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H_2 model reduction for SISO systems *

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Abstract

We consider the problem of approximating the impulse response of a discrete time linear time invariant single-input single-output system by one of lower order in a least squares sense. Using Lagrange multipliers, we derive a set of nonlinear equations which can be interpreted as a 'nonlinear' generalized singular value decomposition, in which the elements of the weights are quadratic functions of the components of the singular vectors. We analyse the problem both in the time domain and the z-domain. An algorithm is derived that is inspired by inverse iteration.

Keywords: Balanced realization, balanced gains, singular value decomposition, stochastic realization, inverse iteration.

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1 Introduction

In this paper, we consider the approximation of a given n-dimensional discrete linear time invariant (LTI) single-input single-output (SISO) system with proper transfer function

$$\frac{b(z)}{a(z)} = \frac{b_{n-1}z^{n-1} + \ldots + b_1 z + b_0}{z^n + \ldots + a_1 z + a_0}$$

by an LTI SISO system of order $q < n$

$$\frac{\tilde{b}(z)}{\tilde{a}(z)} = \frac{\tilde{b}_{q-1}z^{q-1} + \ldots + \tilde{b}_1 z + \tilde{b}_0}{z^q + \ldots + \tilde{a}_1 z + \tilde{a}_0}$$

such that $\sum_{i=1}^{p+q}(h_i - \tilde{h}_i)^2$ is minimized. Here $h_i, i = 1, \ldots, p+q$ are the first $p+q$ Markov parameters (impulse response samples) of the given system while $\tilde{h}_i$ are the impulse response samples of the approximating system. The number p is a user specified time horizon, which may be ∞. The H_2-criterion is particularly relevant in a statistical sense for $p \to \infty$: The obtained optimal reduced model delivers an output that will be closest in a least mean squares sense to the output of the original given system when the applied input is white noise. Earlier references are [2] [3] [19], all the references included in [15] and more recently, the work in [16] [17] [14]. A numerical analysis approach can be found in [12]. The case of noisy data (for which our time domain iteration also applies) is treated in [1] [6] [22] [23] [24]. From a differential geometry point of view, the H_2 model reduction problem is analysed in [4] [5].

2 Derivation in the time domain

Let $\tilde{H}$ be a $p \times (q+1)$ Hankel matrix that contains the approximating impulse responses $\tilde{h}_i$ up to sample $p+q$. The fact that the approximating system is of order q may be expressed in the form of $p+1$ constraints as $\tilde{H}y = 0$ and $y^t y = 1$. The first constraint requires the existence of a vector in the null space of the matrix $\tilde{H}$, which should be non-trivial. This is ensured by the second constraint. We will also use the vectors h and $\tilde{h}$, which contain the first $p+q$ impulse response samples of the original system and the approximating one. Introducing p Lagrange multipliers $l_i, i = 1, \ldots, p$ for the first p constraints (which will be collected in a vector l) and a Lagrange multiplier λ for the last constraint, we can write the

Lagrangean $\mathcal{L}$ of the optimization problem as $\mathcal{L}(\tilde{h},y,l)=\sum_{i=1}^{p+q}(h_i-\tilde{h}_i)^2+l^t\tilde{H}y+\lambda(y^ty-1)$. Setting all derivatives to zero results in the set of equations (a convolution is denoted by a '$\star$'):

$$h-\tilde{h}=l\star y \quad , \quad \tilde{H}^tl=y\lambda \quad , \quad y^ty=1 \quad , \quad \tilde{H}y=0$$

There are $2p+2q+1$ unknowns (the elements of $\tilde{h},l$, y and λ) and exactly $2p+2q+1$ equations. It is straightforward to find that $\lambda=0$ because $l^t\tilde{H}y=\lambda=0$.

It is a tedious though straightforward exercise to write out the difference of the two Hankel matrices $H-\tilde{H}$ as a function of the components of the vectors l and y. One can then eliminate $\tilde{H}$ by post multiplying this expression with y to get $Hy=D_yl$. Post multiplication of $H^t-\tilde{H}^t$ with l results in $H^tl=D_ly$. Here, D_y and D_l are positive definite symmetric Toeplitz matrices, the elements of which are quadratic functions of the elements of y resp. l. Actually, the element at position $|i-j|$ is just the $|i-j|$-th element of the autocorrelation of the sequences y resp. l (appropriately padded with zeros) (see [9] for more details). If we normalize l so that $l/\|l\|=x$ and $\|l\|=\sigma$, we have the equations

$$\begin{aligned} Hy&=D_yx\sigma & x^tx&=1 \\ H^tx&=D_xy\sigma & y^ty&=1 \end{aligned} \tag{1}$$

It can be verified that the object function is given by $\sum_{i=1}^{p+q}(h_i-\tilde{h}_i)^2=l^tHy=x^tHy\sigma=x^tD_yx\sigma=y^tD_xy\sigma$.

