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On the Calculation of the Euclidean Parameter Margin *

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Abstract

In recent papers El Ghaoui introduced the q-parameter stability margin and the q-parameter performance measure for parameter dependent systems. For the Euclidean norm, a quadratically convergent scheme was proposed too, for systems with only two real parameters. In this paper, we describe how the Euclidean parameter stability margin and the Euclidean parameter performance measure can be computed, when more than two real parameters are present. A new method to solve structured total least squares problems is used.

Keywords: linear systems, robustness analysis, real parameter perturbations, l_2 approximation problems, inverse iteration.

1 Introduction

Robustness of dynamic systems against parameter variations is a major research topic. In this paper only real parameter variations are considered. In [2], El Ghaoui introduced the q-parameter stability margin for parameter dependent linear dynamic systems. This is the largest norm of the parameter vector, such that stability is ensured. The norm of the parameter is measured in the q-norm. