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The quotient SVD and H_∞ norms

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Abstract

In this paper we show the use of the quotient singular value decomposition QSVD in the analysis of aspects of multivariable systems relative to one another. The $L_{Q-\infty}$-norm is introduced as a measure for the largest ratio of the output signals of 2 multiple input-multiple output systems. It is shown how the $L_{Q-\infty}$-norm can be computed. Using the QSVD the direction of the worst case input signal can be found.

Keywords: QSVD, ∞-norm, multivariable systems

Introduction

In this paper we will analyse some "directional" aspects of multivariable systems relative to one another. The term 'directionality' refers to the fact that multivariable systems possess properties that vary spatially as well as with frequency ([5]). For one multiple input-multiple output system (MIMO), this concept of directionality can be made precise by the *singular value decomposition* (SVD) (see e.g. [6], as was shown in e.g. [2]).

We will show how a generalisation of the SVD for two matrices, called the *quotient SVD* (QSVD), can be used to study directional aspects of 2 transfer functions, relative to one another.

The definition of the QSVD is given in section 2. Its use in analysing the signal-to-signal ratio of 2 transfer matrices is explained in section 3. In section 4 we introduce the $L_{Q-\infty}$-norm and the $H_{Q-\infty}$-norm and show how it can be calculated. The conclusions are in section 5.

The QSVD

The QSVD states that any pair of complex matrices with the same number of columns, say, $A \in C^{l\times m}$ and $B \in C^{p\times m}$ can be jointly factorized as

$$A = U_a S_a X^{-1}$$
$$B = U_b S_b X^{-1}$$

where $U_a \in C^{l\times l}$ and $U_b \in C^{p\times p}$ are unitary, $X \in C^{m\times m}$ is a non-singular square matrix and $S_a \in R^{l\times m}$ and $S_b \in R^{p\times m}$ are matrices with the following structure:

$$S_a = \begin{array}{c} r_{a|b}-r_b \\ r_a+r_b-r_{a|b} \\ l-r_a \end{array} \overset{\begin{array}{cccc} r_{a|b}-r_b & r_a+r_b-r_{a|b} & r_{a|b}-r_a & m-r_{a|b} \end{array}}{\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & D_a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}$$

$$S_b = \begin{array}{c} p-r_b \\ r_a+r_b-r_{a|b} \\ r_{a|b}-r_a \end{array} \overset{\begin{array}{cccc} r_{a|b}-r_b & r_a+r_b-r_{a|b} & r_{a|b}-r_a & m-r_{a|b} \end{array}}{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & D_b & 0 & 0 \\ 0 & 0 & I & 0 \end{pmatrix}}$$

where $r_a = rank(A)$, $r_b = rank(B)$ and $r_{a|b} = rank\begin{pmatrix} A \\ B \end{pmatrix}$. D_a and D_b are diagonal matrices with strictly positive elements smaller than 1. Call the diagonal elements of S_a, $\alpha_1, \ldots, \alpha_{\min(l,m)}$ and those of S_b, $\beta_1, \ldots, \beta_{\min(p,m)}$. The pairs of real numbers (α_i, β_i), $i = 1, \ldots, \min(l,p,m)$ are called the quotient singular value pairs and by convention, we put

$$\alpha_i^2 + \beta_i^2 = 1$$

and order the pairs so that

$$\frac{\alpha_1}{\beta_1} \geq \frac{\alpha_2}{\beta_2} \geq \ldots$$

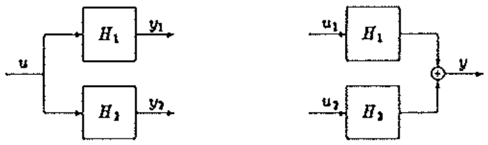

Figure 1: Two interpretations of the oriented signal-to-signal ratio for transfer matrices.

These ratios will be called the *quotient singular values*. Observe that when A is singular, they may be 0 and when B is singular, they may be infinite. The quotient singular values (or Q-singular values) will be denoted as σ_{qi}, where σ_{qi} is the i^{th} Q-singular value.