3 A 'non-linear' generalized singular value problem

Note that, if D_y is invertible, we can eliminate the vector $x=D_y^{-1}Hy/\sigma$ and write $(H^tD_y^{-1}H)y=D_xy\sigma^2$ which is a generalized eigenvalue problem in which the weights D_x and D_y are quadratic functions of the elements of the eigenvectors. If D_x is invertible, we might even convert this generalized eigenvalue problem into a symmetric eigenvalue problem as $[T_{xy}][D_x^{1/2}y]=[D_x^{-1/2}H^tD_y^{-1}HD_x^{-1/2}][D_x^{1/2}y]=[D_x^{1/2}y]\sigma^2$. If D_x and D_y would be constant matrices, the equations (1) would be very much related to a singular value decomposition, with positive definite weights in the column and row space. Let us explore this connection in some more detail. Suppose u,v and τ are solutions of the problem

$$\begin{aligned} Hv &= D_v u\tau \qquad & u^t D_v u = 1 \\ H^t u &= D_u v\tau \qquad & v^t D_u v = 1 \end{aligned} \tag{2}$$

The only difference is here that we use another normalization for both u and v. If D_u and D_v would be constant, we would have a singular value decomposition with different inner products in column and row space (see e.g.[8]). We can scale these equations as $H\frac{v}{\|v\|} = (D_v/\|v\|^2)\frac{u}{\|u\|}(\tau\|u\|\|v\|)$ and $H^t\frac{u}{\|u\|} = (D_u/\|u\|^2)\frac{v}{\|v\|}(\tau\|u\|\|v\|)$. If we now put $x = u/\|u\|$, $y = v/\|v\|$ and $\sigma = \tau\|u\|\|v\|$, then

$$x^t H y\sigma = \frac{u^t}{\|u\|}H\frac{v}{\|v\|}(\tau\|u\|\|v\|) = \frac{u^t}{\|u\|}\frac{D_v}{\|v^2\|}\frac{u}{\|u\|}(\tau\|u\|\|v\|)^2 = \tau^2$$

This implies that the minimum we are after is exactly equal to the minimal singular value squared of the weighted SVD problem (2). The conclusion is that the only difference between (1) and (2) lies in the normalization of the vectors and one can always go from a solution of (1) to one of (2) and the other way around.

4 Analysis using the QR-decomposition

Let the QR decomposition of H be:

$$H = \begin{pmatrix} \underbrace{Q_1}_{p\times(q+1)} & \underbrace{Q_2}_{p\times(p-q-1)} \end{pmatrix} \begin{pmatrix} \underbrace{R}_{(q+1)\times(q+1)} \\ 0 \end{pmatrix} \tag{3}$$

Then we can decompose $l = x\sigma$ as $l = Q_1 v + Q_2 w$ for certain vectors v and w. Hence, from (1) (recall that $l = x\sigma$), we find

$$\begin{pmatrix} R^t & 0 & 0 \\ Q_2^t D_y Q_1 & Q_2^t D_y Q_2 & 0 \\ Q_1^t D_y Q_1 & Q_1^t D_y Q_2 & -R \end{pmatrix} \begin{pmatrix} v \\ w \\ y \end{pmatrix} = \begin{pmatrix} D_l y \\ 0 \\ 0 \end{pmatrix} \tag{4}$$

5 Algorithms

We will present two algorithms to solve the set of nonlinear equations as in (1), which represents the necessary conditions for a triplet x, σ, y to be the minimizing solution. The iteration number is indexed between square brackets.

5.1 An algorithm based on the generalized eigenvalue problem

The formulation of the problem as a generalized eigenvalue problem naturally lends itself towards implementation in an iterative algorithm as follows:

Iteratively reweighted generalized eigenvalue problem :

Initialization: Choose initial guesses $u^{[0]}$ and $v^{[0]}$ and construct $D_{v^{[0]}}$ and $D_{u^{[0]}}$.

For $k = 1$ till convergence:

Solve $\begin{pmatrix} 0 & H \\ H^t & 0 \end{pmatrix} \begin{pmatrix} u^{[k]} \\ v^{[k]} \end{pmatrix} = \begin{pmatrix} D_{v^{[k-1]}} & 0 \\ 0 & D_{u^{[k-1]}} \end{pmatrix} \begin{pmatrix} u^{[k]} \\ v^{[k]} \end{pmatrix} \tau^{[k]}$ for the smallest eigenvalue $\tau^{[k]}$ with $(u^{[k]})^t D_{v^{[k-1]}} u^{[k]} = 1$, $(v^{[k]})^t D_{u^{[k-1]}} v^{[k]} = 1$.

Results: After convergence, set $x = u/\|u\|, y = v/\|v\|$ and $\sigma = \tau\|u\|\|v\|$.

