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ON THE STRUCTURE OF GENERALIZED SINGULAR VALUE AND QR DECOMPOSITIONS*

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Abstract. This paper analyzes in detail the structure of generalizations of the singular value decomposition and the QR decomposition for any number of matrices. The structure is completely determined as a function of the ranks of the matrices or their products and concatenations.

Key words. ordinary, product, quotient, restricted singular value decomposition, QR decomposition, complete orthogonal factorization

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1. Introduction. In a previous paper [4], we introduced an infinite tree of generalizations of the ordinary singular value decomposition (OSVD) and we derived a constructive proof of it. All decompositions in this tree are considered as generalized singular value decompositions (GSVDs) and it was shown in [4] how all of them can be labeled with a sequence of the letters P and Q, where P stands for *product* and Q stands for *quotient*. In [5], we introduced a corresponding set of generalizations of the QR decomposition, which could be denoted by appropriate enumerations of the letters L (lower) and U (upper). It is the purpose of this paper to discuss in more detail the structure of these generalizations. In particular, we shall derive formulas for the dimensions of the blocks in the quasi-diagonal matrices of the GSVDs of [4] (Theorem 1.1 of this paper), or the triangular matrices in the GQRDs (generalized QR decompositions) of [5] (Theorem 1.2 of this paper), in terms of the ranks of the matrices involved and concatenations and products of these matrices.

This paper is organized as follows. In the remainder of this section, we summarize the main results on generalized SVDs and QRDs obtained in [4] and [5]. Since there is a one-to-one correspondence between these two generalizations, we will concentrate on the generalizations of the SVD, while the results will apply for the GQRDs as well. In §2, we analyze in detail the structure of a GSVD that only consists of P-steps. In §3, we analyze GSVDs that only contain Q-steps. In §4, we discuss the general case where we exploit the obtained insights from §§2 and 3. Instead of providing rigorous proofs, we have chosen to indicate our methods of deriving these results with illustrative examples.

Let us first state the main result of [4] in the following theorem.

THEOREM 1.1 (GSVDs for k matrices). *Consider a set of k matrices with compatible dimensions: A_1 $(n_0 \times n_1), A_2$ $(n_1 \times n_2), \ldots,$ A_{k-1} $(n_{k-2} \times n_{k-1}), A_k$ $(n_{k-1} \times n_k)$. Then there exist*

- *unitary matrices U_1 $(n_0 \times n_0)$ and V_k $(n_k \times n_k)$;*

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- *matrices $D_j, j = 1, 2, \ldots, (k-1)$ of the form*

$$\underset{n_{j-1}\times n_j}{D_j} = \begin{array}{l} \\ r_j^1 \\ r_{j-1}^1 - r_j^1 \\ r_j^2 \\ r_{j-1}^2 - r_j^2 \\ r_j^3 \\ \ldots \\ \ldots \\ r_j^j \\ n_{j-1} - r_{j-1} - r_j^j \end{array} \begin{array}{c} \begin{array}{ccccccc} r_j^1 & r_j^2 & r_j^3 & \ldots & \ldots & r_j^j & n_j - r_j \end{array} \\ \begin{pmatrix} I & 0 & 0 & \ldots & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & \ldots & 0 & 0 \\ 0 & I & 0 & \ldots & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & \ldots & 0 & 0 \\ 0 & 0 & I & \ldots & \ldots & 0 & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots & I & 0 \\ 0 & 0 & \ldots & \ldots & \ldots & 0 & 0 \end{pmatrix} \end{array},$$

where the integers r_j are the ranks of the matrices A_j, satisfying

$$r_j = \mathrm{rank}(A_j) = \sum_{i=1}^{j} r_j^i;$$

- *a matrix S_k of the form*

$$\underset{n_{k-1}\times n_k}{S_k} = \begin{array}{l} \\ r_k^1 \\ r_{k-1}^1 - r_k^1 \\ r_k^2 \\ r_{k-1}^2 - r_k^2 \\ r_k^3 \\ \ldots \\ \ldots \\ r_k^k \\ n_{k-1} - r_{k-1} - r_k^k \end{array} \begin{array}{c} \begin{array}{ccccccc} r_k^1 & r_k^2 & r_k^3 & \ldots & \ldots & r_k^k & n_k - r_k \end{array} \\ \begin{pmatrix} S_k^1 & 0 & 0 & \ldots & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & \ldots & 0 & 0 \\ 0 & S_k^2 & 0 & \ldots & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & \ldots & 0 & 0 \\ 0 & 0 & S_k^3 & \ldots & \ldots & 0 & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots & S_k^k & 0 \\ 0 & 0 & \ldots & \ldots & \ldots & 0 & 0 \end{pmatrix} \end{array}.$$

The $r_k^i \times r_k^i$ matrices S_k^i are diagonal with positive diagonal elements.

