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The quotient SVD and directional aspects of 2 transfer matrices

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Abstract

In this paper directional aspects of 2 transfer matrices relative to one another are studied. It is shown that the quotient singular value decomposition is the appropriate tool for it. It is also shown how the relative directional gain can be shown graphically using a signal-to-signal plot. The directions of maximal and minimal relative gain can easily be computed using the quotient singular value decomposition. They are normally not orthogonal, *in contrast with the results for one transfer matrix.*

1 Introduction

It is well known that directional aspects of one transfer matrix can be studied using the singular value decomposition (**SVD**) (see e.g. [6], as was shown in e.g. [2]). The term directionality refers to the fact that multivariable systems possess properties that vary spatially as well as with frequency. In this paper, directional aspects of multiple input-multiple output systems relative to one another are studied. Therefore the quotient singular value decomposition (**QSVD**), a generalization of the **SVD**, will be used.

In section 2 the definition and some properties of the **QSVD** are stated. In section 3 the concepts of indicator ellipsoid and oriented energy plot are discussed. It will be shown how the oriented energy plot allows for a better visualization of the directional gain. In section 4 the oriented signal-to-signal ratio is introduced. It will be shown how it can be computed using the **QSVD**. In section 5 some applications in control are mentioned and the conclusions are summarized.

2 The quotient singular value decomposition

The **QSVD** states that any pair of complex matrices with the same number of columns, say, $A \in \mathbb{C}^{l\times m}$ and $B \in \mathbb{C}^{p\times m}$ can be jointly factorized as

$$\begin{aligned} A &= U_a S_a X^{-1} \\ B &= U_b S_b X^{-1} \end{aligned}$$

where $U_a \in \mathbb{C}^{l\times l}$ and $U_b \in \mathbb{C}^{p\times p}$ are unitary, $X \in \mathbb{C}^{m\times m}$ is a nonsingular square matrix and $S_a \in \mathbb{R}^{l\times m}$ and $S_b \in \mathbb{R}^{p\times m}$ are matrices with the following structure:

$$S_a = \begin{array}{c} \\ r_{a|b}-r_b \\ r_a+r_b-r_{a|b} \\ l-r_a \end{array} \begin{array}{c} \begin{array}{cccc} r_{a|b}-r_b & r_a+r_b-r_{a|b} & r_{a|b}-r_a & m-r_{a|b} \end{array} \\ \left(\begin{array}{cccc} I & 0 & 0 & 0 \\ 0 & D_a & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

$$S_b = \begin{array}{c} \\ p-r_b \\ r_a+r_b-r_{a|b} \\ r_{a|b}-r_a \end{array} \begin{array}{c} \begin{array}{cccc} r_{a|b}-r_b & r_a+r_b-r_{a|b} & r_{a|b}-r_a & m-r_{a|b} \end{array} \\ \left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & D_b & 0 & 0 \\ 0 & 0 & I & 0 \end{array}\right) \end{array}$$

where $r_a = rank(A)$, $r_b = rank(B)$ and $r_{a|b} = rank\begin{pmatrix} A \\ B \end{pmatrix}$.

D_a and D_b are diagonal matrices with strictly positive elements smaller than 1. Call the diagonal elements of S_a, $\alpha_1, \ldots, \alpha_{\min(l,m)}$ and those of S_b, $\beta_1, \ldots, \beta_{\min(p,m)}$. The pairs of real numbers (α_i, β_i), $i = 1, \ldots, \min(l,p,m)$ are called the quotient singular value pairs and by convention, we put

$$\alpha_i^2 + \beta_i^2 = 1$$

and order the pairs so that

$$\frac{\alpha_1}{\beta_1} \geq \frac{\alpha_2}{\beta_2} \geq \cdots$$

These ratios will be called the *quotient singular values*. Observe that when A is singular, some of them will be 0 and when B is singular, some of them will be infinite. The quotient singular values (or Q-singular values) will be denoted as σ_{qi}, where σ_{qi} is the i^{th} Q-singular value. The **QSVD** used to be called the *generalized SVD*. Originally it was formulated by Paige and Saunders ([7]) and Van Loan ([9]). The **QSVD** is only one member of an infinite set of generalizations of the **SVD** as shown in [4].

