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GENERALIZATIONS OF THE SINGULAR VALUE AND QR DECOMPOSITIONS*

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We dedicate this paper to Gene Golub, a true source of inspiration for our work, but also a genuine friend, on the occasion of his 60th birthday

Abstract. This paper discusses multimatrix generalizations of two well-known orthogonal rank factorizations of a matrix: the generalized singular value decomposition and the generalized QR- (or URV-) decomposition. These generalizations can be obtained for any number of matrices of compatible dimensions. This paper discusses in detail the structure of these generalizations and their mutual relations and gives a constructive proof for the generalized QR-decompositions.

Key words. singular value decomposition, QR-factorization, URV-decomposition, complete orthogonal decomposition

AMS(MOS) subject classifications. 15A09, 15A18, 15A21, 15A24, 65F20

1. Introduction.

In this paper, we present *multimatrix generalizations* of some well-known orthogonal rank factorizations. We show how the idea of a QR-decomposition (QRD), a URV-decomposition (URVD), and a singular value decomposition (SVD) for one matrix can be generalized to any number of matrices. While generalizations of the SVD for any number of matrices have been derived in [9], one of the main contributions of this paper is the constructive derivation of a generalization for the QRD (or URVD) for any number of matrices of compatible dimensions. The idea is to reduce the set of matrices $A_1, A_2, \ldots, A_k$ to a simpler form using unitary transformations only. Hereby, we avoid explicit products and inverses of the matrices that are involved. We show that these *generalized* QR-*decompositions* (GQRD) can be considered as a preliminary reduction for any *generalized singular value decomposition* (GSVD). The reason is that there is a certain one-to-one relation between the structure of a GQRD and the "corresponding" GSVD, which is explained in detail below.

This paper is organized as follows. In §2, we provide a summary of *orthogonal rank factorizations* for one matrix. We briefly review the SVD, the QRD, and the URVD as special cases. In §3, we give a survey of existing generalizations of the SVD and QRD for two or three matrices. In §4, we summarize the results on GSVDs for any number of matrices of compatible dimensions. Section 5, which contains the main new contribution of this paper, describes a generalization of the QRD and the URVD for any number of matrices. We derive a constructive, inductive proof which shows that a GQRD can be used as a preliminary reduction for a corresponding GSVD. In §6, we analyze in detail the structure of the GQRDs and GSVDs and show that there is a one-to-one relation between the two generalizations. This relation is elaborated in more detail in §7, where we illustrate how a GQRD can be used as a preliminary step in the derivation of a corresponding GSVD.

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While all results in this paper are stated for complex matrices, they can be specialized to the real case without much difficulty. This can be done in much the same way as with the SVD for complex and real matrices. In particular, it suffices to restate most results using the term *real orthonormal* instead of *unitary* and to replace a superscript "$*$" (which denotes the complex conjugate transpose of a matrix) by a superscript "t" (which is the transpose of a matrix).

2. Orthogonal rank factorizations. Any matrix $A \in C^{m\times n}$ can be factorized as

$$A = Q\begin{pmatrix} R \\ 0 \end{pmatrix}\Pi, \tag{1}$$

where $R \in C^{n\times n}$ is upper trapezoidal and Π is a real $n \times n$ permutation matrix that permutes the columns of A so that the first $r_a = \text{rank}(A)$ columns are linearly independent. The matrix $Q \in C^{m\times m}$ is unitary and can be partitioned as

$$Q = \begin{matrix} & r_a & m-r_a \\ (& Q_1 & Q_2 &). \end{matrix}$$

If we partition R accordingly as $R = (R_{11} \quad R_{12})$, where $R_{11} \in C^{r_a\times r_a}$ is upper triangular and nonsingular, we obtain

$$A = Q_{11}(\ R_{11} \quad R_{12}\ )\Pi$$

which is sometimes called *the* QR-*factorization* of A.

If we rewrite (1) as

$$Q^*A = \begin{pmatrix} R \\ 0 \end{pmatrix}\Pi,$$

we see that Q is an orthogonal transformation that compresses the rows of A. Therefore, it is called a *row compression.* A similar construction exists, of course, for a *column compression.* A *complete orthogonal factorization* of an $m \times n$ matrix A is any factorization of the form

$$A = U\begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}V^*, \tag{2}$$

where T is $r_a \times r_a$ square nonsingular and $r_a = \text{rank}(A)$. One particular case is the SVD, which has become an important tool in the analysis and numerical solution of numerous problems, especially since the development of numerically robust algorithms by Golub and his coworkers [15], [16], [17]. The SVD is a complete orthogonal factorization where the matrix T is diagonal with positive diagonal elements:

$$A = U\Sigma V^*.$$

Here $U \in C^{m\times m}$ and $V \in C^{n\times n}$ are unitary and $\Sigma \in \Re^{m\times n}$ is of the form [1]

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 & 0 \\ 0 & \sigma_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \sigma_{r_a} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

[1] In this paper, we use the convention that zero blocks may be "empty" matrices, i.e., certain block dimensions may be 0.

The positive numbers $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{r_a} > 0$ are called the *singular values* of A, while the columns of U and V are the *left* and *right singular vectors.*

In applications where $m \gg n$, it is often a good idea to use the QRD of the matrix as a preliminary step in the computation of its SVD. The SVD of A is obtained via the SVD of its triangular factor as

$$A = QR = Q(U_r \Sigma_r V_r^*) = (QU_r)\Sigma_r V_r^*.$$

This idea of combining the QRD and the SVD of the triangular matrix, in order to compute the SVD of the full matrix, is mentioned in [22, p. 119] and more fully analyzed in [3]. In [18] the method is referred to as R-bidiagonalization. Its flop count is (mn^2+n^3), as compared to $(2mn^2-2/3n^3)$ for a bidiagonalization of the full matrix. Hence, whenever $m \geq 5/3n$, it is more advantageous to use the R-bidiagonalization algorithm.

There exist still other complete orthogonal factorizations of the form (2) where only T is required to be triangular (upper or lower) (see, e.g., [18]). Such a factorization was called a URV-*decomposition* in [27]. Here

$$A = U \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} V^*,$$

where $U \in C^{m\times m}$, $V \in C^{n\times n}$ are unitary matrices and $R \in C^{r_a\times r_a}$ is square nonsingular upper triangular.

It is well known that the QR-factorization of a singular matrix A and of its transpose A^* can be used for finding the image and kernel of A (URV-decompositions actually give both at once). In this paper, we try to extend these ideas to several matrices. Suppose we have a sequence of matrices $A_i, i = 1, \ldots, k$, and we want to know the kernels (or null spaces) of each partial product $A_1 \cdot A_2 \ldots A_j$. We could form these products and compute QR-decompositions of each of them. That can, in fact, be avoided, as shown below. Let us take the "special" example $A_i = A$, $i = 1, 2, 3$, with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is well known that the null spaces of A^i in fact give the Jordan structure of A, and this structure is already obvious from the form of A. But let us reconstruct it from a sequence of QR-decompositions (in fact we need here RQ-decompositions of A). The first one is, of course, a column compression of A_1, for which we use the permutation of columns 2 and 4 (denoted by the matrix P_{24}):

$$A_1 P_{24} = \left[\begin{array}{cc|ccc} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The separation line here indicates that the first two columns of P_{24} (i.e., e_1 and e_4) span the kernel of $A = A_1$. For the kernel of $A^2 = A_1 A_2$ we do not form this product, but apply the inverse of the orthogonal transform P_{24} (which is again P_{24}) to the rows of $A_2 = A$:

$$P_{24}A_2 = \left[\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Since $A_1A_2 = (A_1P_{24})(P_{24}A_2)$, it is clear that the kernel of A_1A_2 is also the kernel of the bottom part of $P_{24}A_2$. The following column compression of $P_{24}A_2$ actually yields the kernel of both A_2 and the product A_1A_2. Perform indeed the orthogonal transformation $P_{24}P_{35}$:

$$P_{24}A_2P_{24}P_{35} = \left[\begin{array}{cc|cc|c} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