- *Nonsingular matrices X_j $(n_j \times n_j)$ and Z_j, $j = 1, 2, \ldots, (k-1)$, where Z_j is either $Z_j = X_j^{-*}$ or either $Z_j = X_j$ (i.e., both choices are always possible), such that the given matrices can be factorized as*

$$\begin{aligned} A_1 &= U_1 D_1 X_1^{-1}, \\ A_2 &= Z_1 D_2 X_2^{-1}, \\ A_3 &= Z_2 D_3 X_3^{-1}, \\ \cdots &= \cdots, \\ A_i &= Z_{i-1} D_i X_i^{-1}, \\ \cdots &= \cdots, \\ A_k &= Z_{k-1} S_k V_k^*. \end{aligned}$$

Expressions for the integers r_j^i are given below; they are ranks of certain matrices in the constructive proof of this theorem [4].

Observe that the matrices D_j and S_k are generally not diagonal. Their only nonzero blocks however are diagonal block matrices. Observe that we always take the last factor in every factorization as the inverse of a nonsingular matrix, which is only a matter of convention. (Another convention would result in a modified definition of the matrices Z_i.) As to the name of a certain GSVD, we propose to adopt the following convention.

DEFINITION 1 (Nomenclature for GSVDs). If $k = 1$ in Theorem 1.1, then the corresponding factorization of the matrix A_1 will be called the OSVD. If for a matrix pair $A_i, A_{i+1}, 1 \leq i \leq k-1$ in Theorem 1.1, we have that $Z_i = X_i$ then, the factorization of the pair will be said to be of P-type. If, on the other hand, for a matrix pair $A_i, A_{i+1}, 1 \leq i \leq k-1$ in Theorem 1.1, we have that $Z_i = X_i^{-*}$ the factorization of the pair will be said to be of Q-type. The name of a GSVD of the matrices A_i, $i = 1, 2, \ldots, k > 1$ as in Theorem 1.1, is then obtained by simply enumerating the different factorization types.

We now give some examples.

Example 1. Consider two matrices A_1 $(n_0 \times n_1)$ and A_2 $(n_1 \times n_2)$. Then, we have the following two possible GSVDs.

	P-type	Q-type
A_1	$U_1 D_1 X_1^{-1}$	$U_1 D_1 X_1^{-1}$
A_2	$X_1 S_2 V_2^*$	$X_1^{-*} S_2 V_2^*$

The P-type factorization corresponds to the PSVD (product singular value decomposition) as in [9] (called ΠSVD there) and in [1] and [3], while the Q-type factorization is nothing else than the QSVD (quotient singular value decomposition) in [8], [10], and [11] (called generalized SVD there). A P-type factorization is precisely the kind of transformation that occurs in the PSVD while a Q-type factorization occurs in the QSVD.

Example 2. Let us write down the PQQP-SVD for five matrices:

$$\begin{aligned}
A_1 &= U_1 D_1 X_1^{-1}, \\
A_2 &= X_1 D_2 X_2^{-1}, \\
A_3 &= X_2^{-*} D_3 X_3^{-1}, \\
A_4 &= X_3^{-*} D_4 X_4^{-1}, \\
A_5 &= X_4 S_5 V_5^*.
\end{aligned}$$

In [5], we derived the following generalization of the QR decomposition for a chain of k matrices.

THEOREM 1.2 (Generalized QR decompositions for k matrices). *Given k complex matrices A_1 $(n_0 \times n_1)$, A_2 $(n_1 \times n_2)$, $\ldots$, A_k $(n_{k-1} \times n_k)$. There always exist unitary matrices $Q_0, Q_1, \ldots, Q_k$ such that $\tilde{T}_i = Q_{i-1}^* A_i Q_i$, where $\tilde{T}_i$ is a lower triangular or upper triangular matrix (both cases are always possible) with the following structure.*

Lower triangular (which will be denoted by a superscript l):

$$\tilde{T}_i^l = \begin{array}{c} \\ r_{i-1}^1 \\ r_{i-1}^2 \\ \vdots \\ r_{i-1}^i \end{array} \!\!\begin{array}{c} \begin{array}{ccccc} r_i^1 & r_i^2 & \ldots & r_i^i & r_i^{i+1} \end{array} \\ \begin{pmatrix} R_{i,1} & 0 & \ldots & 0 & 0 \\ * & R_{i,2} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & \ldots & R_{i,i} & 0 \end{pmatrix} \end{array}, \quad \text{where } R_{i,j} = \begin{pmatrix} 0 \\ R_{i,j}^l \end{pmatrix},$$

and $R_{i,j}^l$ is a square nonsingular lower triangular matrix.

Upper triangular (which will be denoted by a superscript u):

$$\tilde{T}_i^u = \begin{array}{c} \\ r_{i-1}^1 \\ r_{i-1}^2 \\ \vdots \\ r_{i-1}^i \end{array} \begin{array}{c} \begin{array}{ccccc} r_i^{i+1} & r_i^1 & \dots & r_i^{i-1} & r_i^i \end{array} \\ \begin{pmatrix} 0 & R_{i,1} & * & \dots & * \\ 0 & 0 & R_{i,2} & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & R_{i,i} \end{pmatrix} \end{array}, \quad \textit{where } R_{i,j} = \begin{pmatrix} R_{i,j}^u \\ 0 \end{pmatrix},$$

and $R_{i,j}^u$ is a square nonsingular upper triangular matrix. The block dimensions r_i^j coincide with those of Theorem 1.1.

As to the nomenclature of these GQRDs, we propose the following definition.

DEFINITION 2 (Nomenclature for GQRD). The name of a GQRD of k matrices of compatible dimensions is generated by enumerating the letters L (for lower) and U (for upper), according to the lower or upper triangularity of the matrices $\tilde{T}_i, i = 1, \dots, k$ in the decomposition of Theorem 1.2.

For k matrices, there are 2^k different sequences with two letters. For instance, for $k = 3$, there are eight GQRDs (LLL, LLU, LUL, LLU, ULL, ULU, UUL, UUU).

The relation between the two generalizations, the GSVDs and the GQRDs, is the following:

(i) A pair of identical letters, i.e., L-L or U-U that occurs in the factorization of A_i, A_{i+1} corresponds to a P-type factorization of the pair;

(ii) A pair of alternating letters, i.e., L-U or U-L that occurs in the factorization of A_i, A_{i+1} corresponds to a Q-type factorization of the pair.

As an example, for a PQP-SVD of four matrices, there are two possible corresponding GQRDs, namely, an LLUL decomposition and an UULU decomposition. As with the GSVD, we can also introduce the convention to use powers of (a sequence of) letters. For instance, for a P^3Q^2-SVD (which is short for a PPPQQ-SVD), there are two QR decompositions, namely, an L^4UL-QR and an U^4LU-QR.

2. Structure of a GSVD with only P-steps. The main purpose of this section is to derive expressions for the block dimensions r_p^q when all steps in the GSVD are P-steps. These block dimensions will be expressed as a function of the ranks of products of the form

$$\text{rank}(A_i A_{i+1}, \dots, A_{j-1} A_j),$$

which will be denoted by $r_{i(i+1)\dots(j-1)j}$. This will be done in two steps. First, we derive an implicit characterization of the block dimensions. This leads directly to an explicit determination of these block dimensions.

LEMMA 2.1. *The rank of the product of the matrices $D_i, D_{i+1}, \dots, D_j$ that appears in a P^{k-1}-SVD (or the rank of the product $A_i A_{i+1}, \dots, A_j$ in an L^k-QR or a U^k-QR) is given by*

$$\text{rank}(D_i D_{i+1} \dots D_j) = r_{(i)(i+1)\dots(j)} = r_j^1 + r_j^2 + \dots + r_j^i.$$

As the examples will reveal, the following theorem follows directly from this lemma.

THEOREM 2.2. *Consider a P^{k-1}-SVD of the matrices $A_1, A_2, \dots, A_k$. Then, the block dimensions r_p^q, $p = 1, \dots, k$, $q = 1, \dots, p$ are given by*

$$r_j^1 = r_{(1)(2)\dots(j)},$$
$$r_j^i = r_{i(i+1)\dots(j)} - r_{(i-1)(i)\dots(j)},$$

with $r_{(i)\dots(j)} = r_i$ *if* $i = j$.

Let us analyze an example from which we will see the general result.

Example 3 (P^3-SVD). Let us derive expressions for the block dimensions $r_4^1, r_4^2, r_4^3, r_4^4$ of the matrix S_4 in terms of $r_1, r_2, r_3, r_{(3)(4)}, r_{(2)(3)(4)}, r_{(1)(2)(3)(4)}$. The matrices D_1, D_2, D_3, S_4 have the following structure:

$$D_1 = \begin{array}{c} \\ r_1^1 \\ n_0 - r_1 \end{array} \!\!\begin{array}{c} \begin{array}{cc} r_1^1 & n_1 - r_1^1 \end{array} \\ \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \end{array}, \qquad D_2 = \begin{array}{c} \\ r_2^1 \\ r_1^1 - r_2^1 \\ r_2^2 \\ n_1 - r_2^2 - r_1^1 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} r_2^1 & r_2^2 & n_2 - r_2 \end{array} \\ \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array},$$

$$D_3 = \begin{array}{c} \\ r_3^1 \\ r_2^1 - r_3^1 \\ r_3^2 \\ r_2^2 - r_3^2 \\ r_3^3 \\ n_2 - r_2 - r_3^3 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} r_3^1 & r_3^2 & r_3^3 & n_3 - r_3 \end{array} \\ \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array},$$

$$S_4 = \begin{array}{c} \\ r_4^1 \\ r_3^1 - r_4^1 \\ r_4^2 \\ r_3^2 - r_4^2 \\ r_4^3 \\ r_3^3 - r_4^3 \\ r_4^4 \\ n_3 - r_3 - r_4^4 \end{array} \!\!\begin{array}{c} \begin{array}{ccccc} r_4^1 & r_4^2 & r_4^3 & r_4^4 & n_4 - r_4 \end{array} \\ \begin{pmatrix} S_4^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & S_4^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & S_4^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & S_4^4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