3 The indicator ellipsoid and oriented energy plot

In this section the definitions of the indicator ellipsoid and the oriented energy are stated. They are first illustrated on matrices and subsequently their definition is extended to transfer matrices.

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3.1 The indicator ellipsoid

A complete picture of the action of a matrix $A \in C^{l\times m}$ is provided by the image under A of the unit sphere of C^m:

Theorem 1 **Indicator ellipsoid of a matrix** $A \in C^{l\times m}$ *[2]*
Let $A \in C^{l\times m}$ be of rank r and consider the linear transformations $y = Ax$. The domain and codomain transformations induced by its **SVD** *$A = U\Sigma V^*$, where $\Sigma = diag(\sigma_i)$ contains the singular values σ_i, are $x = V\xi$ and $y = U\eta$. Let $S_m = \{ x \in C^m \mid \|x\| = 1 \}$ denote the unit sphere of C^m. Let $A[S_m]$ denote the image under A of S_m. Then the following properties hold:*

If $r = l = m$: $A[S_m] = \{ y \in C^l \mid y = U\eta \, , \sum_{i=1}^{r}(\eta_i/\sigma_i)^2 = 1 \}$

If $r = l < m$: $A[S_m] = \{ y \in C^l \mid y = U\eta \, , \sum_{i=1}^{r}(\eta_i/\sigma_i)^2 \leq 1 \}$

If $r = m < l$: $A[S_m] = \{ y \in C^l \mid y = U\eta \, , \sum_{i=1}^{r}(\eta_i/\sigma_i)^2 = 1, \eta_{r+1} = \ldots = \eta_l = 0 \}$

If $r < min(l,m)$: $A[S_m] = \{ y \in C^l | y = U\eta, \sum_{i=1}^{r}(\eta_i/\sigma_i)^2 \leq 1, \eta_{r+1} = \ldots = \eta_l = 0 \}$

The matrix A maps the unit sphere S_m of C^m onto an r-dimensional ellipsoid in $R(A)$ with as principal axes the left singular vectors u_i of U that correspond to nonzero singular values $\sigma_i > 0$. If A is not of full column rank, the points in $R(A)$ interior to the ellipsoid have to be included. If A has full row rank, the ellipsoid has l principal axes with positive length. Note that the four cases of the theorem are mutually exclusive.

Let $A \in C^{l\times m}$, $x \in C^m$ and $y = Ax$. Then

$$max_{\|x\|=1} \|y\| = \sigma_1$$
$$min_{\|x\|=1} \|y\| = \sigma_{min(l,m)}$$

The Frobenius norm of A is defined as $\|A\|_F^2 = \sum_{i=1}^{l}\sum_{j=1}^{m} |a_{ij}|^2$. It is easy to show that $\|A\|_F^2 = trace(A^*A) = trace(AA^*)$. Using the fact that $\|UA\|_F = \|AV\|_F = \|A\|_F$ for arbitrary unitary matrices U and V (i.e. the Frobenius norm is *unitarily invariant*), we find from the **SVD** of A that

$$\|A\|_F^2 = \|U\Sigma V^*\|_F^2 = \|U^*U\Sigma V^*V\|_F^2 = \|\Sigma\|_F^2 = \sigma_1^2 + \ldots + \sigma_r^2$$

Summarizing, we have $\|A\|_F^2 = \sum_{i=1}^{r}\sigma_i^2 = trace(A^*A) = trace(AA^*)$. We have also the following statistical result, which puts the indicator ellipsoid in a statistical framework.