We see that the kernel of A_2 comprises the first two columns of $P_{24}P_{35}$ (i.e., e_1 and e_4 as before) and the kernel of A_1A_2 comprises the first four columns of $P_{24}P_{35}$, i.e., e_1, e_2, e_4, and e_5. An additional step of this procedure finally shows that the kernel of the product $A_1A_2A_3 = (A_1P_{24})(P_{24}A_2P_{24}P_{35})(P_{35}P_{24}A_3)$ is that of the bottom part of the matrix

$$P_{35}P_{24}A_3 = \left[\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

which is a zero matrix. Hence the kernel of A^3 is the whole space, as expected. The interesting part of this simple example is the fact that we have not formed the intermediate products to get their corresponding kernels. The case treated here of equal matrices A_i is a simple one (and could be solved using the results of [19]), but in the next few sections we show how this can also be done for arbitrary sequences of matrices. The key idea is that at each step we do a number of QR-factorizations on the blocks of a partitioned matrix (column blocks in our case). This then induces a new partitioning on the rows of this matrix, on the columns of the next matrix, and so on.

3. Generalizations for two or three matrices. In the last decade or so, several generalizations for the SVD have been derived. The motivation is basically the necessity to avoid the explicit formation of products and matrix quotients in the computation of the SVD of products and quotients of matrices. Let A and B be nonsingular square matrices and assume that we need the SVD of $AB^{-*} = USV^*$.[2] It is well known that the explicit calculation of B^{-1}, followed by the computation of the product, may result in loss of numerical precision (digit cancellation), even before any factorization is attempted! The reason is the finite machine precision of

[2] The notation B^{-*} refers to the complex conjugate transpose of the inverse of the matrix B.

any calculator. Therefore, it seems more appropriate to come up with an implicit combined factorization of A and B separately, such as

$$\begin{aligned} A &= UD_1X^{-1}, \\ B &= X^{-*}D_2V^*, \end{aligned} \tag{3}$$

where U and V are unitary and X nonsingular. The matrices D_1 and D_2 are real but "sparse" (*quasi-diagonal*), and the product $D_1D_2^{-t}$ is diagonal with positive diagonal elements. Then we find

$$AB^{-*} = UD_1X^{-1}XD_2^{-t}V^* = U(D_1D_2^{-t})V^*.$$

A factorization as in (3) is always possible for two square nonsingular matrices. In fact, it is always possible for two matrices $A \in C^{m\times n}$ and $B \in C^{n\times p}$ (as long as the number of columns of A is the same as the number of rows of B, which we refer to as a *compatibility condition*). In general, the matrices A and B may even be rank deficient. The combined factorization (3) is called the *quotient singular value decomposition* (QSVD) and was first suggested in [32] and refined in [23] (it was originally called the generalized SVD, but we have suggested a standardized nomenclature in [6]).

A similar idea might be exploited for the SVD of the product of two matrices $AB = USV^*$, via the so-called *product singular value decomposition* (PSVD)

$$\begin{aligned} A &= UD_1X^{-1}, \\ B &= XD_2V^*, \end{aligned} \tag{4}$$

so that $AB = U(D_1D_2)V^*$, which is an SVD of AB. The combined factorization (4) was proposed in [13] as a formalization of ideas in [21]. In the general case, for two compatible matrices A and B (which may be rank deficient), the PSVD of (4) always exists and provides the SVD of AB without the explicit construction of the product. Similarly, if A and B are compatible, the QSVD always exists. However, it does not always deliver the SVD of $AB^\dagger$ when B is rank deficient ($B^\dagger$ is the pseudoinverse of B).

Another generalization, this time for three matrices, is the *restricted singular value decomposition* (RSVD). It was proposed in [35], and numerous applications were reviewed in [7]. It was soon found that all of these generalized SVDs for two or three matrices are special cases of a general theorem, presented in [9]. The main result is that there exist GSVDs for any number of matrices $A_1, A_2, \ldots, A_k$ of compatible dimensions. The general structure of these GSVDs was further analyzed in [10]. The dimensions of the blocks that occur in any GSVD can be expressed as ranks of the matrices involved and as certain products and concatenations of these. We present a summary of the results below.

As for generalizations of the QRD, it is mainly Paige [25] who pointed out the importance of generalized QRDs for two matrices as a basic conceptual and mathematical tool. The motivation is that in some applications, we need the QRD of a product of two matrices AB where $A \in \Re^{m\times n}$ and $B \in \Re^{n\times p}$. For general matrices A and B such a computation avoids forming the product explicitly, and transforms A and B separately to obtain the desired results. Paige [25] refers to such a factorization as a *product* QR *factorization.* Similarly, in some applications we need the QR-factorization of AB^{-1} where B is square and nonsingular. A general numerically robust algorithm would not compute the inverse of B nor the product explicitly, but would transform A and B separately. Paige [25] proposed calling such a combined decomposition of two matrices a *generalized* QR *factorization*, following [20]. We propose here to reserve the name *generalized* QRD for the *complete* set of generalizations of the QR-decompositions, which are developed in this paper. We also propose a novel nomenclature, as we did for the generalizations of the SVD in [6].

Stoer [28] appears to be the first to have given a reliable computation of this type of generalized QR-factorization for two matrices (see [14]). Computational methods for producing the two types of generalized QR factorizations for two matrices, as described above, have appeared regularly in the literature as (intermediate) steps in the solution of some problems. In this paper, we derive a constructive proof of generalizations of the QRD for any number of matrices. As we see below, our generalized QRDs can also be considered the appropriate generalization of the URVD of a matrix.

4. Generalized singular value decompositions. In this section, we present a general theorem that can be considered the appropriate generalization for any number of matrices of the SVD of one matrix. It contains the existing generalizations of the SVD for two matrices (i.e., the PSVD and the QSVD) and three matrices (i.e., the RSVD) as special cases. A constructive proof can be found in [9].

THEOREM 4.1 (generalized singular value decompositions for k matrices). *Consider a set of k matrices with compatible dimensions: A_1 $(n_0 \times n_1), A_2$ $(n_1 \times n_2), \dots,$ A_{k-1} $(n_{k-2} \times n_{k-1}), A_k$ $(n_{k-1} \times n_k)$. Then there exist*

—Unitary matrices U_1 $(n_0 \times n_0)$ and V_k $(n_k \times n_k)$.