Signal-to-signal ratio

First we recapitulate the signal-to-signal ratio for matrices. The definition of the signal-to-signal ratio (SSR) is ([3]):

Definition 1 *The signal-to-signal ratio of 2 matrices is the ratio of the magnitudes of the outputs when the same input unit vector is applied to both:*

$$R_x[A,B] = \frac{\|Ax\|}{\|Bx\|}$$

Consider two matrices A and B with an equal number of columns. Using the QSVD it can be shown that, when $x = x_i$, the i^th column of X, then

$$R_{x_i}[A,B] = \frac{\alpha_i}{\beta_i}$$

The directions of maximal and minimal SSR are given by the columns of the matrix X in the QSVD-decompostion. These directions are *not orthogonal.*

Let us now look at the signal-to-signal ratio of 2 transfer matrices as a function of frequency. Two different situations can be considered (see figure 1), either 2 systems with the same input

$$y_1 = H_1 u \quad y_2 = H_2 u$$

or 2 inputs with the same output

$$y = H_1 u_1 + H_2 u_2$$

In the first case the 2 transfer functions have the same input u but different outputs y_1 and y_2. For an interpretation assume that H_2 is a 'good' transfer matrix while H_1 is a disturbing one. Obviously, we would like the relative influence of u on y_1 to be small compared to its influence on y_2. Calculate the QSVD at each frequency:

$$y(s) = \begin{pmatrix} H_1(s) \\ H_2(s) \end{pmatrix} u(s) = \begin{pmatrix} U_1(s)S_1(s)X^{-1}(s) \\ U_2(s)S_2(s)X^{-1}(s) \end{pmatrix} u(s)$$

The signal-to-signal ratio will be good if the Q-singular values are small for all frequencies. The QSVD shows the size and the direction of the largest signal-to-signal ratio at each frequency. If in control application the output of the 'bad' transfer function has to be minimized relatively to the output of the 'good' transfer function, the control engineer has to design a controller that makes the Q-singular values as small as possible over all frequencies. This is discussed further.

In the second situation there are different inputs u_1 and u_2 and one output y. Then we can write that

$$y = y_1 + y_2 = H_1 u_1 + H_2 u_2$$

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The SSR is in this case a ratio of input signals. Suppose u_1 and u_2 are vectors of sinusoidal signals with the same frequency ω. Call u_{1y} the input signal that produces a unit norm output signal in the direction y, when no input u_2 is present. Define a similar u_{2y}. Then the SSR in the output direction y, is the ratio of the norms of the input signals u_{2y} and u_{1y}. A SSR value larger than 1 shows that the corresponding direction y is controlled relatively more by u_1 than by u_2. Or the same output signal y can be generated by a smaller signal u_1 than u_2.

For a control interpretation, suppose that u_1 is the 'good' input while u_2 is the disturbing one. If y has to be controlled mainly by u_1, a controller has to be designed that minimizes the Q-singular values of the pair $(H_1(j\omega), H_2(j\omega))$ at all frequencies. maximize

The $L_{Q-\infty}$ and $H_{Q-\infty}$-norm

Now we are able to generalise the L_∞-norm and H_∞-norm for the ratio of the outputs of 2 transfer functions. A level set algorithm to calculate the $L_{Q-\infty}$-norm is also described.

Definition

Definition 2 *If $H_1(s)$ and $H_2(s)$ have an equal number of inputs, $H_1(s)$ has no imaginary poles and $H_2(s)$ is of full column rank on the $j\omega$-axis, then the $L_{Q-\infty}$-norm of the transfer matrices is defined as*

$$\left\| \frac{H_1}{H_2} \right\|_{Q-\infty} = \sup_{\omega \in \mathbb{R}} \sigma_{q1}\left(H_1(j\omega), H_2(j\omega)\right)$$

Definition 3 *If $H_1(s)$ and $H_2(s)$ have an equal number of inputs, $H_1(s)$ has no poles in the right half plane and $H_2(s)$ is of full column rank on the right half plane, then the $H_{Q-\infty}$-norm of the transfer matrices is defined as*

$$\left\| \frac{H_1}{H_2} \right\|_{Q-\infty} = \sup_{\Re(s)>0} \sigma_{q1}\left(H_1(s), H_2(s)\right)$$

When H_1 is stable and all the zeros of H_2 are in the open left half plane, then the $L_{Q-\infty}$ and $H_{Q-\infty}$-norm coincide.