Theorem 2
Let $x \in C^m$ be a random vector, which is independently and identically normally distributed with mean zero and covariance matrix I_m. Then $y = Ax$ is normally distributed with mean zero and covariance matrix AA^* and $\mathbf{E}\|y\|^2 = (\sigma_1^2 + \ldots + \sigma_r^2)m$.

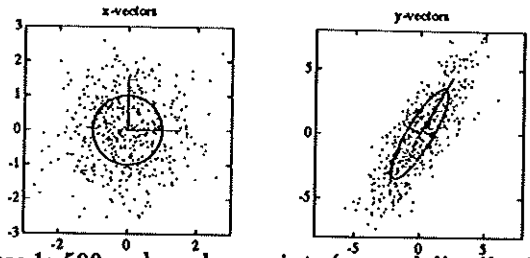

Figure 1: 500 real random points (normal distribution, zero mean, covariance I_2) are shown in the left plot. The unit circle is generated by normalizing these random points to have norm one. For each random point x, we compute $y = Ax$ and show the obtained points y in the right plot. The ellipsoid is obtained by plotting the image under A of the points on the unit circle. Observe that its principal axes lie in the direction of the left singular vectors, with half lenghts equal to the singular values.

Example: Consider the 2×2 matrix A with its **SVD**:

$$A = \begin{pmatrix} 2 & -0.866 \\ 3.464 & 0.5 \end{pmatrix} = \begin{pmatrix} 0.5 & -0.866 \\ 0.866 & 0.5 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Figure 1 and 2 show the indicator ellipsoid of the matrix A. Figure 1 illustrates how random unit vectors x are mapped on the indicator ellipsoid. The principal axes are in the direction of the left singular vectors with half lenghts equal to the singular values. Figure 2 shows how on average the points are more attracted to the left singular vector corresponding to the largest singular value.

3.2 Oriented energy of a matrices

The geometrical interpretation of the static transfer performed by a matrix A can also be shown in another way. Let x be a unit vector in R^m and $y = Ax \in R^l$. An *oriented energy plot* can be obtained by plotting the vector

$$x \sqrt{x^t A^t A x}$$

in each direction x. Such a plot tells us how much a vector x will get amplified when the matrix A is applied to it. The quantity $\|Ax\|$ is called the oriented energy in the direction x. In general, there is one maximum (in the direction of the first right singular vector) and one minimum (in the direction of the m-th right singular vector). The other extrema, which are saddle points, occur at the remaining right singular vectors. An illustration can be found in figure 3. There is an important difference between the *indicator ellipsoid* and an *oriented energy plot*: The indicator ellipsoid is a 'picture' in the output space R^l of the vectors y while the oriented energy plot is a 'picture' in the domain R^m of the input space.
The oriented energy plot shows the size of the output generated by the input directions. If large outputs are to be avoided, the oriented energy plot shows which

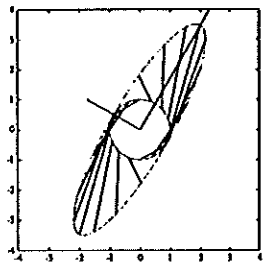

Figure 2: The small circle is generated by 500 random points on the unit circle. Also shown are the 500 corresponding points on the indicator ellipsoid, some of which are connected by a line with their corresponding point on the unit circle. The right singular vectors are in the directions $(1,0)$ for $\sigma_1 = 4$ and $(0,1)$ for $\sigma_2 = 1$. The directions of the left singular vectors are given by the long lines. It can be observed that on the average points are more 'attracted' to the left singular vector corresponding to the largest singular value, especially in the neighborhood of the right singular vector corresponding to the smallest singular value.

input directions have to be suppressed, e.g. by controller design. Hence its importance in control.

3.3 Oriented energy of transfer matrices

Let us now consider complex transfer matrices, that are function of frequency. The SVD of a transfer matrix is defined as:

$$y(s) = H(s)u(s) = U(s)S(s)V(s)^* u(s)$$

where $U(s), S(s)$ and $V(s)$ are such that at each s $U(s)S(s)V(s)^*$ is the SVD of $H(s)$.

Here we are only interested in the frequencies at the imaginary axis $s = j\omega$. If $H(s)$ has no poles on the imaginary axis, the singular values $\sigma_i(j\omega)$ are continuous, non-differentiable functions of ω. However, the next theorem relates the singular values on the imaginary axis with real analytic functions:

Theorem 3 *There are real analytic functions f_i : $R \to \mathbf{R}, i = 1, \ldots, m$, such that for all $\omega \in \mathbf{R}$,*

$$\{\sigma_1(H(j\omega)), \ldots, \sigma_m(H(j\omega))\} = \{|f_1(\omega)|, \ldots, |f_m(\omega)|\}$$

The functions $f(.)$ form an *unsigned singular value decomposition* (**USVD**) of the transfer matrix $H(s)$. Further details can be found in ([1]).

It is also possible to show the oriented energy as a function of frequency. This is illustrated with an example.

Example:

Consider a 2 by 2 transfer function, real on the imaginary axis:

$$H(j\omega) = \begin{pmatrix} 5 & \frac{8-\omega^2}{16.5-\omega^2} \\ \frac{19-\omega^2}{16.5-\omega^2} & \frac{8-\omega^2}{1.4-\omega^2} \end{pmatrix}$$

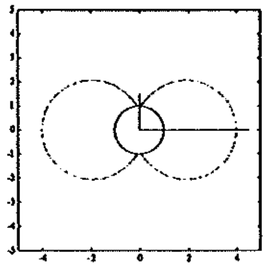

Figure 3: Shown is the unit circle obtained by normalizing 500 random points (normal distribution, zero mean, covariance I_2) to have norm 1. For each normalized point x, the corresponding point $x\|Ax\|$ is also plotted, generating the oriented energy plot. The minimum and maximum occur at the directions of the right singular vectors.

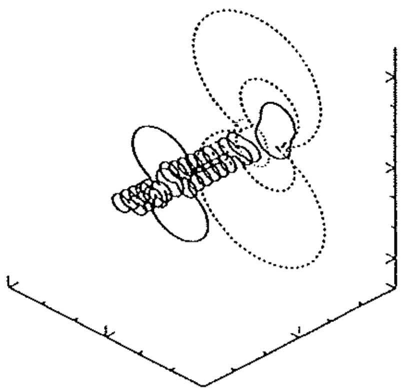

Figure 4: The oriented energy plot of a 2×2 transfer function.

In figure 7, the oriented energy plot $u\|H(j\omega)u\|$ is plotted where u is a unit input vector. In this case the vector represents the amplitudes and phases of sinus waves with frequency ω at the inputs. Putting all these plots one behind the other we obtain the 3-dimensional plot shown in figure 4.

At $\omega = \sqrt{8} = 2.828$, H has a zero, which is obtained as a number $s \in \mathbf{C}$ where $H(s)$ drops rank. As can be seen on the corresponding plot, this does not mean that there cannot be an output. The output is zero only when the correct input direction is applied. In all other cases there is still an output signal. This directional aspect does not occur with SISO transfer functions which themselves become zero at a system zero.

4 Signal-to-signal ratio

Until now we have been looking at the properties of only 1 matrix or transfer function. In this section, we look at the properties of one matrix *relative* to another one.

4.1 The signal-to-signal ratio for matrices

The definition of the signal-to-signal (**SSR**) is:

Definition 1 *The signal-to-signal ratio of 2 matrices A and B in the direction x is the ratio of the oriented energies of A and B in the direction x:*

$$R_x[A,B] = \frac{\|Ax\|}{\|Bx\|}$$

Consider two matrices A and B with an equal number of columns and let their **QSVD** be:

$$\begin{cases} A = U_a S_a X^{-1} \\ B = U_b S_b X^{-1} \end{cases}$$

Let us now demonstrate how the **QSVD** provides a canonical decomposition for the signal-to-signal ratios just as the **SVD** does for the oriented energy. Hereto, consider the **QSVD** of $[A, B]$. Then,

$$R_x^2[A,B] = \frac{x^t A^t A x}{x^t B^t B x} = \frac{(x^t X^{-t}) S_a^t S_a (X^{-1} x)}{(x^t X^{-t}) S_b^t S_b (X^{-1} x)}$$

Obviously, when $x = x_i$, the i^{th} column of X, then

$$R_{x_i}[A,B] = \frac{\alpha_i}{\beta_i}$$

Hence the following properties hold:

$$\max_{x,\|x\|=1} R_x[A,B] = \frac{\alpha_1}{\beta_1}$$
$$\min_{x,\|x\|=1} R_x[A,B] = \frac{\alpha_m}{\beta_m}$$

A signal-to-signal ratio plot is obtained by plotting the vector $x\frac{\|Ax\|}{\|Bx\|}$ for each unit direction x. The **SSR**-plot is again a picture in the input space.

Example:
As an example the signal-to-signal ratio is plotted for 2 pairs of matrices: $[A, B_1]$ and $[A, B_2]$

$$A = \begin{pmatrix} .25 & 0 \\ 0 & 2 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 1 & -1.5 \\ 0 & 1.5 \end{pmatrix} \qquad B_2 = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

For several unit vectors, $R_x[A, B_1]x$ and $R_x[A, B_2]x$ are plotted, the signal-to-signal ratios in the direction x. The directions of maximal and minimal **SSR** are shown and also the circle with radius $\frac{\alpha_1}{\beta_1}$ and $\frac{\alpha_2}{\beta_2}$.

The maximal **SSR** is given by $\frac{\alpha_1}{\beta_1}$ and the minimal by $\frac{\alpha_2}{\beta_2}$. The directions of maximal and minimal **SSR** are given by the columns of the matrix X in the **QSVD**-decomposition. These directions are *not* orthogonal as in the case of the oriented energy. The direction and size of the maximal **SSR** is dependent on the 2 matrices, as can be seen in the figure.

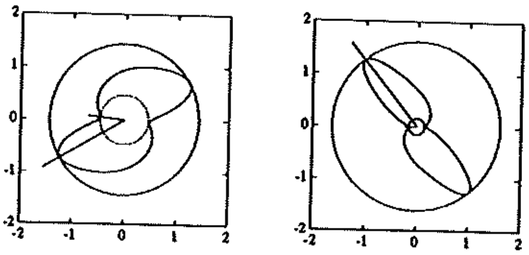

Figure 5: The signal to signal ratio of 1 matrix A and 2 different matrices B. The maximal **SSR** is $\frac{\alpha_1}{\beta_1}$ and the minimal $\frac{\alpha_2}{\beta_2}$. The directions of maximal and minimal **SSR** are *not orthogonal*. They are given by the columns of the X in the **QSVD** decomposition.

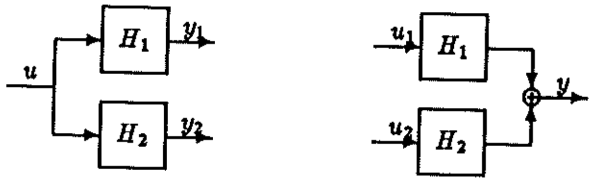

Figure 6: Two interpretations of the oriented signal-to-signal ratio for matrices.