—Matrices $D_j, j = 1, 2, \dots, (k-1)$ of the form

$$
(5)\qquad \underset{n_{j-1}\times n_j}{D_j} = \begin{array}{l} \\ r_j^1 \\ r_{j-1}^1 - r_j^1 \\ r_j^2 \\ r_{j-1}^2 - r_j^2 \\ r_j^3 \\ \dots \\ \dots \\ r_j^j \\ n_{j-1} - r_{j-1} - r_j^j \end{array}
\begin{array}{c} \begin{array}{ccccccc} r_j^1 & r_j^2 & r_j^3 & \dots & \dots & r_j^j & n_j - r_j \end{array} \\
\left(\begin{array}{ccccccc}
I & 0 & 0 & \dots & \dots & 0 & 0 \\
0 & 0 & 0 & \dots & \dots & 0 & 0 \\
0 & I & 0 & \dots & \dots & 0 & 0 \\
0 & 0 & 0 & \dots & \dots & 0 & 0 \\
0 & 0 & I & \dots & \dots & 0 & 0 \\
0 & \dots & \dots & \dots & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & \dots & \dots & \dots & \dots & I & 0 \\
0 & 0 & \dots & \dots & \dots & 0 & 0
\end{array}\right) \end{array},
$$

where

$$
(6)\qquad r_0 = 0, \qquad r_j = \sum_{i=1}^{j} r_j^i = \operatorname{rank}(A_j).
$$

—A matrix S_k of the form

$$
(7)\qquad \underset{n_{k-1}\times n_k}{S_k} = \begin{array}{l} \\ r_k^1 \\ r_{k-1}^1 - r_k^1 \\ r_k^2 \\ r_{k-1}^2 - r_k^2 \\ r_k^3 \\ \dots \\ \dots \\ r_k^k \\ n_{k-1} - r_{k-1} - r_k^k \end{array}
\begin{array}{c} \begin{array}{ccccccc} r_k^1 & r_k^2 & r_k^3 & \dots & \dots & r_k^k & n_k - r_k \end{array} \\
\left(\begin{array}{ccccccc}
S_k^1 & 0 & 0 & \dots & \dots & 0 & 0 \\
0 & 0 & 0 & \dots & \dots & 0 & 0 \\
0 & S_k^2 & 0 & \dots & \dots & 0 & 0 \\
0 & 0 & 0 & \dots & \dots & 0 & 0 \\
0 & 0 & S_k^3 & \dots & \dots & 0 & 0 \\
0 & \dots & \dots & \dots & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & \dots & \dots & \dots & \dots & S_k^k & 0 \\
0 & 0 & \dots & \dots & \dots & 0 & 0
\end{array}\right) \end{array},
$$

where

$$r_k = \sum_{i=1}^{k} r_k^i = \operatorname{rank}(A_k) \tag{8}$$

and the $r_k^i \times r_k^i$ matrices S_k^i are diagonal with positive diagonal elements. Expressions for the integers r_j^i are given in §6.[3]

—Nonsingular matrices X_j ($n_j \times n_j$) and Z_j, $j = 1, 2, \ldots, (k-1)$ where Z_j is either $Z_j = X_j^{-*}$ or $Z_j = X_j$ (i.e., both choices are always possible), such that the given matrices can be factorized as

$$\begin{aligned} A_1 &= U_1 D_1 X_1^{-1}, \\ A_2 &= Z_1 D_2 X_2^{-1}, \\ A_3 &= Z_2 D_3 X_3^{-1}, \\ \cdots &= \cdots, \\ A_i &= Z_{i-1} D_i X_i^{-1}, \\ \cdots &= \cdots, \\ A_k &= Z_{k-1} S_k V_k^*. \end{aligned}$$

Observe that the matrices D_j in (5) and S_k in (7) are generally not diagonal. Their only nonzero blocks, however, are diagonal block matrices. We propose to label them as *quasi-diagonal* matrices. The matrices $D_j, j = 1, \ldots, k-1$ are quasi-diagonal, their only nonzero blocks being identity matrices. The matrix S_k is quasi-diagonal and its nonzero blocks are diagonal matrices with positive diagonal elements. Observe that we always take the last factor in every factorization as the inverse of a nonsingular matrix, which is only a matter of convention (another convention would result in a modified definition of the matrices Z_i). As for the name of a certain GSVD, we propose to adopt the following convention (see also [9]).

DEFINITION 4.2 (the nomenclature for GSVDs). If $k = 1$ in Theorem 4.1, then the corresponding factorization of the matrix A_1 will be called the (ordinary) singular value decomposition. If for a matrix pair $A_i, A_{i+1}, 1 \le i \le k-1$ in Theorem 4.1, we have $Z_i = X_i$, then the factorization of the pair is said to be of P type. If, on the other hand, for a matrix pair $A_i, A_{i+1}, 1 \le i \le k-1$ in Theorem 4.1, we have $Z_i = X_i^{-*}$, then the factorization of the pair is said to be of Q type. The name of a GSVD of the matrices A_i, $i = 1, 2, \ldots k > 1$ as in Theorem 4.1, is then obtained by simply enumerating the different factorization types.

Let us give some examples.

Example. Consider two matrices A_1 ($n_0 \times n_1$) and A_2 ($n_1 \times n_2$). Then, we have two possible GSVDs:

	P type	Q type
A_1	$U_1 D_1 X_1^{-1}$	$U_1 D_1 X_1^{-1}$
A_2	$X_1 S_2 V_2^*$	$X_1^{-*} S_2 V_2^*$

The P-type factorization is called the PSVD (see [8] and references therein), while the Q-type factorization is called the QSVD.

[3] In [9], these block dimensions follow from the constructive proof.

Example. Let us write a PQQP-SVD for five matrices:

$$\begin{aligned} A_1 &= U_1 D_1 X_1^{-1}, \\ A_2 &= X_1 D_2 X_2^{-1}, \\ A_3 &= X_2^{-*} D_3 X_3^{-1}, \\ A_4 &= X_3^{-*} D_4 X_4^{-1}, \\ A_5 &= X_4 S_5 V_5^{*}. \end{aligned}$$

We also introduce a notation using powers that symbolize a certain repetition of a letter or of a sequence of letters:

- $\mathrm{P}^3\mathrm{Q}^2$-SVD = PPPQQ-SVD,
- $(\mathrm{PQ})^2\mathrm{Q}^3(\mathrm{PPQ})^2$-SVD = PQPQQQQPPQPPQ-SVD.

Despite the fact that there are 2^{k-1} different sequences of letters P and Q at level $k > 1$, not all of these sequences correspond to different GSVDs. The reason for this is that, for instance, the QP-SVD of (A^1, A^2, A^3) can be obtained from the PQ-SVD of $((A^3)^*, (A^2)^*, (A^1)^*)$. Similarly, the $\mathrm{P}^2(\mathrm{QP})^3$-SVD of $(A^1, \ldots, A^9)$ is essentially the same as the $(\mathrm{PQ})^3\mathrm{P}^2$-SVD of $((A^9)^*, \ldots, (A^1)^*)$. The number of different factorizations for k matrices is, in fact, $\frac{1}{2}(2^{k-1} + 2^{k/2})$ for k even and $\frac{1}{2}(2^{k-1} + 2^{(k-1)/2})$ for k odd.

A possible way to visualize Theorem 4.1 is to build a tree with all different factorizations for 1, 2, 3, etc $\cdots$ matrices as follows:

1					**O**					
2				**P**		**Q**				
3			$\mathbf{P^2}$		**PQ**		$\mathbf{Q^2}$			
4	$\mathbf{P^3}$	$\mathbf{P^2Q}$		**PQP**		$\mathbf{PQ^2}$		**QPQ**	$\mathbf{Q^3}$	
$\cdots$	$\cdots$	$\cdots$	$\cdots$		$\cdots$		$\cdots$	$\cdots$		$\cdots$

5. Generalized URVDs. In this section, we derive a generalization for several matrices, of the URVD of one matrix. We proceed in several stages. First, we show how k matrices can be reduced to block triangular matrices using unitary transformations only. Next, we show how the block triangular factors can be triangularized further to triangular factors.