There are several possible convergence tests. A natural one is to measure the difference between two iterates as $\|v^{[k]} - v^{[k-1]}\|$ and then stop when decreases below a prespecified tolerance. We will however not discuss this in detail.

5.2 Inverse iteration

A possible algorithm to calculate the smallest eigenvalue and corresponding eigenvector of a symmetric matrix is by inverse iteration. Instead however of calculating the minimal eigenvalue in each step, we could also perform only one step of an inverse iteration scheme and then update the weighting matrices D_x and D_y. This is achieved in the following iteration which is nothing else than an iterative way of solving equation (4) [1].

Inverse iteration algorithm :

[1] As a matter of fact, there are several other possibilities of solving this set of equations iteratively (like 'Gauss-Seidel'-like or 'SOR'-like variants), but we only analyse one particular version here...

Initialization: Choose $x^{[0]}, y^{[0]}, \sigma^{[0]}$ and normalize $\|x^{[0]}\| = 1$, $\|y^{[0]}\| = 1$.

For $k = 1$ till convergence:

1. $v^{[k]} = R^{-t} D_{x^{[k-1]}} y^{[k-1]}$.
2. $w^{[k]} = -(Q_2^t D_{y^{[k-1]}} Q_2)^{-1} (Q_2^t D_{y^{[k-1]}} Q_1) v^{[k]}$.
3. $x^{[k]} = Q_1 v^{[k]} + Q_2 w^{[k]}$.
4. $x^{[k]} = x^{[k]} / \|x^{[k]}\|$.
5. $z^{[k]} = R^{-1} Q_1^t D_{y^{[k-1]}} x^{[k]}$.
6. $\sigma^{[k]} = \|z^{[k]}\|$.
7. $y^{[k]} = z^{[k]} / \sigma^{[k]}$.
8. Convergence test using $x^{[k]}, y^{[k]}, \sigma^{[k]}$.

Result: The approximating model can be obtained from x, y and σ using the formulas of section 2.

The asymptotic convergence rate of this algorithm, which is linear, is discussed in Figure 1 below. The matrix Q_2 is a $p \times (p-q-1)$ matrix. Hence as p is usually much larger than q, using Q_2 is quite time consuming. It is however possible to eliminate Q_2 (and also x completely) when for $p \to \infty$ the z-transform is used, as will be shown below. For finite p, a good initial guess might be provided by the singular triplet corresponding to the smallest singular value of H. Other initial estimates might be provided by a balanced realization or a realization based on Kabamba's balanced gains [15]. But, as we will see below, starting from an initial guess which is good with respect to the initial H_2 error, does not necessarily imply that convergence to the global minimum will occur. A natural convergence test is to monitor the difference between two consecutive iterates, as e.g $\|y^{[k]} - y^{[k-1]}\|$.

6 Properties in the z-domain

Since the residual signal $h - \tilde{h}$ is a convolution of the sequences l and y, we easily find that[2] $h(z) - \tilde{h}(z) = l(z)y(z)$. If $p \to \infty$, we put $h(z) = b(z)/a(z)$ and $\tilde{h}(z) = \tilde{b}(z)/\tilde{a}(z)$ so

[2]The z-transform of a sequence a is denoted by $\mathcal{Z}[a] = a(z) = \sum_{k=-\infty}^{+\infty} a_k z^{-k}$. For a causal sequence, we have $a(z) = \sum_{k=0}^{\infty} a_k z^{-k}$. The inverse z-transform is $\mathcal{Z}^{-1}$. The causal part of a sequence is indicated

that

$$\frac{b(z)}{a(z)} - \frac{\tilde{b}(z)}{\tilde{a}(z)} = l(z)y(z) \tag{5}$$

Denote by $\tilde{a}_r(z)$ the reverse polynomial of $\tilde{a}(z)$: $\tilde{a}_r(z) = \tilde{a}_0 z^q + \tilde{a}_1 z^{q-1} + \ldots + 1$. Let the components of y be $y = (\ y_0 \ \ y_1 \ \ \ldots \ \ y_q\ )^t$. From the fact that $\tilde{H}y = 0$ and $y^t y = 1$, it follows easily that $y(z) = \eta \tilde{a}_r(z)/z^q = \eta \tilde{a}(z^{-1})$ where η is a normalization constant which ensures that $y^t y = 1$. It also follows from (5) that in the optimum, $l(z)$ will be rational with poles given by the zeros of $a(z)\tilde{a}(z)$.

