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H_2 controller design with an H_∞ bounded controller

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Abstract

In this paper the minimization of an H_2 norm is considered, when the controller is restricted to be linear, stable, finite dimensional and H_∞-norm bounded. It is also shown how this can be used in the design of a mixed H_2/H_∞ controller.

1 Introduction

In recent years there has been a lot of interest in the mixed H_2/H_∞ design problem (e.g. [1]-[6]). This problem is stated as minimizing an H_2-norm subject to the constraint that an H_∞-norm inequality has to be satisfied. In most papers, however, not the real H_2-norm is minimized but an upper bound. In this paper an other problem is considered first. The H_2-norm of a transfer function should be minimized using a linear, stable, finite dimensional controller that satisfies an H_∞-bound. Necessary conditions for this problem are derived, using a parameterization of this set of controllers of Steinbuch and Bosgra [5]. This can then be used to find controllers of a certain finite dimension that minimize a H_2-norm subject to a H_∞-norm constraint, as in the case of the general mixed H_2/H_∞ design problem.

2 Parameterization of H_∞ norm bounded transfer functions

In [5] Steinbuch and Bosgra describe a parameterization for strictly proper, stable, norm-bounded finite dimensional transfer functions:
Let the set Ω^*_γ be defined as

$$\Omega^*_\gamma = \{F(s) \mid \|F(s)\|_\infty < \gamma, F(s) \text{ stable, real rational,}$$

$$\text{strictly proper and of McMillan degree} \leq n\}$$

and consider the set Ω_γ

$$\Omega_\gamma = \{C(sI-A)^{-1}B | A = A_s + A_k, A_s = -\frac{1}{2}\gamma^{-2}BB^T$$

$$-\frac{1}{2}C^TC,\ A_k = -A_k^T \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times n_w}, C \in \mathbb{R}^{n_z\times n}\}$$

Proposition 1 *[5]:* $\Omega_\gamma = \Omega^*_\gamma$.

[1]The following text presents research results obtained within the framework of the Belgian programme on interuniversity attraction poles initiated by the Belgian State - Prime Minister's Office - Science Policy Programming. The scientific responsibility is assumed by its authors.
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3 H_2 optimization with an H_∞ bounded controller

In this section necessary conditions are derived for a H_∞-norm bounded controller that minimizes an H_2-norm. We will call this problem a constrained H_2 problem.
Consider the following linear, time invariant plant:

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{11} w + D_{12} u \\ y &= C_2 x + D_{21} w + D_{22} u \end{aligned} \quad (1)$$

Now find a dynamic, stable, strictly proper, H_∞-norm bounded controller K, $u = Ky$, that stabilizes the closed loop and such that the H_2-norm of the closed loop transfer function from w to z is minimized. K has to belong to Ω_γ. Without loss of generality γ can be taken $\gamma = 1$. The state space realization of the controller is then:

$$\begin{aligned} \dot{x}_c &= A_c x_c + B_c y \\ u &= C_c x_c \end{aligned} \quad (2)$$

To make the 2-norm of the closed loop finite, D_{11} has to be 0. For notational reasons, we also assume $D_{22} = 0$. This is not a limitation, as the controller is strictly proper. This is shown e.g. in the paper of Glover and Doyle on H_∞ control [7].
From (1) and (2) the state space realization of the closed loop can be derived ($D_{11} = 0$ and $D_{22} = 0$):

$$\begin{pmatrix} \dot{x} \\ \dot{x}_c \end{pmatrix} = \begin{pmatrix} A & B_2C_c \\ B_cC_2 & A_c \end{pmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix} + \begin{pmatrix} B_1 \\ B_cD_{21} \end{pmatrix} w$$
$$z = \begin{pmatrix} C_1 & D_{12}C_c \end{pmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix}$$

Define

$$\overline{A} = \begin{pmatrix} A & B_2C_c \\ B_cC_2 & A_c \end{pmatrix} \quad \overline{B} = \begin{pmatrix} B_1 \\ B_cD_{21} \end{pmatrix}$$
$$\overline{C} = \begin{pmatrix} C_1 & D_{12}C_c \end{pmatrix}$$

The control objective can be expressed as $\min_{K\in\Omega_1}$ trace$\{\overline{C}^t\overline{C}S\}$ where S is the solution of $\overline{A}S + S\overline{A}^t + \overline{B}\overline{B}^t = 0$. Necessary conditions for this problem are given in the following lemma:

Lemma 1 *Necessary conditions for the constrained H_2 problem.*

Given the state space realization of (1), a stable, finite dimensional, strictly proper controller K, with H_∞-norm smaller than 1, that stabilizes the closed loop

and minimizes the H_2-norm of the closed loop transfer function from w to z satisfies the following equations:

$$\begin{aligned} \overline{A}S + S\overline{A}^t + \overline{B}\overline{B}^t &= 0 \\ P\overline{A} + \overline{A}^t P + \overline{C}^t\overline{C} &= 0 \end{aligned}$$

$$\begin{pmatrix} P_{12}^t & P_{22} \end{pmatrix} \begin{pmatrix} S_{12} \\ S_{22} \end{pmatrix} = \begin{pmatrix} S_{12}^t & S_{22} \end{pmatrix} \begin{pmatrix} P_{12} \\ P_{22} \end{pmatrix}$$