4.2 Signal-to-signal ratio of transfer matrices

Let us now look at the signal-to-signal ratio of 2 transfer matrices as a function of frequency. Two different situations can be considered (see figure 6), either 2 systems with the same input

$$y_1 = H_1 u \quad y_2 = H_2 u$$

or 2 inputs with the same output

$$y = H_1 u_1 + H_2 u_2$$

In the first case the 2 transfer functions have the same input u but different outputs y_1 and y_2. For an interpretation assume that H_2 is a 'good' transfer matrix while H_1 is a disturbing one. Obviously, we would like the relative influence of u on y_1 to be small compared to its influence on y_2. Calculate the **QSVD** at each frequency:

$$y(s) = \begin{pmatrix} y_1(s) \\ y_2(s) \end{pmatrix} = \begin{pmatrix} H_1(s) \\ H_2(s) \end{pmatrix} u(s) = \begin{pmatrix} U_1(s)S_1(s)X^{-1}(s) \\ U_2(s)S_2(s)X^{-1}(s) \end{pmatrix} u(s)$$

The signal-to-signal ratio will be good if the Q-singular values are small for all frequencies. The **QSVD** shows the size and the direction of the largest signal-to-signal ratio at each frequency. If in control application the output of the 'disturbing' transfer function has to be minimized relatively to the output of the 'good' transfer function, the control engineer has to design a controller that minimizes the

Q-singular values over all frequencies.

In the second situation there are different inputs u_1 and u_2 and one output y. Then we can write that

$$y = y_1 + y_2 = H_1 u_1 + H_2 u_2$$

using the **QSVD** this can be rewritten as

$$y = Y \begin{pmatrix} I & 0 & 0 \\ 0 & D_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} U^t u_1 + Y \begin{pmatrix} 0 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{pmatrix} V^t u_2$$

Put

$$\begin{aligned} u_1 &= U_1 p_1 + U_2 p_2 + U_3 p_3 \\ u_2 &= V_1 q_1 + V_2 q_2 + V_3 q_3 \end{aligned}$$

Then

$$y = \begin{pmatrix} Y_1 & Y_2 D_1 & 0 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} + \begin{pmatrix} 0 & Y_2 D_2 & Y_3 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

or

$$\begin{aligned} y &= Y_1 p_1 + Y_2 (D_1 p_2 + D_2 q_2) + Y_3 q_3 \\ &= \begin{pmatrix} Y_1 & Y_2 & Y_3 \end{pmatrix} \begin{pmatrix} p_1 \\ D_1 p_2 + D_2 q_2 \\ q_3 \end{pmatrix} \end{aligned}$$

From this we can conclude that there are directions in the output space which can only be influenced by the first input, in these directions $\frac{\alpha_i}{\beta_i}$ is infinite. Other directions can only be controlled by the second input. In these directions $\frac{\alpha_i}{\beta_i}$ is 0. Other directions are controlled by both inputs.

Suppose now that the input signal u_1 is known that produces an output signal in the direction y, when $u_2 = 0$. Similarly, also the input signal u_2 that produces the same output signal, with $u_1 = 0$ is known. Then the **SSR** in the direction y is here the ratio of the magnitudes of the input signals u_1 and u_2. The Q-singular values indicate then the relative influence of u_1 compared to u_2. Thus if the output y has to be influenced primarily by the input u_1, the Q-singular values have to be small.

Example:
As an example, take 2 transfer matrices in the first configuration (with the same input u):

$$H_1(s) = \begin{pmatrix} 5 & \frac{s^2+8}{s^2+16.5} \\ \frac{s^2+19}{s^2+16.5} & \frac{s^2+8}{s^2+1.4} \end{pmatrix}$$

$$H_2(s) = \begin{pmatrix} \frac{1}{s^2+2.5} & \frac{s^2+3}{s^2+5} \\ \frac{2}{s^2+13} & \frac{s^2+5}{s^2+10} \\ \frac{s^2+2}{s^2+7} & \frac{s^2+11}{s^2+19} \end{pmatrix}$$

The signal-to-signal ratio is plotted for different values of ω. The plots are shown in figure 8. The maximal and minimal signal-to-signal ratio and direction vary quite drastically with changing ω. Notice that when H_2 becomes small, the signal-to-signal ratio becomes large (around $\omega = 1.25$). If H_1 has a zero on the imaginary axis, the **SSR** becomes zero in one direction.