THEOREM 5.1. *Given k complex matrices* A_1 $(n_0 \times n_1)$, A_2 $(n_1 \times n_2)$, $\ldots$, A_k $(n_{k-1} \times n_k)$, *there always exist unitary matrices* $Q_0, Q_1, \ldots, Q_k$ *such that*

$$T_i = Q_{i-1}^* A_i Q_i,$$

where T_i *is a block lower triangular or block upper triangular matrix* (*both cases are always possible*) *with the following structures:*

—*Lower block triangular* (*denoted by a superscript* l) :

$$T_i^l = \begin{array}{c} \\ r_{i-1}^1 \\ r_{i-1}^2 \\ \vdots \\ r_{i-1}^i \end{array} \begin{array}{c} \begin{array}{cccccc} r_i^1 & r_i^2 & \ldots & r_i^{i-1} & r_i^i & r_i^{i+1} \end{array} \\ \begin{pmatrix} T_{i,1} & 0 & \ldots & 0 & 0 & 0 \\ * & T_{i,2} & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & \ldots & * & T_{i,i} & 0 \end{pmatrix} \end{array}. \tag{9}$$

—*Upper block triangular (denoted by a superscript u):*

$$(10) \qquad T_i^u = \begin{array}{c} \\ r_{i-1}^1 \\ r_{i-1}^2 \\ \vdots \\ r_{i-1}^i \end{array} \begin{array}{c} \begin{array}{cccccc} r_i^1 & r_i^2 & \dots & r_i^{i-1} & r_i^i & r_i^{i+1} \end{array} \\ \begin{pmatrix} T_{i,1} & * & \dots & * & * & 0 \\ 0 & T_{i,2} & \dots & * & * & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & T_{i,i} & 0 \end{pmatrix} \end{array},$$

where $T_{i,j}$, $j = 1, \dots, i$ are full column rank matrices and each $*$ represents a nonzero block. The block dimensions coincide with those of Theorem 4.1. *In particular,*

$$r_0^1 = n_0,$$
$$r_i^{i+1} = \text{nullity}(A_i) = n_i - r_i,$$

and

$$\sum_{j=1}^{i} r_i^j = \text{rank}(A_i) = r_i,$$
$$\sum_{j=1}^{i} r_{i-1}^j = n_{i-1}.$$

Our proof of Theorem 5.1 is inductive: We obtain the required factorization of A_i from that of A_{i-1}.

Proof. The induction is initialized for $i = 1$ as follows. First, take the case where T_1 is to be lower block triangular. Use a unitary column compression matrix Q_1 to reduce the matrix A_1 to

$$T_1^l = A_1 Q_1 = r_0^1 \begin{array}{c} \begin{array}{cc} r_1^1 & r_1^2 \end{array} \\ \begin{pmatrix} T_{1,1} & 0 \end{pmatrix} \end{array},$$

where

$$r_1^1 = \text{rank}(T_{1,1}) = \text{rank}(A_1),$$
$$r_1^2 = \text{nullity}(A_1) = n_1 - r_1,$$

and

$$r_0^1 = n_0.$$

The case where T_1 is required to be upper block triangular is similar:

$$T_1^u = A_1 Q_1 = r_0^1 \begin{array}{c} \begin{array}{cc} r_1^1 & r_1^2 \end{array} \\ \begin{pmatrix} T_{1,1} & 0 \end{pmatrix} \end{array}.$$

Observe that we have taken $Q_0 = I_{n_0}$.

Now, we can start our induction. Assume that we have the required factorization for the first $i - 1$ matrices

$$T_1 = Q_0^* A_1 Q_1,$$
$$\vdots = \vdots,$$
$$T_{i-1} = Q_{i-2}^* A_{i-1} Q_{i-1},$$

where the matrices $T_j, j = 1, \ldots, i-1$ have the block structure as in Theorem 5.1. We now want to find a unitary matrix Q_i such that $T_i = Q_{i-1}^* A_i Q_i$ is either lower or upper block triangular. First, consider the case where T_i is to be lower block triangular. The matrix $Q_{i-1}^* A_i$ can be partitioned according to the dimensions of the block columns of T_{i-1} as

$$Q_{i-1}^* A_i = \begin{array}{c} \\ r_{i-1}^1 \\ r_{i-1}^2 \\ \vdots \\ r_{i-1}^i \end{array} \begin{array}{c} n_i \\ \begin{pmatrix} * \\ * \\ \vdots \\ * \end{pmatrix} \end{array}. \tag{11}$$

It is always possible to construct a unitary matrix Q_i to compress the columns of each of the block rows to the left as

$$T_i^l = Q_{i-1}^* A_i Q_i = \begin{array}{c} \\ r_{i-1}^1 \\ r_{i-1}^2 \\ \vdots \\ r_{i-1}^i \end{array} \begin{array}{c} \begin{array}{ccccc} r_i^1 & r_i^2 & \ldots & r_i^i & r_i^{i+1} \end{array} \\ \begin{pmatrix} T_{i,1} & 0 & \ldots & 0 & 0 \\ * & T_{i,2} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & \ldots & T_{i,i} & 0 \end{pmatrix} \end{array}, \tag{12}$$

where the subblocks $T_{i,j}$ are of full column rank, denoted by r_i^l, $l = 1, \ldots, i$ and $r_i^{i+1} = \text{nullity}(A_i)$. Hereto, we first compress the first block row of (11) to the left with unitary column transformations applied to the full matrix. Then we proceed with the second block row in the deflated matrix (i.e., without modifying the previous block column). By repeating this procedure i times, we find the required form (12).

Obviously,

$$r_i^l \leq r_{i-1}^l, \qquad l = 1, \ldots, i. \tag{13}$$

The construction of T_i when it is required to be upper block triangular is similar. Construct a unitary matrix Q_i that compresses the columns of the block rows of $Q_{i-1}^* A_i$ to the right. The only difference is that we now start from the bottom to find that

$$T_i^u = Q_{i-1}^* A_i Q_i = \begin{array}{c} \\ r_{i-1}^1 \\ r_{i-1}^2 \\ \vdots \\ r_{i-1}^i \end{array} \begin{array}{c} \begin{array}{ccccc} r_i^{i+1} & r_i^1 & r_i^2 & \ldots & r_i^i \end{array} \\ \begin{pmatrix} 0 & T_{i,1} & * & \ldots & * \\ 0 & 0 & T_{i,2} & \ldots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & \ldots & 0 & T_{i,i} \end{pmatrix} \end{array}. \tag{14}$$

We can now apply an additional (block) column permutation to the right of the matrix T_i^u so as to find the matrix of (10). This completes the proof. □

We now demonstrate that the matrices $T_{i,j}$ can always be further reduced to triangular form using unitary transformations into

$$\begin{pmatrix} 0 \\ R_{i,j}^l \end{pmatrix}$$

in the case when T_i is lower block triangular. Here, $R^l_{i,j}$ is a lower triangular matrix. Similarly, we can always reduce $T_{i,j}$ to

$$\begin{pmatrix} R^u_{i,j} \\ 0 \end{pmatrix}$$

in the case where T_i is upper block triangular. Here $R^u_{i,j}$ is an upper triangular matrix. In order to demonstrate this, we need the following result.