The inverse iteration algorithm of the previous section becomes inefficient as $p \to \infty$. It is however possible to transform the iterative solution of the set of equations (4) using z-transforms into an iteration in the z-domain with a finite number of parameters. Instead of using the QR-decomposition of H, we will use the QR-decomposition of the $p \times n$ Hankel matrix H_n ($p \to \infty$):

$$H_n = \begin{pmatrix} \underbrace{Z_1}_{p\times(n)} & \underbrace{Z_2}_{p\times(p-n)} \end{pmatrix} \begin{pmatrix} \underbrace{S_1}_{(n)\times(n)} \\ 0 \end{pmatrix} \tag{6}$$

$$= \begin{pmatrix} \underbrace{Z_{11}}_{p\times(q+1)} & \underbrace{Z_{12}}_{p\times(n-q-1)} & \underbrace{Z_2}_{p\times(p-n)} \end{pmatrix} \begin{pmatrix} \underbrace{S_{11}}_{(q+1)\times(q+1)} & \underbrace{S_{12}}_{(q+1)\times(n-q-1)} \\ 0 & \underbrace{S_{22}}_{(n-q-1)\times(n-q-1)} \\ 0 & 0 \end{pmatrix} \tag{7}$$

Observe that $Q_1 = Z_{11}$ and $R = S_{11}$ because H is given by the first $q+1$ columns of H_n. Let $v_\star \in \mathbb{R}^{n\times 1}$ and $w_\star \in \mathbb{R}^{(p-n)\times 1}$ be defined by decomposing l as $l = Z_1 v_\star + Z_2 w_\star$ (Recall that also $l = Q_1 v + Q_2 w = Z_{11} v + (\ Z_{12} \ \ Z_2\ ) w$). Using this new QR-decomposition, we can rewrite equation (4) as

$$\left(\begin{array}{c|cc|c} R^t & 0 & 0 & 0 \\ \hline Z_{12}^t D_y Z_{11} & Z_{12}^t D_y Z_{12} & Z_{12}^t D_y Z_2 & 0 \\ Z_2^t D_y Z_{11} & Z_2^t D_y Z_{12} & Z_2^t D_y Z_2 & 0 \\ \hline Z_{11}^t D_y Z_{11} & Z_{11}^t D_y Z_{12} & Z_{11}^t D_y Z_2 & -R \end{array}\right) \begin{pmatrix} v \\ w \\ y \end{pmatrix} = \begin{pmatrix} D_l y \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

as $[.]_+$ while the causal part of the z-transform $a(z)$ is denoted by $[a(z)]_+$. All sequences are considered to be zero where ever they are not explicitly defined. When we only want to retain the first $q+1$ components of a sequence, this is denoted by $[.]_{0,\ldots,q}$

Changing the order of the block rows from $1-2-3-4$ to $1-3-4-2$, allows us to write

$$\begin{pmatrix} H^t Z_1 & 0 & 0 \\ Z_2^t D_y Z_1 & Z_2^t D_y Z_2 & 0 \\ Z_1^t D_y Z_1 & Z_1^t D_y Z_2^t & -Z_1^t H \end{pmatrix} \begin{pmatrix} v_\star \\ w_\star \\ y \end{pmatrix} = \begin{pmatrix} D_l y \\ 0 \\ 0 \end{pmatrix} \tag{8}$$

which implies that $(v^t \ \ w^t)^t = (v_\star^t \ \ w_\star^t)^t$. The conclusion is that an iteration with the set of equations (8) delivers in each iteration step the same results as an iteration with the set of equations (4). While the dimensions of the blocks in (4) permit to solve the system by bäcksubstitution, this is not the case for (8). The difference is however that the iterative solution of (8) can be converted to the z-domain. Let's now do just that. First note that in each iteration, $y^{[k]}(z) = \tilde{a}^{[k]}(z^{-1})\eta = \eta \tilde{a}_r^{[k]}(z)/z^q$.

Lemma 1 *The z-transform of $l^{[k]}$ is of the form $\mathcal{Z}[l^{[k]}] = l^{[k]}(z) = \frac{z^q f^{[k]}(z)}{a(z)\tilde{a}^{[k-1]}(z)}$ where $f(z)$ is a polynomial of degree $n-1$.*

Proof: Since $Z_2^t D_{y^{[k]}} l^{[k]} = 0$, it follows that $D_{y^{[k]}} l^{[k]}$ is in the column space of H_n, so that its z-transform must be of the form $\xi(z)/a(z)$ ($\xi(z)$ denotes an unknown polynomial). So we find $[\mathcal{Z}[D_{y^{[k-1]}} l(k)]]_+ = [\eta^2 \tilde{a}^{[k-1]}(z)\tilde{a}_r^{[k-1]}(z) l(z)/z^q]_+ = \xi(z)/a(z)$. It follows that $l(z)$ must cancel the poles at 0. Now recall that in the equilibrium $l(z)$ contains $a(z)$ and $\tilde{a}(z)$ in the denominator and that it is stable. Hence, in iteration step k, we include $\tilde{a}^{[k-1]}(z)$ in the denominator of $l^{[k]}(z)$. Furthermore, it must contain the poles in $a(z)$. This implies the Lemma. □