The directions and sizes of the maximum and minimum **SSR** can be obtained using the **QSVD**. Notice again that these directions are in general *not* orthogonal.

5 Conclusions

In this paper, several ways of presenting directional aspects of matrices and transfer functions are discussed. In our opinion the oriented energy is best suited for studying the directional gains. The extension to directional aspects of 2 matrices or transfer matrices is the signal-to-signal ratio. The directions and gains can easily be computed using the QSVD.

In control system design, this can e.g. be used in cases where not the actual output of a system is important but the sizes of 2 outputs relative to one another. A ∞-norm can be defined for the largest ratio over all frequencies. Controllers that minimize or maximize certain relative properties might also be developed.

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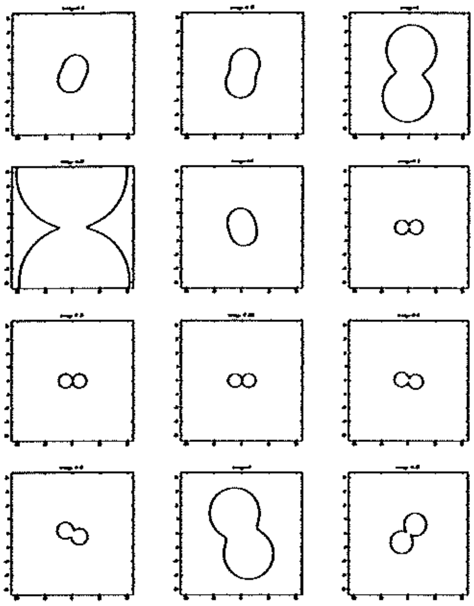

Figure 7: The oriented energy plots of $H(j\omega)$ for different values of ω. The direction and size of the largest amplification changes drastically. Observe also that even at a zero of the transfer matrix on the imaginary axis the output is non-zero unless the input has the correct orientation.

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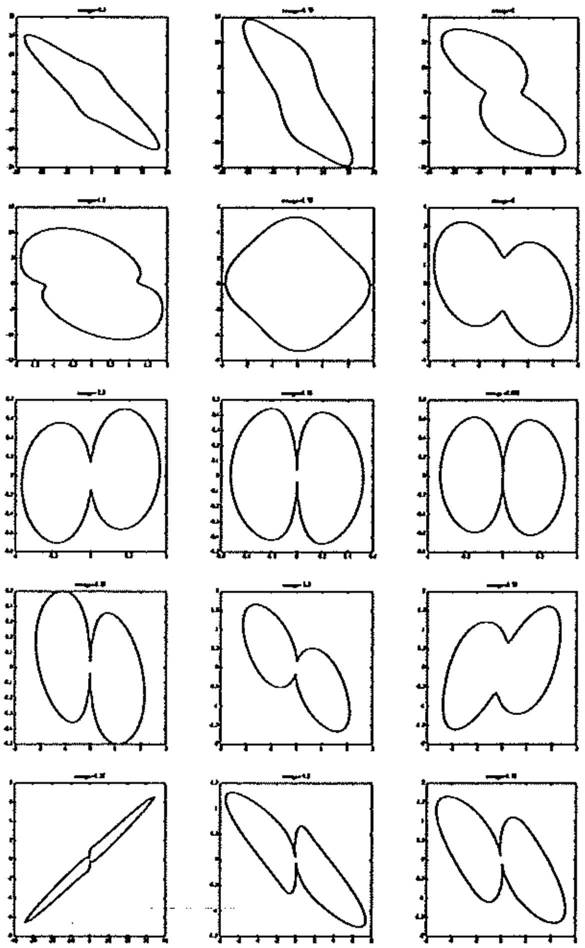

Figure 8: The **SSR** of two the transfer matrices with 2 equal inputs at different frequencies ω. The direction and size of the **SSR** can change drastically with the frequency. Notice that the directions of maximal and minimal **SSR** are, in general, not orthogonal.