LEMMA 5.2. *Let $P_1, \dots, P_k$ be k given complex matrices where P_i has dimensions $p_{i-1} \times p_i$, $p_{i-1} \geq p_i$ and* $\operatorname{rank}(P_i) = p_i$. *Then there always exist unitary matrices $Q_0, Q_1, \dots, Q_k$ such that*

$$R_i = Q^*_{i-1} P_i Q_i,$$

where R_i is either of the form

$$R_i = \begin{matrix} p_{i-1} - p_i \\ p_i \end{matrix} \overset{p_i}{\begin{pmatrix} 0 \\ R^l_i \end{pmatrix}} \tag{15}$$

with R^l_i a lower triangular matrix, or

$$R_i = \begin{matrix} p_i \\ p_{i-1} - p_i \end{matrix} \overset{p_i}{\begin{pmatrix} R^u_i \\ 0 \end{pmatrix}} \tag{16}$$

with R^u_i upper triangular. For every $i = 1, \dots, k$, both choices, (15) *and* (16), *are always possible.*

Proof. Again, the proof is by induction, but now for *decreasing* index i. For the initialization, start with $i = k$ and obtain a QR-decomposition of P_k with either an upper or a lower triangular factor as required. This defines the unitary matrix Q_{k-1}. We take $Q_k = I_{p_k}$. Hence, we find

—Lower triangular:

$$P_k = Q_{k-1} R_k = Q_{k-1} \begin{pmatrix} 0 \\ R^l_k \end{pmatrix};$$

—Upper triangular:

$$P_k = Q_{k-1} R_k = Q_{k-1} \begin{pmatrix} R^u_k \\ 0 \end{pmatrix}.$$

We can now start the induction for $i = k-1, k-2, \dots, 1$. Therefore, assume that we have the required factorizations for the matrices $P_k, P_{k-1}, \dots, P_{i+1}$:

$$\begin{aligned} R_k &= Q^*_{k-1} P_k Q_k, \\ R_{k-1} &= Q^*_{k-2} P_{k-1} Q_{k-1}, \\ \vdots &= \vdots, \\ R_{i+1} &= Q^*_i P_{i+1} Q_{i+1}. \end{aligned}$$

Then, if R_i is to be lower triangular, obtain a QR-decomposition of the product $P_i Q_i$ as

$$P_i Q_i = Q_{i-1} R_i = Q_{i-1} \begin{pmatrix} 0 \\ R_i^l \end{pmatrix},$$

so that

$$R_i = \begin{pmatrix} 0 \\ R_i^l \end{pmatrix} = Q_{i-1}^* P_i Q_i.$$

If R_i is required to be upper triangular, obtain a QR-decomposition as

$$P_i Q_i = Q_{i-1} R_i = Q_{i-1} \begin{pmatrix} R_i^u \\ 0 \end{pmatrix},$$

so that

$$R_i = \begin{pmatrix} R_i^u \\ 0 \end{pmatrix} = Q_{i-1}^* P_i Q_i.$$

This completes the construction. □

We now repeatedly apply Lemma 5.2 on the full rank blocks in the matrices T_i in (9) and (10). First, we apply Lemma 5.2 to the sequence of k subblocks

$$T_{1,1} \ T_{2,1} \ \dots \ T_{k,1}.$$

Next, we apply it to the sequence of the $k-1$ subblocks

$$T_{2,2} \ T_{3,2} \ \dots \ T_{k,2}.$$

In general, we apply Lemma 5.2 k times to the k sequences of subblocks

$$T_{j,j}, T_{j+1,j}, \dots, T_{k,j} \quad \text{for } j = 1, \dots, k.$$

In applying Lemma 5.2 to the jth of these sequences, we can find a sequence of unitary matrices $Q_0^{[j]}, Q_1^{[j]}, \dots, Q_{k-j+1}^{[j]}$ and matrices $R_{i,j}$ such that

$$T_{i,j} = Q_{i-j}^{[j]} R_{i,j} Q_{i-j+1}^{[j]*}, \qquad i = j, \dots, k,$$

where

$$R_{i,j} = \begin{pmatrix} 0 \\ R_{i,j}^l \end{pmatrix}$$

or

$$R_{i,j} = \begin{pmatrix} R_{i,j}^u \\ 0 \end{pmatrix}.$$

We now define the *unitary* matrices $\tilde{Q}_i$ for $i = 0, \dots, k$, which are block diagonal with blocks

$$\tilde{Q}_i = \operatorname{diag}(Q_i^{[1]}, Q_{i-1}^{[2]}, \dots, Q_1^{[i]}, Q_0^{[i+1]}), \qquad i = 0, \dots, k,$$

with

$$Q_0^{[k+1]} = I.$$

Next we define

$$\tilde{T}_i = \tilde{Q}_{i-1}^* T_i \tilde{Q}_i, \qquad i = 0, \ldots, k.$$

Then, it can be verified that for the lower triangular case we obtain

$$\tilde{T}_i = \tilde{T}_i^l = \begin{array}{c} \\ r_{i-1}^1 \\ r_{i-1}^2 \\ \vdots \\ r_{i-1}^i \end{array} \begin{array}{c} \begin{array}{ccccc} r_i^1 & r_i^2 & \ldots & r_i^i & r_i^{i+1} \end{array} \\ \left(\begin{array}{ccccc} R_{i,1} & 0 & \ldots & 0 & 0 \\ * & R_{i,2} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & \ldots & R_{i,i} & 0 \end{array} \right) \end{array}, \tag{17}$$

and for the upper triangular case we find that

$$\tilde{T}_i = \tilde{T}_i^u = \begin{array}{c} \\ r_{i-1}^1 \\ r_{i-1}^2 \\ \vdots \\ r_{i-1}^i \end{array} \begin{array}{c} \begin{array}{ccccc} r_i^1 & \ldots & r_i^{i-1} & r_i^i & r_i^{i+1} \end{array} \\ \left(\begin{array}{ccccc} R_{i,1} & * & \ldots & * & 0 \\ 0 & R_{i,2} & \ldots & * & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & R_{i,i} & 0 \end{array} \right) \end{array}. \tag{18}$$

If we now combine (9)–(17) and (10)–(18), we obtain a combined factorization of the form

$$\tilde{T}_i = (Q_{i-1}\tilde{Q}_{i-1})^* A_i (Q_i \tilde{Q}_i).$$

Hence, we have proved the following theorem.

THEOREM 5.3 (generalized URVDs). *Given k complex matrices A_1 $(n_0 \times n_1)$, A_2 $(n_1 \times n_2)$, $\ldots$, A_k $(n_{k-1} \times n_k)$, there always exist unitary matrices $Q_0, Q_1, \ldots, Q_k$ such that*

$$\tilde{T}_i = Q_{i-1}^* A_i Q_i,$$

where $\tilde{T}_i$ is a lower triangular or upper triangular matrix (both cases are always possible) with the following structures:

—Lower triangular (denoted by a superscript l):

$$\tilde{T}_i^l = \begin{array}{c} \\ r_{i-1}^1 \\ r_{i-1}^2 \\ \vdots \\ r_{i-1}^i \end{array} \begin{array}{c} \begin{array}{ccccc} r_i^1 & r_i^2 & \ldots & r_i^i & r_i^{i+1} \end{array} \\ \left(\begin{array}{ccccc} R_{i,1} & 0 & \ldots & 0 & 0 \\ * & R_{i,2} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & \ldots & R_{i,i} & 0 \end{array} \right) \end{array},$$

where

$$R_{i,j} = \left(\begin{array}{c} 0 \\ R_{i,j}^l \end{array} \right),$$

and $R^l_{i,j}$ is a square nonsingular lower triangular matrix.