Lemma 2 *The z-transform of $l^{[k]}(z)$ can be written as $\mathcal{Z}[l^{[k]}] = \frac{c^{[k]}(z)}{a(z)} + \frac{a_r(z) e^{[k]}(z)}{a(z)\tilde{a}^{[k-1]}(z)}$ where $c^{[k]}(z)$ is a polynomial of degree $n-1$ and $e^{[k]}(z)$ is a polynomial of degree $q-1$.*

Proof: Obviously, $Z_1 v_\star^{[k]}$ is a finite dimensional signal since it is generated by a linear combination of the columns of H_n. Hence $\mathcal{Z}[Z_1 v_\star^{[k]}]$ is of the form $c^{[k]}(z)/a(z)$ for a certain polynomial $c^{[k]}(z)$. Therefore, because $l^{[k]} = Z_1 v_\star^{[k]} + Z_2 w_\star^{[k]}$, $\mathcal{Z}[Z_2 w_\star^{[k]}]$ must have in its denominator $a(z)\tilde{a}^{[k-1]}(z)$. The form of its numerator follows from $H^t Z_2 w_\star^{[k]} = 0$. Indeed, the null space of H^t is spanned by the columns of a band Toeplitz matrix in which each column of the band is just the vector a which contains the coefficients of $a(z)$. Hence, $Z_2 v_\star^{[k]}$ can be written as a linear combination of the columns of this matrix. This matrix-vector multiplication can be interpreted as a convolution of a FIR filter (the coefficients of a) with an infinite signal (the weights of the linear combination) so that with z-transforms we find that the numerator of $\mathcal{Z}(Z_2 w_*^{[k]})$ should contain a factor $a_r(z)$. □.

Lemma 3 *There exists a polynomial $s(z)$ of degree n with $s_0 = 0$ so that*

$$[\frac{b(z)}{a(z)}\frac{c^{[k]}(z^{-1})}{a(z^{-1})}]_+ + \frac{s^{[k]}(z)}{a(z)\tilde{a}^{[k-2]}(z)} = [\frac{f^{[k-1]}(z)z^q}{a(z)\tilde{a}^{[k-2]}(z)}\frac{f^{[k-1]}(z^{-1})z^{-q}}{a(z^{-1})\tilde{a}^{[k-2]}(z^{-1})}\frac{\eta\tilde{a}_r^{[k-1]}(z)}{z^q}]_+$$

Proof: The z-transform of the equation $H^t Z_1 v_*^{[k]} = D_{l[k-1]} y^{[k-1]}$ results in $[\mathcal{Z}^{-1}[\frac{b(z)}{a(z)}\frac{c^{[k]}(z^{-1})}{a(z^{-1})}]]_{0,\dots,q} = [\mathcal{Z}^{-1}[l^{[k-1]}(z)l^{[k-1]}(z^{-1})y^{[k-1]}(z)]]_{0,\dots,q}$. The fact that the first $q+1$ coefficients are equal, does not imply that the causal parts of z-transforms are equal. Therefore, one needs to introduce a polynomial $s(z)$ of degree n. Let's now show that $s_0 = 0$. Define $g(z) = g_n z^n + \dots + g_0$ via $[\frac{b(z)c(z^{-1})}{a(z)a(z^{-1})}]_+ = \frac{g(z)}{a(z)}$ and $m(z)$ of degree $n+q$ via $[l(z)l(z^{-1})y(z)]_+ = \frac{m(z)}{a(z)\tilde{a}(z)}$. We can now show that $g_0 = 0$ and $m_0 = 0$ by introducing also the anti-causal parts $\frac{b(z)c(z^{-1})}{a(z)a(z^{-1})} = \frac{g(z)}{a(z)} + \frac{h(z)}{a(z^{-1})}$ where $h(z)$ is a polynomial of degree $n-1$. If we equate equal powers in the numerator, we find that $g_0 = 0$. A similar trick works for $m_0 = 0$. That $s_0 = 0$ now follows directly from $\frac{g(z)}{a(z)} + \frac{s(z)}{a(z)\tilde{a}(z)} = \frac{m(z)}{a(z)\tilde{a}(z)}$. □

Lemma 4

$$[\frac{b(z)}{a(z)}\tilde{a}^{[k]}(z)]_+ = [\frac{\tilde{a}_r^{[k-1]}(z)\eta f^{[k]}(z)}{a(z)}]_+$$

Proof: This equality follows directly by taking z-transforms of the time-domain equation $Hy^{[k-1]} = D_{y[k-1]} l^{[k]}$. For instance, $\mathcal{Z}[D_{y[k]} l^{[k]}] = \tilde{a}^{[k-1]}(z)\tilde{a}^{[k-1]}(z^{-1})\eta^2 \frac{f^{[k]}(z)z^q}{a(z)\tilde{a}^{[k-1]}(z)}$. □