From the structure and dimensions of these matrices, we see that (we only show block dimensions that are relevant)

$$D_3S_4 = \begin{array}{c} \\ r_4^1 \\ r_3^1 - r_4^1 \\ r_2^1 - r_3^1 \\ r_4^2 \\ r_3^2 - r_4^2 \\ r_2^2 - r_3^2 \\ r_4^3 \\ r_3^3 - r_4^3 \\ n_2 - r_2 - r_3^3 \end{array} \!\!\begin{array}{c} \begin{array}{ccccc} r_4^1 & r_4^2 & r_4^3 & & \end{array} \\ \begin{pmatrix} S_4^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & S_4^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & S_4^3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \qquad D_2D_3S_4 = \begin{array}{c} \\ r_4^1 \\ \\ \\ \\ r_4^2 \\ \\ \\ \\ \end{array} \!\!\begin{array}{c} \begin{array}{ccccc} r_4^1 & r_4^2 & & & \end{array} \\ \begin{pmatrix} S_4^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & S_4^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array},$$

$$D_1D_2D_3S_4 = \begin{array}{c} \\ r_4^1 \\ \\ \\ \\ \\ \end{array} \!\!\begin{array}{c} \begin{array}{ccccc} r_4^1 & & & & \end{array} \\ \begin{pmatrix} S_4^1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

We see by inspection that

$$\begin{aligned} r_{(1)(2)(3)(4)} &= r_4^1, \\ r_{(2)(3)(4)} &= r_4^1 + r_4^2, \\ r_{(3)(4)} &= r_4^1 + r_4^2 + r_4^3, \\ r_4 &= r_4^1 + r_4^2 + r_4^3 + r_4^4, \end{aligned}$$

from which it follows that

$$\begin{aligned} r_4^1 &= r_{(1)(2)(3)(4)}, \\ r_4^2 &= r_{(2)(3)(4)} - r_{(1)(2)(3)(4)}, \\ r_4^3 &= r_{(3)(4)} - r_{(2)(3)(4)}, \\ r_4^4 &= r_4 - r_{(3)(4)}. \end{aligned}$$

The same expressions apply for the blocks in the corresponding U^4- or L^4-QR.

Observe that in the product D_3 and S_4, only the diagonal blocks of S_4 with dimensions r_4^1, r_4^2, r_4^3 survive. In the product $D_2D_3S_4$ only the blocks with dimensions r_4^1 and r_4^2 survive, and in $D_1D_2D_3S_4$ only the block with dimension r_4^1 survives. This observation can easily be generalized to the following survival rule for a pure PSVD, which is the essence of the proof of Lemma 2.1.

In a product of matrices $D_i, D_{i+1}, \ldots, D_j$ (or S_k) only the blocks with block dimensions $r_j^1, r_j^2, \ldots, r_j^i$ survive.

Once this observation has been established, a proof of Theorem 3.2 is straightforward.

3. A GSVD with only Q-steps. Let us now look closer at the structure of a GSVD with only Q-steps. We will see that we can derive expressions for the block dimensions $r_p^q, p = 1, \ldots, k, q = 1, \ldots, p$ in two steps. First, we obtain an implicit formula where the required block dimensions are unknowns in a set of linear equations. In a second step, these are solved to obtain an expression for the block dimensions r_p^q in terms of the ranks of the block matrices

$$\begin{pmatrix} A_i & 0 & 0 & \ldots & 0 & 0 & 0 \\ A_{i+1}^* & A_{i+2} & 0 & \ldots & 0 & 0 & 0 \\ 0 & A_{i+3}^* & A_{i+4} & \ldots & 0 & 0 & 0 \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ 0 & \ldots & \ldots & \ldots & A_{j-3}^* & A_{j-2} & 0 \\ 0 & \ldots & \ldots & \ldots & 0 & A_{j-1}^* & A_j \end{pmatrix}.$$

Their rank is denoted by $r_{i|i+1|\ldots|j-1|j}$.

We will proceed in the same way as in §2. Instead of proving our results rigorously, we prefer to reveal the mechanisms by some clarifying examples. First, we obtain the following implicit characterization.