—Upper triangular (denoted by a superscript u):

$$\tilde{T}^u_i = \begin{array}{c} r^1_{i-1} \\ r^2_{i-1} \\ \vdots \\ r^i_{i-1} \end{array} \overset{\begin{array}{ccccc} r^1_i & \dots & r^{i-1}_i & r^i_i & r^{i+1}_i \end{array}}{\begin{pmatrix} R_{i,1} & * & \dots & * & 0 \\ 0 & R_{i,2} & \dots & * & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & R_{i,i} & 0 \end{pmatrix}},$$

where

$$R_{i,j} = \begin{pmatrix} 0 \\ R^u_{i,j} \end{pmatrix},$$

and $R^u_{i,j}$ is a square nonsingular upper triangular matrix. The block dimensions coincide with those of Theorem 4.1.

As for the nomenclature of these generalized URVDs, we propose the following definition.

DEFINITION 5.4 (nomenclature for generalized URV). The name of a generalized URVD of k matrices of compatible dimensions is generated by enumerating the letters L (for lower) and U (for upper), according to the lower or upper triangularity of the matrices $T_i, i = 1, \dots, k$ in the decomposition of Theorem 5.3.

For k matrices, there are 2^k different sequences with two letters. For instance, for $k = 3$, there are eight generalized URVD (LLL, LLU, LUL, LUU, ULL, ULU, UUL, UUU).

Remarks. The decompositions in Theorems 5.1 and 5.3 both use column and row compressions of a matrix as a cornerstone for the rank determination of the individual blocks. As already pointed out in §2, the rank determination can be done via an ordinary SVD (OSVD), but a more economical method uses the QRD as initial step, since typically the matrices involved here have many more columns than rows or vice versa. A further alternative would be to replace the OSVD of the triangular matrix resulting from the initial QRD by a rank-revealing QRD. Since the time of the initial paper drawing attention to this [5], much progress has been made in this area, and we only want to stress here that such alternatives can only benefit our decomposition.

The overall complexity of this GQRD is easily seen to be comparable to that of performing two QRDs of each matrix A_i involved. For each A_i we indeed apply the left transformation Q^*_{i-1} derived from the previous matrix and then apply a "special" compression Q_i of the resulting matrix while respecting its block structure. Both steps have a complexity comparable to a QRD of a matrix of the same dimensions. For parallel machines we can check that the "block" algorithms [18] for one-sided orthogonal transformations such as the QRD can also be applied to the present decomposition, and that they will yield satisfactory speedups. The main reason for this is that the two-sided orthogonal transforms applied to each A_i are done separately, and hence they can essentially be considered one-sided for parallellization purposes.

6. On the structure of the GSVD and the GQRD. In this section, we first point out how for each GSVD there are two generalized URVDs, and we clarify the correspondence between the two types of generalized decompositions. Next, we give a summary of expressions for the block dimensions r^j_i in Theorems 4.1 and 5.1, in terms of the ranks of the matrices $A_1, \dots, A_k$ and concatenations and products thereof. These expressions were derived in [10].

Recall the nomenclature for the generalized URVDs (Definition 5.4) and the GSVDs (Definition 4.2). The relationship between these two definitions is as follows. A pair of identical letters, i.e., L-L or U-U, that occurs in the factorization of A_i, A_{i+1} corresponds to a P-type factorization of the pair. A pair of alternating letters, i.e., L-U or U-L, that occurs in the factorization of A_i, A_{i+1} corresponds to a Q-type factorization of the pair. As an example, for a PQP-SVD of four matrices, there are two possible corresponding generalized URVDs, namely an LLUL-decomposition and a UULU-decomposition. As with the GSVD, we can also introduce the convention to use powers of (a sequence of) letters. For instance, for a $\mathrm{P}^3\mathrm{Q}^2$-SVD, there are two GURVs, namely, an L^4UL-URV and a U^4LU-URV.

We now derive expressions for the block dimensions r_i^j.[4] Let us first consider the case of a GSVD that consists only of P-type factorizations. Denote the rank of the product of the matrices A_i,A_{i+1}, ..., A_j with $i \le j$ by

$$r_{i(i+1)\cdots(j-1)j} = \mathrm{rank}(A_i A_{i+1} \dots A_{j-1} A_j).$$

THEOREM 6.1 (on the structure of P^{k-1}-SVD, L^k-URV, and U^k-URV). *Consider any of the factorizations above for the matrices $A_1, A_2, \dots, A_k$. Then, the block dimensions r_j^i that appear in Theorems* 4.1, 5.1, *and* 5.3 *are given by:*

$$r_j^1 = r_{(1)(2)\cdots(j)}, \tag{19}$$

$$r_j^i = r_{i(i+1)\cdots(j)} - r_{(i-1)(i)\cdots(j)}, \tag{20}$$

with $r_j^i = r_i$ if $i = j$.

Next, consider the case of a GSVD that only consists of Q-type factorizations. Denote the rank of the block bidiagonal matrix

$$\begin{pmatrix} A_i & 0 & 0 & \dots & 0 & 0 & 0 \\ A_{i+1}^* & A_{i+2} & 0 & \dots & 0 & 0 & 0 \\ 0 & A_{i+3}^* & A_{i+4} & \dots & 0 & 0 & 0 \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & A_{j-3}^* & A_{j-2} & 0 \\ 0 & \dots & \dots & \dots & 0 & A_{j-1}^* & A_j \end{pmatrix} \tag{21}$$

(by $r_{i|i+1|\dots|j-1|j}$).

THEOREM 6.2 (on the structure of Q^{k-1}-SVD, $(\mathrm{LU})^{k/2}$-URV (k even), $(\mathrm{UL})^{k/2}$-URV (k even), $(\mathrm{UL})^{(k-1)/2}U$-URV (k odd), and $(\mathrm{LU})^{(k-1)/2}L$-URV (k odd)). *Consider any of the above factorizations for the matrices $A_1, A_2, \dots, A_k$. Then,*

- *If $j - i$ is even,*

$$r_{i|\cdots|j} = r_{i|\cdots|j-1} + (r_j^1 + r_j^2 + \cdots + r_j^i) + r_j^{i+2} + r_j^{i+4} + \cdots + r_j^{j-2} + r_j^j;$$

- *If $j - i$ is odd,*

$$r_{i|\cdots|j} = r_{i|\cdots|j-1} + (r_j^{i+1} + r_j^{i+3} + \cdots + r_j^{j-2} + r_j^j).$$

[4] Recall that the subscript i refers to the ith matrix, while the superscript j refers to the jth block in that matrix.

For the general case, we need a mixture of the two preceding notations for block bidiagonal matrices, the blocks of which can be products of matrices, such as

$$\begin{pmatrix} A_{i_0}A_{i_0+1}\dots A_{i_1-1} & 0 & 0 & \dots & 0 \\ (A_{i_1}\dots A_{i_2-1})^* & A_{i_2}\dots A_{i_3-1} & 0 & \dots & 0 \\ 0 & (A_{i_3}\dots A_{i_4-1})^* & A_{i_4}\dots A_{i_5-1} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & A_{i_l}\dots A_j \end{pmatrix},$$

where $1 \le i_0 < i_1 < i_2 < i_3 < \dots < i_l \le j \le k$. Their rank is denoted by

$$r_{(i_0)\cdots(i_1-1)|i_1\cdots(i_2-1)|\cdots|i_l\cdots(j)}.$$

For instance, the rank of the matrix

$$\begin{pmatrix} A_2A_3 & 0 & 0 \\ A_4^* & A_5A_6A_7 & 0 \\ 0 & (A_8A_9)^* & A_{10} \end{pmatrix}$$

is represented by $r_{(2)(3)|4|(5)(6)(7)|(8)(9)|(10)}$.

