j^{k+1} > 0$ or $s_j^{k+1} < 0$, indicating that S_j^k lies in either of the two halfspaces defined by $(a^{k+1})^t x = 0$.

The construction of the extreme rays S^{k+1}, can then proceed as follows:

Case 1. If an extreme ray in S^k lies in the hyperplane $(a^{k+1})^t x = 0$ it is also an extreme ray of S^{k+1}.

Case 2 and 3. The extreme face generated by any two adjacent extreme rays that lie on either side of hyperplane $(a^{k+1})^t x = 0$, intersects the hyperplane. This intersection is an extreme ray of S^{k+1}: If $s_j^{k+1} < 0$ and $s_l^{k+1} > 0$ and S_j^k and S_l^k are adjacent (cf. Definition 2), then $|s_j^{k+1}| \cdot S_l^k + |s_l^{k+1}| \cdot S_j^k \in S^{k+1}$. The adjacency tests are described in Theorem 2 and Corollary 2.

The validity of these operations is an immediate consequence of Theorems 1 and 2:

– An extreme ray of S^k, lying in the hyperplane, will still satisfy the condition for extremity in Theorem 1 after updating.

– Theorem 2 implies that for any convex combination of two adjacent extreme rays that lies in the hyperplane $(a^{k+1})^t x = 0$, the condition in Theorem 1 will be fulfilled after updating.

The GLCP algorithm is now obvious from a combination of

- the inductive algorithm for the solution of (11),
- Theorem 3 which allows to eliminate at each stage those vertices of S^k that do not satisfy the complementarity condition. In other words, in Case 2 and 3 of the algorithm, an extra requirement will be added: two extreme rays that lie on either side of the hyperplane are to be combined only if they are *adjacent* and *cross-complementary*.

The matrix S^k then contains at each stage the solution of the rectangular $k \times n$ GLCP, formed by the first k rows of (12).

We summarize the algorithm for the solution of (12)–(13) where $M \in \mathbb{R}^{m \times n}$:

GLCP-algorithm.

- $S_0 = I_n$, the $n \times n$ identity matrix,
- **for** $k = 1$ **to** m

  compute $(s^k)^t = (m^k)^t S^{k-1}$.

  – for all $s_j^k = 0$, add S_j^{k-1} to S^k;

  – for all $s_j^k < 0$ and all $s_l^k > 0$, if S_j^{k-1} and S_l^{k-1} are adjacent and cross-complementary vertices of S^{k-1}, add $|s_j^k| S_l^{k-1} + |s_l^k| S_j^{k-1}$ to S^k;

  end
- Let S^m be the final set of solution vertices. The geometrical description of the solution set can be obtained by detecting all maximal subsets of vertices that are cross-complementary (=enumerating all cliques).

Let us conclude this section with the following remarks:

– For large problems the computationally most demanding step is the verification whether all pairs of vertices lie at opposite sides of the hyperplane, especially when the necessary and sufficient condition for adjacency stated in Theorem 2 is used. One might consider to replace this condition by the much simpler necessary condition of Corollary 2, at risk of leaving a number of redundant vectors in the solution set. These redundant vectors can be eliminated from the final solution set by applying Theorem 1 and its corollary.

– In this context, the importance of the complementarity tests in each step should be stressed. It has been experienced that, depending on the order in which the equations are taken, the complementarity tests cause a dramatic decrease of the number of vertices as will be demonstrated in the numerical examples below.

– Another important aspect that influences the computational complexity is the *sparsity* of the matrix M of the GLCP. The sparser this matrix, the smaller the number of intermediate vertices that is to be processed in each step. This will clearly be demonstrated in the examples below.

– Algorithms for the clique enumeration problem can be found in the literature (see, e.g., [15]). Although the enumeration of all cliques in a given graph can be a formidable task in general, it is not the most demanding step in the present application. In many cases the solutions to the GLCP will even be isolated and the clique enumeration will be trivial.