7 Interpolation conditions

Starting from Lemma 1-4, we can now derive interpolation conditions in the roots $\alpha_i, i = 1,\dots,n$ of $a(z)$ and the roots $\tilde{\alpha}_i^{[k]}$, $i = 1,\dots,q$ of $\tilde{a}^{[k]}(z)$. These will give us a set of equations to be solved in each iteration. First, we consider Lemma 3 in the n roots of $a(z)$, i.e. $\forall i = 1,\dots,n$:

$$b(\alpha_i)\tilde{a}^{[k-2]}(\alpha_i)c^{[k]}(1/\alpha_i) + a(1/\alpha_i)s^{[k]}(\alpha_i) = \frac{f^{[k-1]}(\alpha_i)f^{[k-1]}(1/\alpha_i)}{\tilde{a}^{[k-2]}(1/\alpha_i)}\eta\tilde{a}^{[k-1]}(1/\alpha_i)$$

These n equations in $2n$ unknowns will be written as $M_1^{[k]}c^{[k]} + M_2^{[k]}s^{[k]} = F_1^{[k]}$. Now apply Lemma 3 with the q zeros of $\tilde{a}^{[k-2]}$:

$$a(1/\tilde{\alpha}_i^{[k-2]})s^{[k]}(\tilde{\alpha}_i^{[k-2]}) = \frac{f^{[k-1]}(\tilde{\alpha}_i^{[k-2]})f^{[k-1]}(1/\tilde{\alpha}_i^{[k-2]})}{\tilde{a}^{[k-2]}(1/\tilde{\alpha}_i^{[k-2]})}\eta\tilde{a}^{[k-1]}(1/\tilde{\alpha}_i^{[k-1]}) \qquad i = 1,\ldots,q$$

These q equations in n unknowns are denoted by $M_3^{[k]}s^{[k]} = F_2^{[k]}$. Next we combine Lemma 1-2. Since there must be q zeros at zero, we have q linear equations for the coefficients of c and e as $M_4^{[k]}c^{[k]} + M_5^{[k]}e^{[k]} = 0$. From equating the remaining n coefficients in that expression, we find also an expression for the coefficients of $f(z)$ as $M_6^{[k]}c^{[k]} + M_7^{[k]}e^{[k]} = f^{[k]}$. Finally, we evaluate Lemma 4 in the n roots of $a(z)$ to find

$$b(\alpha_i)\tilde{a}^{[k]}(\alpha_i) = \eta\tilde{a}_r^{[k-1]}(\alpha_i)f^{[k]}(\alpha_i) \qquad i = 1,\ldots,n$$

which will be denoted as $M_8^{[k]}\tilde{a}^{[k]} = M_9^{[k]}f^{[k]}$. Together these equations form a set of $(3n+2q)$ equations in $(3n+2q)$ unknowns which can be written as

$$\begin{pmatrix} M_1 & M_2 & 0 & 0 & 0 \\ 0 & M_3 & 0 & 0 & 0 \\ M_4 & 0 & M_5 & 0 & 0 \\ M_6 & 0 & M_7 & -I_n & 0 \\ 0 & 0 & 0 & -M_9 & M_8 \end{pmatrix}^{[k]} \begin{pmatrix} c \\ s \\ e \\ f \\ \tilde{a} \end{pmatrix}^{[k]} = \begin{pmatrix} F_1 \\ F_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}^{[k]} \qquad (9)$$

which is the z-domain version of equation (8). The iteration with this set of equations, will converge linearly towards an equilibrium point. The approximating system can be easily reconstructed from these results.

8 Some more properties of the minimum

In this section, we derive some additional properties.

Theorem 1 *The z-transform of the residual signal $h - \tilde{h}$ satisfies*

$$\frac{b(z)}{a(z)} - \frac{\tilde{b}(z)}{\tilde{a}(z)} = \frac{[\tilde{a}_r(z)]^2 r(z)}{\tilde{a}(z)a(z)}$$

where $r(z) = r_{n-q-1}z^{n-q-1} + r_{n-q-2}z^{n-q-2} + \ldots + r_1 z + r_0$ is a non-monic polynomial.

Proof: Another way of stating it, is that $l(z) = \frac{z^q \tilde{a}_r(z) r(z)}{\eta a(z) \tilde{a}(z)}$. This structure follows directly from the orthogonality $\tilde{H}^t l = 0$ in the optimum. Indeed, a null space of $\tilde{H}^t$ is generated by the columns of a band Toeplitz matrix, of which each column of the band is the vector y. Hence, the vector l can be written as a linear combination of the columns of this Toeplitz matrix. Because of the Toeplitz structure, this gives a convolution in the time domain of a FIR filter (the coefficients of y) with an infinite signal t (the weights of the linear combinations). In the z-domain we then have $l(z) = \tilde{a}_r(z)t(z)\eta/z^q$ The structure of $l(z)$ now follows directly from $b(z)/a(z) - \tilde{b}(z)/\tilde{a}(z) = l(z)y(z)$. □