THEOREM 6.3 (on the structure of a GSVD and a GURV). *The rank* $r_{(i_0)(i_0+1)\dots(i_1-1)|i_1\dots(i_2-1)|\dots|i_l\dots j}$ *can be derived as follows:*

1. *Calculate the following* $l+1$ *integers* s_j^i, $i = 1, 2, \dots, l+1$*:*

$$\begin{aligned} s_j^1 &= r_j^1 + r_j^2 + \dots + r_j^{i_0}, \\ s_j^2 &= r_j^{i_0+1} + r_j^{i_0+2} + \dots + r_j^{i_1}, \\ \dots &= \dots, \\ s_j^{l+1} &= r_j^{i_{l-1}+1} + r_j^{i_{l-1}+2} + \dots + r_j^{i_l}. \end{aligned}$$

2. *Depending on* l *even or odd there are two cases:*
—l *even:*

$$r_{i_0\cdots i_1-1|i_1\cdots i_2-1|\cdots|i_l\cdots j} = r_{i_0\cdots i_1-1|i_1\cdots i_2-1|\cdots|i_{l-1}\cdots i_l-1} + s_j^1 + s_j^3 + \dots + s_j^{l+1};$$

—l *odd:*

$$r_{i_0\cdots i_1-1|i_1\cdots i_2-1|\cdots|i_l\cdots j} = r_{i_0\cdots i_1-1|i_1\cdots i_2-1|\cdots|i_{l-1}\cdots i_l-1} + s_j^2 + s_j^4 + \dots + s_j^{l+1}.$$

Observe that Theorems 6.1 and 6.2 are special cases of Theorem 6.3. While Theorem 6.1 provides a direct expression of the dimensions r_j^i in terms of differences of ranks of products, Theorems 6.2 and 6.3 do so only implicitly. This is illustrated in the following examples.

Example. Let us determine the block dimensions of the quasi-diagonal matrix S_4 in a QPP-SVD of the matrices A_1, A_2, A_3, A_4 (which are also the block dimensions of an LUUU- or a ULLL-decomposition). From Theorem 6.2 we find that

$$\begin{aligned} r_4^4 &= r_4 - r_{34}, \\ r_3^4 &= r_{34} - r_{234}. \end{aligned}$$

From Theorem 6.3, we find that

$$s_4^1 = r_4^1, \qquad s_4^2 = r_4^2,$$

and

$$r_{(1)|(2)(3)(4)} = r_1 + s_4^2,$$

so that

$$r_4^2 = r_{1|(2)(3)(4)} - r_1.$$

Finally, since $r_4 = r_4^1 + r_4^2 + r_4^3 + r_4^4$, we find that

$$r_4^1 = r_1 + r_{(2)(3)(4)} - r_{1|(2)(3)(4)}.$$

Observe that this last relation can be interpreted geometrically as the dimension of the intersection between the row spaces of A_1 and $A_2A_3A_4$:

$$r_4^1 = \dim \operatorname{span}_{\text{row}}(A_1) + \dim \operatorname{span}_{\text{row}}(A_2A_3A_4) - \dim \operatorname{span}_{\text{row}} \begin{pmatrix} A_1 \\ A_2A_3A_3 \end{pmatrix}.$$

Example. Consider the determination of $r_5^1, r_5^2, r_5^3, r_5^4, r_5^5$ in a PQ^3-SVD of five matrices A_1, A_2, A_3, A_4, A_5 with Theorem 6.3, which coincides with the structure of a UULUL-URV or an LLULU-URV (see Table 1).

TABLE 1

	s_5^i
$r_{4\|5}$	$s_5^1 = r_5^1 + r_5^2 + r_5^3 + r_5^4$ $s_5^2 = r_5^5$
$r_{3\|4\|5}$	$s_5^1 = r_5^1 + r_5^2 + r_5^3$ $s_5^2 = r_5^4$ $s_5^3 = r_5^5$
$r_{2\|3\|4\|5}$	$s_5^1 = r_5^1 + r_5^2$ $s_5^2 = r_5^3$ $s_5^3 = r_5^4$ $s_5^4 = r_5^5$
$r_{(1)(2)\|3\|4\|5}$	$s_5^1 = r_5^1$ $s_5^2 = r_5^2 + r_5^3$ $s_5^3 = r_5^4$ $s_5^4 = r_5^5$

These relations can be used to set up a set of equations for the unknowns r_5^1, r_5^2, r_5^3, r_5^4, r_5^5, using Theorem 6.3 as

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} r_5^1 \\ r_5^2 \\ r_5^3 \\ r_5^4 \\ r_5^5 \end{pmatrix} = \begin{pmatrix} r_5 \\ r_{4|5} - r_4 \\ r_{3|4|5} - r_{3|4} \\ r_{2|3|4|5} - r_{2|3|4} \\ r_{(1)(2)|3|4|5} - r_{(1)(2)|3|4} \end{pmatrix},$$

the solution of which is

$$r_5^1 = r_{3|4|5} - r_{3|4} + r_{(1)(2)|3|4} - r_{(1)(2)|3|4|5},$$

$$
\begin{aligned}
r_5^2 &= r_{(1)(2)|3|4|5} - r_{(1)(2)|3|4} - r_{2|3|4|5} + r_{2|3|4},\\
r_5^3 &= r_{2|3|4|5} - r_{2|3|4} - r_{4|5} + r_4,\\
r_5^4 &= r_5 - r_{3|4|5} + r_{3|4},\\
r_5^5 &= r_{4|5} - r_4.
\end{aligned}
$$

7. A further block diagonalization of the GQRD. In this section, we note that a further block diagonalization of a GQRD can be interpreted as a preliminary step towards the corresponding GSVD. We proceed in two stages. First, we observe that each upper or lower triangular matrix in the generalized URVD of Theorem 5.3 can be block diagonalized. Next, we show how these block diagonalizations can be propagated backward through the GQRD. The first step is the factorization of the upper and lower triangular matrices $\tilde{T}_i$ of Theorem 5.3 into an upper or lower triangular matrix and a block diagonal matrix. For lower triangular matrices $\tilde{T}_i = \tilde{T}_i^l$, we can obtain a factorization of the form

$$\tilde{T}_i^l = L_i \tilde{D}_i^l,$$

where

$$
L_i = \begin{array}{c} r_{i-1}^1 \\ r_{i-1}^2 \\ \vdots \\ r_{i-1}^i \end{array}
\overset{\begin{array}{ccccc} r_{i-1}^1 & r_{i-1}^2 & \dots & r_{i-1}^{i-1} & r_{i-1}^i \end{array}}{\begin{pmatrix} I & 0 & \dots & 0 & 0 \\ * & I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & \dots & * & I \end{pmatrix}},
$$

$$
\tilde{D}_i^l = \begin{array}{c} r_{i-1}^1 \\ r_{i-1}^2 \\ \vdots \\ r_{i-1}^i \end{array}
\overset{\begin{array}{ccccc} r_i^1 & r_i^2 & \dots & r_i^i & r_i^{i+1} \end{array}}{\begin{pmatrix} R_{i,1} & 0 & \dots & 0 & 0 \\ 0 & R_{i,2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & R_{i,i} & 0 \end{pmatrix}}.
$$