LEMMA 3.1. *Consider a* Q^{k-1}-SVD *of the matrices* $A_1, A_2, \ldots, A_k$. *Then if* $j - i$ *even*

$$r_{i|\ldots|j} = r_{i|\ldots|j-1} + (r_j^1 + r_j^2 + \cdots + r_j^i) + r_j^{i+2} + r_j^{i+4} + \cdots + r_j^{j-2} + r_j^j;$$

if $j - i$ *odd*

$$r_{i|\ldots|j} = r_{i|\ldots|j-1} + (r_j^{i+1} + r_j^{i+3} + \cdots + r_j^{j-2} + r_j^j).$$

As will be shown, this lemma leads to the following theorem.

THEOREM 3.2. *Consider a* Q^{k-1}*-SVD of the matrices* $A_1, A_2, \ldots, A_k$. *Then*

$$\begin{aligned}
r_k^1 &= (-1)^{k+1}(r_{1|\ldots|k} - r_{1|\ldots|k-1} - r_{2|\ldots|k} + r_{2|\ldots|k-1}),\\
r_k^j &= (-1)^{j+k+1}(r_{(j+1)|\ldots|k} - r_{(j+1)|\ldots|k-1} - r_{(j-1)|\ldots|k} + r_{(j-1)|\ldots|k-1})\\
&\qquad \textit{for} \ \ 2 \le j \le k-2,\\
r_k^{k-1} &= r_k - r_{k-2|k-1|k} + r_{k-2|k-1},\\
r_k^k &= r_{k-1|k} - r_{k-1}.
\end{aligned}$$

Observe that in all cases, no more than four ranks $r_{i|\ldots|j}$ are involved. Also, the third case may be recognized as Grassman's dimension theorem, giving the dimension of the intersection of the column spaces of the matrices

$$\begin{pmatrix} A_{k-2} \\ A_{k-1}^* \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ A_k \end{pmatrix}.$$

Let us derive the result of Theorem 3.2 by an example.

Example 4 (QQ-SVD).[1] A QQ-SVD of three matrices A_1 $(n_0 \times n_1)$, $A_2 (n_1 \times n_2)$, and A_3 $(n_2 \times n_3)$ takes the form

$$\begin{aligned}
A_1 &= U_1 D_1 X_1^{-1},\\
A_2 &= X_1^{-*} D_2 X_2^{-1},\\
A_3 &= X_2^{-*} S_3 V_3^*,
\end{aligned}$$

where

$$D_1 = \begin{array}{c} \\ r_1^1 \\ n_0 - r_1 \end{array} \begin{array}{c} \begin{array}{cc} r_1^1 & n_1 - r_1^1 \end{array} \\ \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \end{array}, \qquad D_2 = \begin{array}{c} \\ r_2^1 \\ r_1^1 - r_2^1 \\ r_2^2 \\ n_1 - r_2^2 - r_1 \end{array} \begin{array}{c} \begin{array}{ccc} r_2^1 & r_2^2 & n_2 - r_2 \end{array} \\ \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array},$$

$$S_3 = \begin{array}{c} \\ r_3^1 \\ r_2^1 - r_3^1 \\ r_3^2 \\ r_2^2 - r_3^2 \\ r_3^3 \\ n_3 - r_2 - r_3^3 \end{array} \begin{array}{c} \begin{array}{cccc} r_3^1 & r_3^2 & r_3^3 & n_3 - r_3 \end{array} \\ \begin{pmatrix} S_3^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & S_3^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & S_3^3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Observe that

$$\begin{pmatrix} A_1 & 0 \\ A_2^t & A_3 \end{pmatrix} = \begin{pmatrix} U_1 & 0 \\ 0 & X_2^{-*} \end{pmatrix} \begin{pmatrix} D_1 & 0 \\ D_2^t & S_3 \end{pmatrix} \begin{pmatrix} X_1^{-1} & 0 \\ 0 & V_3^* \end{pmatrix}.$$

[1] A complete detailed analysis of the QQ-SVD (which is also called the restricted singular value decomposition (RSVD)) together with numerous applications can be found in [2].

The left and right factors are nonsingular. Hence, we can obtain expressions for all dimensions involved by analyzing the block bidiagonal matrix

$$\left(\begin{array}{c|c} D_1 & 0 \\ \hline D_2^t & S_3 \end{array}\right) = \begin{array}{l} r_3^1 \\ r_2^1 - r_3^1 \\ r_1 - r_2^1 \\ n_0 - r_1 \\ \hline r_3^1 \\ r_2^1 - r_3^1 \\ r_3^2 \\ r_2^2 - r_3^2 \\ r_3^3 \\ n_2 - r_2 - r_3^3 \end{array} \left(\begin{array}{cccccc|cccc} I & 0 & 0 & & & & & & & \\ 0 & I & 0 & & & & & & & \\ 0 & 0 & I & & & & & & & \\ 0 & 0 & 0 & & & & & & & \\ \hline I & 0 & 0 & & & & S_3^1 & 0 & 0 & 0 \\ 0 & I & 0 & & & & 0 & 0 & 0 & 0 \\ & & & I & 0 & 0 & 0 & S_3^2 & 0 & 0 \\ & & & 0 & I & 0 & 0 & 0 & 0 & 0 \\ & & & & & & 0 & 0 & S_3^3 & 0 \\ & & & & & & 0 & 0 & 0 & 0 \end{array}\right),$$