In this section we assume that H_2 (the 'inverted' transfer matrix) is full rank on the $j\omega$-axis, otherwise the $H_{Q-\infty}$-norm would become ∞ or undefined.

A level set algorithm for $L_{Q-\infty}$-norm

In this section, we generalise an algorithm due to Boyd [1] for the computation of the L_∞-norm. Here, we will follow a similar approach. Let (α, β) with $\alpha^2 + \beta^2 = 1$ be a quotient singular value pair of the pair of transfer matrices $H_1(j\omega)$ and $H_2(j\omega)$ at a certain frequency ω where

$$\begin{aligned} H(j\omega) &= \begin{pmatrix} H_1(j\omega) \\ H_2(j\omega) \end{pmatrix} \\ &= \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} (j\omega I_n - A)^{-1} B \end{aligned}$$

Theorem 1 *Put*

$$\begin{aligned} M_\gamma &= \begin{pmatrix} A^t & 0 & 0 \\ 0 & A^t & 0 \\ 0 & 0 & -A \end{pmatrix} + \begin{pmatrix} C_1^t & 0 & 0 \\ 0 & C_2^t & 0 \\ 0 & 0 & -B \end{pmatrix} \\ &\times \begin{pmatrix} -\alpha & 0 & D_1 \\ 0 & -\beta & D_2 \\ D_1^t\beta & -D_2^t\alpha & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 & -C_1 \\ 0 & 0 & -C_2 \\ -B^t\beta & B^t\alpha & 0 \end{pmatrix} \end{aligned}$$

Suppose $\gamma = \frac{\alpha}{\beta}$ is not a Q-singular value of the matrix pair (D_1, D_2), then $-j\omega$ is an eigenvalue of M_γ if and only if for some i, $\sigma(H_1(j\omega), H_2(j\omega)) = \gamma$.

In case of discrete transfer matrices, a generalized eigenvalue problem has to be solved. This is stated in the next theorem:

Theorem 2 *Define the next two matrices:*

$$\begin{aligned} V_\gamma &= \begin{pmatrix} I & 0 & 0 \\ 0 & A^t & 0 \\ 0 & 0 & A^t \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ C_1^t & 0 & 0 \\ 0 & C_2^t & 0 \end{pmatrix} \\ &\times \begin{pmatrix} \alpha I & 0 & -D_1 \\ 0 & \beta I & -D_2 \\ D_1^t\beta & -D_2^t\alpha & 0 \end{pmatrix}^{-1} \begin{pmatrix} C_1 & 0 & 0 \\ C_2 & 0 & 0 \\ 0 & -B^t\beta & B^t\alpha \end{pmatrix} \\ W_\gamma &= \begin{pmatrix} A & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} + \begin{pmatrix} 0 & 0 & B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &\times \begin{pmatrix} \alpha I & 0 & -D_1 \\ 0 & \beta I & -D_2 \\ D_1^t\beta & -D_2^t\alpha & 0 \end{pmatrix}^{-1} \begin{pmatrix} C_1 & - & 0 \\ C_2 & 0 & 0 \\ 0 & -B^t\beta & B^t\alpha \end{pmatrix} \end{aligned}$$

Suppose (α, β) is not a Q-singular value pair of the matrix pair (D_1, D_2).

Then $e^{j\omega T}$ is a generalized eigenvalue of $V_\gamma z = e^{j\omega T} W_\gamma z$ if and only if for some i, $\sigma_{qi}(H_1(e^{j\omega T}), H_2(e^{j\omega T})) = \gamma$.

These theorems can be used to develop a quadratic convergent algorithm for computing the $L_{Q-\infty}$-norm as in [1]. First experiments in MATLAB show good convergence. For the sake of conciseness, the details and further applications are omitted.

Conclusions

In this article we showed how the QSVD can be used to study directional aspects of transfer matrices, relative to other transfer matrices. Using the $L_{Q-\infty}$-norm we can find the largest signal-to-signal ratio as a function of frequency. This norm can be obtained with a level set algorithm. When the frequency at which this occurs is found, the direction of the worst case input signal can also be determined, using the QSVD.

The QSVD is only one member of a whole set of decompositions of two and more matrices ([4]). So, one might also define other norms for more than 2 transfer matrices. However, in that case the interpretation of the result gets complicated.

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