Since the diagonal blocks $R_{i,j}$ are of full column rank, such a factorization is always possible. In a similar way, for upper triangular matrices $\tilde{T}_i = \tilde{T}_i^u$, we find a factorization of the form

$$\tilde{T}_i^u = U_i \tilde{D}_i^u,$$

with U_i an upper triangular block matrix with identity matrices on the block diagonal:

$$
U_i = \begin{array}{c} r_{i-1}^1 \\ r_{i-1}^2 \\ \vdots \\ r_{i-1}^i \end{array}
\overset{\begin{array}{ccccc} r_{i-1}^1 & r_{i-1}^2 & \dots & r_{i-1}^{i-1} & r_{i-1}^i \end{array}}{\begin{pmatrix} I & * & \dots & * & * \\ 0 & I & \dots & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & I \end{pmatrix}},
$$

$$
\tilde{D}_i^u = \begin{array}{c} r_{i-1}^1 \\ r_{i-1}^2 \\ \vdots \\ r_{i-1}^i \end{array}
\overset{\begin{array}{ccccc} r_i^1 & r_i^2 & \dots & r_i^i & r_i^{i+1} \end{array}}{\begin{pmatrix} R_{i,1} & 0 & \dots & 0 & 0 \\ 0 & R_{i,2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & R_{i,i} & 0 \end{pmatrix}}.
$$

Now suppose that we have done this for all matrices $\tilde{T}_i, i = 1, \ldots, k$ in a GQRD of Theorem 5.3. We show how we can propagate a further block diagonalization backward through the GQRD, in a way that is completely consistent with the corresponding GSVD of Theorem 4.1. To simplify the notation, we simply replace $\tilde{T}_i$ by T_i and $\tilde{D}_i$ by D_i in the following.

First, assume that T_k is lower block triangular. It then follows from the previous section that we can factorize T_k as

$$T_k^l = L_k D_k.$$

Depending on whether T_{k-1} is upper or lower triangular, we have two cases:

—$T_{k-1} = T_{k-1}^l$ lower triangular. In this case, the product $T_{k-1}^l L_k$ is lower triangular as well, and we can obtain a similar decomposition

$$T_{k-1}^l L_k = L_{k-1} D_{k-1},$$

where L_{k-1} is again lower triangular and D_{k-1} has the same diagonal blocks $R_{i,j}$ as T_{k-1}^l.

—$T_{k-1} = T_{k-1}^u$ upper triangular. In this case, the product $T_{k-1}^u L_k^{-*}$ is upper triangular, and we can obtain a factorization

$$T_{k-1}^u L_k^{-*} = U_{k-1} D_{k-1},$$

where U_{k-1} is upper triangular and D_{k-1} has the same diagonal blocks $R_{i,j}$ as T_{k-1}^u.

It is easily verified that when T_k is upper triangular, similar conclusions can be obtained.

In general, let T_i be lower triangular and assume that it is factorized as

$$T_i^l = L_i D_i Z_i.$$

Assume that T_{i-1} is lower triangular. Then T_{i-1} can be factored as

$$T_{i-1}^l = L_{i-1} D_{i-1} L_i^{-1}.$$

If T_{i-1} is upper triangular, it can be factored as

$$T_{i-1}^u = U_{i-1} D_{i-1} L_i^*,$$

where U_{i-1} is upper triangular. The cases with T_i upper triangular are similar. Table 2 summarizes all possibilities.

TABLE 2

T_i Lower triangular $T_i = L_i D_i Z_i$	T_{i-1} Lower triangular	$T_{i-1} = L_{i-1} D_{i-1} L_i^{-1}$
	T_{i-1} Upper triangular	$T_{i-1} = U_{i-1} D_{i-1} L_i^*$
T_i Upper triangular $T_i = U_i D_i Z_i$	T_{i-1} Lower triangular	$T_{i-1} = L_{i-1} D_{i-1} U_i^*$
	T_{i-1} Upper triangular	$T_{i-1} = U_{i-1} D_{i-1} U_i^{-1}$

Example. Let us apply this result to a sequence of four matrices A_1, A_2, A_3, A_4 with compatible dimensions. If the required sequence is ULUU, then

$$\begin{aligned} A_1 &= Q_0 T_1^u Q_1^* = Q_0 (U_1 D_1 L_2^*) Q_1^* = (Q_0 U_1) D_1 (Q_1 L_2)^*, \\ A_2 &= Q_1 T_2^l Q_2^* = Q_1 (L_2 D_2 U_3^*) Q_2^* = (Q_1 L_2) D_2 (Q_2 U_3)^*, \\ A_3 &= Q_2 T_3^u Q_3^* = Q_2 (U_3 D_3 U_4^{-1}) Q_3^* = (Q_2 U_3) D_3 (Q_3 U_4)^{-1}, \\ A_4 &= Q_3 T_4^u Q_4^* = Q_3 (U_4 D_4) Q_4^* = (Q_3 U_4) D_4 Q_4^*. \end{aligned}$$

Note that $U_1 = I_{n_0}$. This follows immediately from the block structure of U_i for $i = 1$. Observe that the relationships between the common factors in the left-hand side of these expressions conform with the requirements for a QQP-SVD. Only the middle factors $D_i, i = 1, 2, 3, 4$ are not quasi-diagonal.

8. Conclusions. In this paper, a constructive proof was given of a multimatrix generalization of the concept of rank factorization. The connection of this new decomposition with the analogous GSVD was also shown. The block structure of both generalizations and the ranks of the individual diagonal blocks in both decompositions were indeed shown to be identical. As is shown in a forthcoming paper, the spaces spanned by certain block columns of the orthogonal transformation matrices Q_i are, in fact, identical to those of the GSVD. The difference lies only in a particular choice of basis vectors for these spaces. The consequences of these connections are still under investigation. We mention the following results here:

- Updating the above decomposition to yield the GSVD requires nonorthogonal transformation. These updating transformations can be chosen block triangular with diagonal block sizes compatible with the index sets derived in Theorem 4.1.
- A modified orthogonal decomposition can be defined where the compound matrix is *not* triangularized but diagonalized. This new factorization is a variant of the above decomposition where now a special coordinate system is chosen for each of the individual orthogonal transformations Q_i. The result is an orthogonal decomposition of the type of Theorem 5.3 where now the generalized singular values can be extracted from the diagonal elements of some triangular blocks. The orthogonal updating needed to obtain this new decomposition can be done with techniques described in [2].
- A geometric interpretation can be given of the bases obtained from the transformation matrices Q_i in Theorem 5.1. As particular examples of these spaces we retrieve the following well-known concepts.

(a) For the case $A_i = (A - \alpha I)$ the GQRD in fact reconstructs the nested null spaces of the matrices $(A - \alpha I)^i$, which reveal the Jordan structure of the matrix A at the eigenvalue α (see also the example in §2).

(b) For the cases $A_{2i} = (A - \alpha B)$ and $A_{2i+1} = B$ the decomposition reconstructs the nested null spaces of the sequences $[B^{-1}(A - \alpha B)]^i$ and $[(A - \alpha B)B^{-1}]^i$, which reveal the Kronecker structure of the pencil $\lambda B - A$ at the generalized eigenvalue α (see [30] and [31]).

(c) For the cases $A_1 = D$ and $A_i = C \cdot A^{i-1} \cdot B, i = 1, \ldots,$ the decomposition reconstructs the invertibility subspaces of the discrete time system

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ y_k &= Cx_k + Du_k. \end{aligned}$$

These are in fact also the spaces constructed by the structure algorithm of Silverman [29], and they play a role in several key problems of geometrical systems theory [34].

Other applications of GSVDs have been described in [7], [8], [11], [13], and [35], while applications of the generalized QR-decompositions are described in [25] and [36].

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