5. Influence of sparsity

In these examples, we investigate the influence of the sparsity of the matrices on the number of vertices that has to be processed in each recursion of our GLCP algorithm. We have generated LCPs of the form $w = Mz + q$ with w, $z \geqslant 0$ and $w^t z = 0$, where a specified number of elements of M and q is zero. The elements of M have a normal distribution and the elements of q are uniformly distributed between 0 and 1. Nonnegativity of q guarantees the existence of at least one solution. Three sets of experiments have been performed for M being of dimensions 10, 15 and 20. For each dimension, we have generated 10 random LCPs for 4 different sparsity percentages: 90% zeros, 60%, 30% and 0% zeros. The results are summarized in Figure 3. The following conclusions can be drawn from these results:

– If the matrix M is very sparse (90%), our GLCP algorithm is quite effective as the number of intermediate vertices on the average decreases as a function of the number of processed rows. This could be an advantage as compared to conventional methods based on pivoting, which tends to destroy sparsity. For a noise level of 60% we see that the intermediate number of vertices increases, an effect that becomes

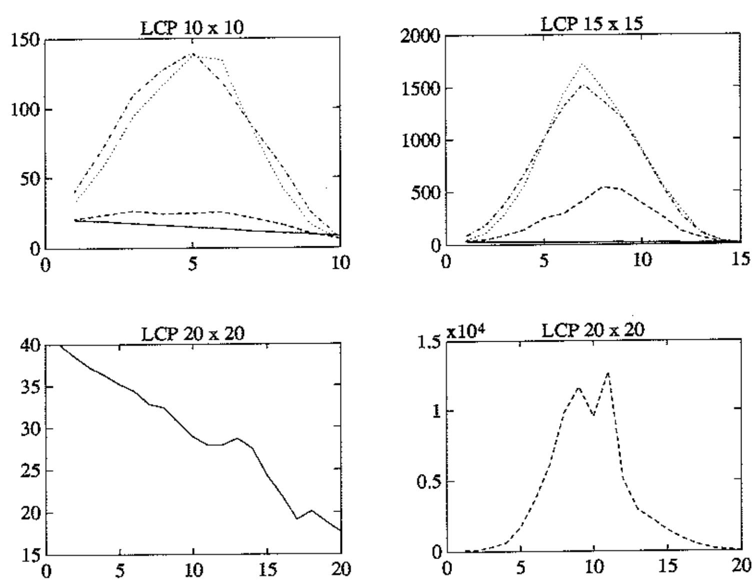

Fig. 3. Average number of vertices as a function of processed rows with full line (90% sparsity); —— (60% sparsity); ··· (30% sparsity); –·–· (0% sparsity); For the LCPs of dimension 20, the lower left plot is the average for the 90% sparsity case over 10 random LCPs, while the lower right is the average for only 2 random LCPs of dimension 20 with sparsity 60%.

more pronounced as the matrices become less sparse. This phenomenon can be explained as follows. In the first stages of the algorithm the intermediate vertices are still relatively sparse and the complementarity constraints are easily satisfied. For increasing k however, there is a gradual fill-in and more and more vertices are discarded as noncomplementary.

– One of the reviewers pointed out that the observation in [10] might also be applicable to our LCP algorithm, namely that the double description method outperforms pivoting methods on small systems of linear inequalities, but that for large problems, pivoting methods are better. We would however like to suggest that the combined effect of *sparsity* and the *presence of complementarity conditions* might enhance the efficiency of the double description method. This is however a subject of current research.

6. Conclusions

The method described in this paper is expected to be particularly useful when one is interested in all solutions to the linear complementarity problem or in those applications where the LCP does not have the classical standard form. Typical problems are large and sparse and have no specific matrix structure nor property (such as P-matrices etc. [12]) which could be exploited. The extensions that we have introduced include rectangular LCPs, "singular" cases where no reordering of the complementary variables w and z yields an LCP in standard form, problems with more general complementarity conditions. Our algorithm to solve the GLCP delivers all solutions to the problem, including those at infinity, which often have physical relevance. Apparently, the efficiency increases with the degree of sparsity. Though some preliminary numerical results have been reported, further research remains to be done on a clear analysis of the computational complexity.

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