where we have used the finest possible subdivision of matrices (i.e., a partitioning based upon the block dimensions r_3^1, r_3^2, r_3^3). All nonzero blocks are diagonal. Elements not shown are zero. First, it is straightforward to see that $r_3 = r_3^1 + r_3^2 + r_3^3$. Next, we concentrate on the submatrix $(\ D_2^t \ \ S_3\ )$. In this matrix, the block columns with the matrices S_3^1 and S_3^2 are linearly dependent on the previous ones. The block column with S_3^3 is linearly independent. Hence $\operatorname{rank}(\ D_2^t \ \ S_3\ ) = r_{2|3} = r_2 + r_3^3$. Next, we will relate the rank $r_{1|2} = \operatorname{rank}\left(D_1^t \quad D_2\right)^t$ to

$$r_{1|2|3} = \operatorname{rank}\begin{pmatrix} D_1 & 0 \\ D_2^t & S_3 \end{pmatrix}.$$

It can be seen that when the block column with S_3^1 is appended to $\left(D_1^t \quad D_2\right)^t$, the rank will increase with

$$r_3^1 \times \left[\operatorname{rank}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - \operatorname{rank}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right] = r_3^1.$$

If the block column with S_3^2 is appended to $\left(D_1^t \quad D_2\right)^t$, the rank will not increase. Finally, if the block column with S_3^1 is appended, then the rank will increase with $r_3^3 \times [\operatorname{rank}(\ 0 \ \ 1\ ) - \operatorname{rank}(0)] = r_3^3$. Hence

$$\begin{aligned} r_{1|2|3} = r_{1|2} &+ r_3^1 \times \left[\operatorname{rank}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - \operatorname{rank}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right] \\ &+ r_3^2 \times [\operatorname{rank}(1 \ \ 1) - \operatorname{rank}(1)] \\ &+ r_3^3 \times [\operatorname{rank}(0 \ \ 1) - \operatorname{rank}(0)] = r_{1|2} + r_3^1 + r_3^3. \end{aligned}$$

We can now set up a set of linear equations as

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} r_3^1 \\ r_3^2 \\ r_3^3 \end{pmatrix} = \begin{pmatrix} r_3 \\ r_{2|3} - r_2 \\ r_{1|2|3} - r_{1|2} \end{pmatrix},$$

which can be solved as

$$\begin{pmatrix} r_3^1 \\ r_3^2 \\ r_3^3 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} r_3 \\ r_{2|3} - r_2 \\ r_{1|2|3} - r_{1|2} \end{pmatrix} = \begin{pmatrix} r_{1|2|3} - r_{1|2} + r_2 - r_{2|3} \\ r_3 + r_{1|2} - r_{1|2|3} \\ r_{2|3} - r_2 \end{pmatrix}.$$

The same expressions will appear in the ULU- or LUL-QR.

Example 5 (Q^7-SVD). The courageous reader may wish to verify that for $k = 7$, the following set of linear equations needs to be solved:

$$\begin{pmatrix} r_7 \\ r_{6|7} - r_6 \\ r_{5|6|7} - r_{5|6} \\ r_{4|5|6|7} - r_{4|5|6} \\ r_{3|4|5|6|7} - r_{3|4|5|6} \\ r_{2|3|4|5|6|7} - r_{2|3|4|5|6} \\ r_{1|2|3|4|5|6|7} - r_{1|2|3|4|5|6} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} r_7^1 \\ r_7^2 \\ r_7^3 \\ r_7^4 \\ r_7^5 \\ r_7^6 \\ r_7^7 \end{pmatrix}.$$

This set of equations can be solved as

$$\begin{pmatrix} r_7^1 \\ r_7^2 \\ r_7^3 \\ r_7^4 \\ r_7^5 \\ r_7^6 \\ r_7^7 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} r_7 \\ r_{6|7} - r_6 \\ r_{5|6|7} - r_{5|6} \\ r_{4|5|6|7} - r_{4|5|6} \\ r_{3|4|5|6|7} - r_{3|4|5|6} \\ r_{2|3|4|5|6|7} - r_{2|3|4|5|6} \\ r_{1|2|3|4|5|6|7} - r_{1|2|3|4|5|6} \end{pmatrix}.$$

The pattern of the inverse matrix now becomes clear. We have a triantidiagonal matrix with a sequence of alternating 1 and -1, ending in a 1 in the top right-hand corner. As a matter of fact, this observation constitutes the essence of a proof of Theorem 3.2.