$$A_s + \frac{1}{2} B_c B_c^t + \frac{1}{2} C_c^t C_c = 0$$

$$B_c = \left[\begin{pmatrix} S_{12}^t & S_{22} \end{pmatrix} \begin{pmatrix} P_{12} \\ P_{22} \end{pmatrix} \right]^{-1} \begin{pmatrix} P_{12}^t & P_{22} \end{pmatrix} \begin{pmatrix} S_{11} \\ S_{12}^t \end{pmatrix} C_2^t$$

$$C_c \begin{pmatrix} S_{12}^t & S_{22} \end{pmatrix} \begin{pmatrix} P_{12} \\ P_{22} \end{pmatrix} - D_{12}^t D_{12} C_c S_{22}$$

$$= D_{12}^t C_1 S_{12} + B_2^t \begin{pmatrix} P_{11}^t & P_{12} \end{pmatrix} \begin{pmatrix} S_{12} \\ S_{22} \end{pmatrix}$$

where $A_c = A_s + A_k$ with $A_s = A_s^t$ and $A_k = -A_k^t$. S and P have to be positive definite $S > 0$ and $P > 0$. The inverse has to exist.

The proof is straight forward using Lagrange multipliers.

4 Application to the mixed H_2/H_∞ problem

Using the above solution for the constrained H_2 problem with dynamic output feedback, the general mixed H_2/H_∞ problem can also be solved.

It is well-known that in general there is a set of solutions to the suboptimal H_∞ control problem [7], [8]. If there exists a solution, of course. Without loss of generality the H_∞ control problem can always be solved such that the H_∞-norm of the closed loop is smaller than 1. The set of controllers satisfying this condition is parameterized by a controller *generator* and a feedback Q. Where Q is a stable, H_∞-norm bounded transfer function.

This is used to solve the mixed H_2/H_∞ control problem. The state space realization is:

$$\begin{aligned} \dot{x} &= Ax + B_{w_1} w_1 + B_{w_2} w_2 + B_u u \\ z_1 &= C_{z_1} x + D_{z_1 w_1} w_1 + D_{z_1 w_2} w_2 + D_{z_1 u} u \quad (3) \\ z_2 &= C_{z_2} x + D_{z_2 w_1} w_1 + D_{z_2 w_2} w_2 + D_{z_2 u} \\ y &= C_y x + D_{y w_1} w_1 + D_{y w_2} w_2 + D_{yu} u \end{aligned}$$

Find a controller such that the closed loop from $\begin{pmatrix} w_1 & w_2 \end{pmatrix}^t$ to $\begin{pmatrix} z_1 & z_2 \end{pmatrix}^t$ is stable, $\|T_{z_1 w_1}\|_2$ is minimized and $\|T_{z_2 w_2}\|_\infty < 1$. The idea is now to calculate first the controller for the H_∞ part of the problem. Thus check if the H_∞ suboptimal problem is solvable, based on w_2, u, z_2 and y. If so, the controller generator, that is also a generalized system, is attached to the plant (3). Now concentrate on the H_2 problem.

Within the set of controllers that stabilize the closed loop and satisfy the H_∞ condition $\|T_{z_2 w_2}\|_\infty < 1$, find the controller that minimizes the H_2-norm of $T_{z_1 w_1}$. Therefore, we search for the transfer function Q, $r = Qv$, that minimizes the H_2-norm from z_1 to w_1.

The transfer function Q, however, should be such that it doesn't destabilize $T_{z_2 w_2}$ and keeps the H_∞-norm less than or equal to 1. Therefore, we know from the H_∞ control theory, that Q has to be a stable transfer function such that $\|Q\|_\infty < 1$.

To obtain a finite H_2-norm the following conditions are needed:

1. $D_{z_1 w_1} = 0$.
2. Q has to be strictly proper. A state space realization of Q is then:

$$\begin{aligned} \dot{x}_q &= A_q x_q + B_q v \\ r &= C_q x_q \end{aligned}$$

Thus Q has to be stable, strictly proper and H_∞-norm bounded $\|Q\|_\infty < 1$. Thus Q has to be an element of Ω_1. So, the procedure explained in section 3 can be applied.

Find $Q \in \Omega_1$, such that $T_{z_1 w_1}$ is stable and $\|T_{z_1 w_1}\|_2$ is minimized. The closed loop will be stable. This is ensured by the H_∞ theory, if there is a stable solution ((A, B_u, C_y) has to be stabilizable and detectable). The H_∞ and H_2 part have the same feedback loop. So, if $T_{z_2 w_2}$ is stabilized so will be $T_{z_1 w_1}$.

From this derivation, it should also be clear that the lowest possible H_∞ norm, is the lowest possible norm that can be achieved for the H_∞-problem for $T_{z_2 w_2}$.

5 Conclusions

In this paper we showed how the mixed H_2/H_∞ problem can be solved over the set of linear finite dimensional controllers. It is possible to solve this problem by first solving an H_∞ problem and then solving a constrained H_2 problem. Necessary conditions for the constrained H_2 problem are derived.

Numerical calculation based on a quasi-Newton optimization give a satisfying result. However, due to space limitations, these are not discussed further.

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