The equation in Theorem 1 can be rewritten as $b(z)\tilde{a}(z) - \tilde{b}(z)a(z) = [\tilde{a}(z)]^2 r(z)$. The same equation in continuous time is found in [16] where it directly follows from certain interpolation conditions, which come from [19]. Note that the residual contains an all-pass transfer function $\tilde{a}_r(z)/\tilde{a}(z)$. It was recently observed in [21] that judicious applications of Beurling's theorem allow to conclude that the residual $h(z) - \tilde{h}(z)$ must contain an all-pass (inner) factor, which is exactly what we find here. Note that if $q = n - 1$, the polynomial $r(z)$ reduces to a scalar r_0, which is also a special case considered in [16]. Let us now show that solving the equation of Theorem 1 is equivalent with solving a set of multivariate polynomials. Note that the coefficients of $\tilde{b}$ and r appear linearly in the equation. Hence we can write $\tilde{A}_1 (\tilde{b}^t \ \ r^t)^t = \tilde{A}_2$ where the elements of $\tilde{A}_1 \in \mathbb{R}^{(n+q)\times n}$ and $\tilde{A}_2 \in \mathbb{R}^{(n+q)\times 1}$ are linear and quadratic functions of the elements of $\tilde{a}$. We can eliminate $\tilde{b}$ and r by requiring that this overdetermined set of equations is consistent, so that we find $rank[\tilde{A}_1] = rank[\tilde{A}_1 \ \ \tilde{A}_2] = n$. This is only satisfied when at least q of the $(n+1)\times(n+1)$ minors vanish, which leads to a set of multivariate polynomials in the coefficients of $\tilde{a}$. For the special case that $q = 1$, we find just that the $(n+1)\times(n+1)$ determinant of $[\tilde{A}_1 \ \ \tilde{A}_2]$ should vanish. Since there is only one unknown coefficient $\tilde{a}_0$, this leads to a polynomial. As an example, for $n = 2$ and $q = 1$, we find a 3rd order polynomial and hence three solutions of our non-linear generalized SVD problem.

Let's finally mention that the error is always orthogonal to the approximant. Indeed, for finite p, $\sum_{i=1}^{p} \tilde{h}_i(h_i - \tilde{h}_i) = 0$ which implies that the vector of residuals (which represents the gradient of the object function) is orthogonal with respect to the approximating impulse response. For $p \to \infty$, one can use z transforms to find that $\oint \tilde{h}(z)(h(z) - \tilde{h}(z))\frac{dz}{z} = 0$.

This orthogonality in the z-domain is exploited in [21] to observe that the residual $E(z) = h(z) - \tilde{h}(z)$ should contain an all-pass transfer function, which is confirmed by our Theorem 1.

9 An example

We consider the four disk system of [18]. The transfer function is

$$h(z) = \frac{0.0448z^5 + 0.2368z^4 + 0.0013z^3 + 0.0211z^2 + 0.2250z + 0.0219}{z^6 - 1.2024z^5 + 2.3675z^4 - 2.0039z^3 + 2.2337z^2 - 1.0420z + 0.8513}$$

The results of the time domain iteration with $p = 100$, $q = 4$ are shown in Figure 1 and 2 for 30 iterations. The reduced system is:

$$\tilde{h}(z) = \frac{0.1747z^3 + 0.1061z^2 - 0.2146z + 0.3144}{z^4 - 1.7100z^3 + 2.3305z^2 - 1.6201z + 0.9148}$$

with an error of 0.4793. Next the z-domain iteration is applied (which is equivalent to the time domain iteration for $p = 100$). Different initial values are considered: With the balanced truncation as initial system (initial error of 0.6898), we find:

$$\tilde{H}(z) = \frac{0.1821z^3 + 0.0933z^2 - 0.2022z + 0.3065}{z^4 - 1.7139z^3 + 2.3333z^2 - 1.6241z + 0.9148}$$

with an error of 0.4779. This system is very similar to the one we found with the time domain iteration (for $p = 100$). With the balanced gains ([15]) as initial system (initial error 0.8082), we find:

$$\tilde{H}(z) = \frac{-0.0167z^3 + 0.1865z^2 + 0.1458z + 0.0069}{z^4 + 0.2205z^3 + 1.7119z^2 + 0.2125z + 0.8793}$$

with an error of 0.2847. The conclusion is that with an initial model that has a larger L_2 error, we converge to an L_2 solution with a lower error.

The error as a function of the reduced order is as follows

Order	5	4	3	2	1
Error	0.2644	0.2847	0.7424	0.7537	1.6340

Since the original plant has 3 complex conjugated pole-pairs (resonances), every time the number of pole pairs is reduced, a big jump in the minimal error is visible (between 3 and 4, and between 1 and 2). On the other hand, when a decaying exponential is deleted, the error stays almost constant (between 4 and 5, and between 2 and 3).