4. On the structure of a GSVD. For the analysis of the structure of a completely general GSVD, in which the letters P and Q can appear in any order, we need a mixture of the two preceding notations for block bidiagonal matrices, the blocks of which can be products of matrices, such as

$$\begin{pmatrix} A_{i_0}A_{i_0+1}\cdots A_{i_1-1} & 0 & 0 & \cdots & 0 \\ (A_{i_1}\cdots A_{i_2-1})^* & A_{i_2}\cdots A_{i_3-1} & 0 & \cdots & 0 \\ 0 & (A_{i_3}\cdots A_{i_4-1})^* & A_{i_4}\cdots A_{i_5-1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & A_{i_l}\cdots A_j \end{pmatrix},$$

where $1 \le i_0 < i_1 < i_2 < i_3 < \cdots < i_l \le j \le k$. Their rank will be denoted by

$$r_{(i_0)\dots(i_1-1)|i_1\dots(i_2-1)|\dots|i_l\dots(j)}.$$

For instance, the rank of the matrix

$$\begin{pmatrix} A_2A_3 & 0 & 0 \\ A_4^* & A_5A_6A_7 & 0 \\ 0 & (A_8A_9)^* & A_{10} \end{pmatrix}$$

will be represented by $r_{(2)(3)|4|(5)(6)(7)|(8)(9)|(10)}$.

In the following theorem, we derive an implicit expression of the block dimensions $r_p^q, p = 1, \dots, k, q = 1, \dots, p$ of a **GSVD** of $A_1, A_2, \dots, A_k$. We proceed in two steps. The first part of the theorem is based on the survival rule described in Lemma 2.1, and the second part is then an application of the pure Q-step SVD in Lemma 3.1.

THEOREM 4.1 (On the structure of a GSVD or GQRD). *The rank*

$$r_{(i_0)(i_0+1)\dots(i_1-1)|i_1\dots(i_2-1)|\dots|i_l\dots j}$$

can be expressed as follows:

1. *Calculate the* $l+1$ *integers* $s_j^i, i = 1, 2, \dots, l+1$.

$$\begin{aligned}
s_j^1 &= r_j^1 + r_j^2 + \cdots + r_j^{i_0},\\
s_j^2 &= r_j^{i_0+1} + r_j^{i_0+2} + \cdots + r_j^{i_1},\\
\cdots &= \cdots,\\
s_j^{l+1} &= r_j^{i_{l-1}+1} + r_j^{i_{l-1}+2} + \cdots + r_j^{i_l}.
\end{aligned}$$

2. *Depending on* l *even or* l *odd there are two cases.*

l *even*:

$$\begin{aligned}
&r_{(i_0)\dots(i_1-1)|(i_1)\dots(i_2-1)|\dots|(i_l)\dots j}\\
&\quad = r_{(i_0)\dots(i_1-1)|(i_1)\dots(i_2-1)|\dots|(i_{l-1})\dots(i_l-1)} + s_j^1 + s_j^3 + \cdots + s_j^{l+1}.
\end{aligned}$$

l *odd*:

$$\begin{aligned}
&r_{(i_0)\dots(i_1-1)|(i_1)\dots(i_2-1)|\dots|(i_l)\dots j}\\
&\quad = r_{(i_0)\dots(i_1-1)|(i_1)\dots(i_2-1)|\dots|(i_{l-1})\dots(i_l-1)} + s_j^2 + s_j^4 + \cdots + s_j^{l+1}.
\end{aligned}$$

Again, we will not give an unreadable algebraic proof of this theorem, but instead we illustrate it with an example.

Example 6 (QPPQ-SVD). A QPPQ-SVD of five matrices A_1, A_2, A_3, A_4, A_5 can be analyzed in terms of the ranks of the matrices

$$\begin{pmatrix} D_1 & 0 \\ (D_2D_3D_4)^t & S_5 \end{pmatrix}, \quad \begin{pmatrix} (D_2D_3D_4)^t & S_5 \end{pmatrix}, \quad \begin{pmatrix} (D_3D_4)^t & S_5 \end{pmatrix}, \quad \begin{pmatrix} D_4^t & S_5 \end{pmatrix}.$$

Let us first consider the first matrix

$$\left(\begin{array}{c||c} D_1 & 0 \\ \hline\hline (D_2D_3D_4)^t & S_5 \end{array}\right) =$$

$$\begin{array}{l}
r_5^1 \\ \\ \\ \\ \\ \\ \hline\hline
r_5^1 \\ \hline r_4^1 - r_5^1 \\ \hline r_5^2 \\ \hline r_4^2 - r_5^2 \\ \hline r_5^3 \\ \\ \hline r_5^4 \\ \\ \hline r_5^5 \\ \\ \hline
\end{array}
\left(\begin{array}{ccccc|ccccc||cccccc}
1 & 0 & 0 & 0 & 0 & & & & & & & & & & & \\
0 & 1 & 0 & 0 & 0 & & & & & & & & & & & \\
0 & 0 & 1 & 0 & 0 & & & & & & & & & & & \\
0 & 0 & 0 & 1 & 0 & & & & & & & & & & & \\
0 & 0 & 0 & 0 & 1 & & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & & & & & & & & & & & \\ \hline\hline
1 & 0 & 0 & 0 & 0 & & & & & & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
 & & & & & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 & & & & & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
 & & & & & & & & & & 0 & 0 & 1 & 0 & 0 & 0 \\
 & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
 & & & & & & & & & & 0 & 0 & 0 & 1 & 0 & 0 \\
 & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
 & & & & & & & & & & 0 & 0 & 0 & 0 & 1 & 0 \\
 & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right).$$

Elements not shown are represented by 0 while 1 represents a nonzero square diagonal matrix. Obviously, $r_5 = r_5^1+r_5^2+r_5^3+r_5^4+r_5^5$. Also, $r_{(2)(3)(4)|5} = r_{(2)(3)(4)}+(r_5^3+r_5^4+r_5^5)$. Using the notation of Theorem 4.1, we have $s_5^1 = r_5^1 + r_5^2$ and $s_5^2 = r_5^3 + r_5^4 + r_5^5$, so that indeed $r_{(2)(3)(4)|5} = r_{(2)(3)(4)} + s_5^2$. Also,

$$\begin{aligned} r_{1|(2)(3)(4)|5} = r_{1|(2)(3)(4)} &+ r_5^1 \times \left[\text{rank}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} - \text{rank}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right] \\ &+ r_5^2 \times [\text{rank}(1 \;\; 1) - \text{rank}(1)] + (r_5^3 + r_5^4 + r_5^5). \end{aligned}$$

With the notation of Theorem 4.1, we have for this case $s_5^1 = r_5^1$, $s_5^2 = r_5^2$, $s_5^3 = r_5^3+r_5^4+r_5^5$, so that indeed $r_{1|234|5} = r_{1|234}+s_5^1+s_5^3$. Up to now, we have three implicit equations for the five unknowns $r_5^1, r_5^2, r_5^3, r_5^4, r_5^5$. The remaining two are found from the matrix $(D_4^t \;\; S_5)$ as $r_{4|5} = r_4 + r_5^5$ and from

$$((D_3D_4)^t \;\; S_5) = \begin{array}{c} r_5^1 \\ r_4^1 - r_5^1 \\ r_5^2 \\ r_4^2 - r_5^2 \\ r_5^3 \\ r_4^3 - r_5^3 \\ r_5^4 \\ \\ r_5^5 \\ \\ \end{array} \left(\begin{array}{ccc|ccc|ccc|c||cccccc} 1 & 0 & 0 & & & & & & & & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline & & & 1 & 0 & 0 & & & & & 0 & 1 & 0 & 0 & 0 & 0 \\ & & & 0 & 1 & 0 & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline & & & & & & 1 & 0 & 0 & & 0 & 0 & 1 & 0 & 0 & 0 \\ & & & & & & 0 & 1 & 0 & & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline & & & & & & & & & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline & & & & & & & & & & 0 & 0 & 0 & 0 & 1 & 0 \\ & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

From this we find that

$$r_{34|5} = r_{3|4} + (r_5^1 + r_5^2 + r_5^3) \times [\text{rank}(1 \;\; 1) - \text{rank}(1)] + r_5^4 + r_5^5 = r_{3|4} + (r_5^4 + r_5^5).$$

With the notation of Theorem 4.1 we have $s_5^1 = r_5^1 + r_5^2 + r_5^3$ and $s_5^2 = r_5^4 + r_5^5$, so that indeed $r_{(3)(4)|5} = r_{3|4} + s_5^2$. From these equations we now find

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} r_5^1 \\ r_5^2 \\ r_5^3 \\ r_5^4 \\ r_5^5 \end{pmatrix} = \begin{pmatrix} r_5 \\ r_{234|5} - r_{234} \\ r_{1|234|5} - r_{1|234} \\ r_{4|5} - r_4 \\ r_{34|5} - r_{34} \end{pmatrix},$$

which, upon solution, results in

$$\begin{aligned} r_5^1 &= r_{1|234|5} - r_{1|234} - r_{234|5} + r_{234}, \\ r_5^2 &= r_5 - r_{1|234|5} + r_{1|234}, \\ r_5^3 &= r_{234|5} - r_{234} - r_{34|5} + r_{34}, \\ r_5^4 &= r_{34|5} - r_{34} - r_{4|5} + r_4, \\ r_5^5 &= r_{4|5} - r_4. \end{aligned}$$

5. Conclusions. In this paper, we have analyzed in detail the structure of some recently introduced generalizations of the singular value and the QR decomposition. The structure is completely determined in terms of the ranks of the involved matrices and other matrices that are formed from products and concatenations of these matrices. Some more examples and details can be found in the technical report [6] and the papers [2]–[5], and [7].

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