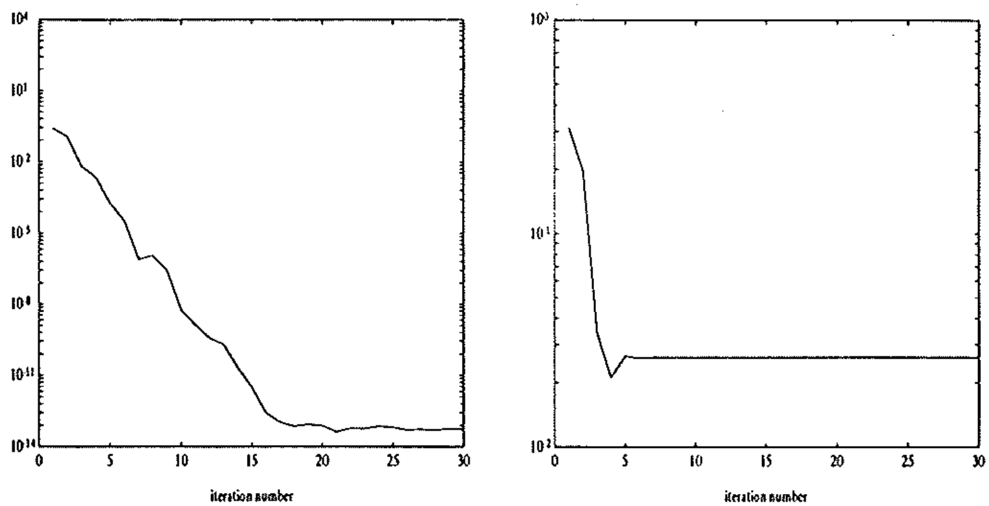

Figure 1: The interpretation as an inverse iteration scheme allows us to estimate the asymptotic convergence rate, which will be governed by the two largest eigenvalues of the matrix T_{xy}^{-1}. The smallest singular value $\sigma_{min}(\tilde{H}^{[k]})$ (shown left) will decrease linearly on a semi-logarithmic scale (asymptotically as the number of iterations $k \rightarrow \infty$), with a slope determined by $\lambda_2^{[k]}/\lambda_1^{[k]}$. This ratio is shown on the right. Observe that already after 5 iterations, we have reached the asymptotic regime. One can also show that asymptotically, the angle between two consecutive iterates $y^{[k-1]}$ and $y^{[k]}$ will be governed by the equation $(\lambda_2^{[k]})^2/(\lambda_1^{[k]})^2 log_{10}((y^{[k]})^t y^{[k-1]})^2 \approx log_{10}((y^{[k+1]})^t y^{[k]})^2$

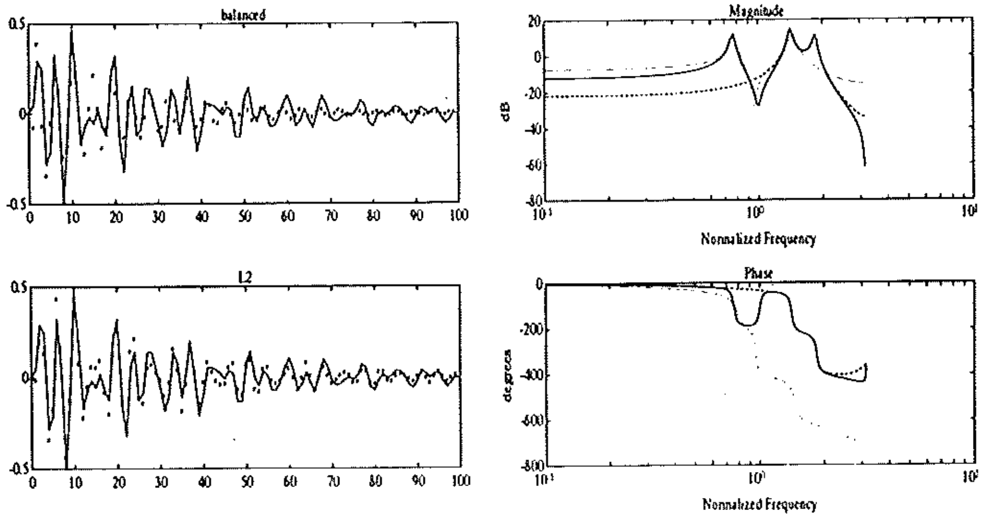

Figure 2: Comparison of the impulse responses. Full line = the original, '*' is the approximant. Top left=balanced, bottom left= L_2. Also the Bode diagrams are compared. Full line = the original, dotted line = balanced, dashed line = L_2. Top right=magnitude, bottom right = phase.

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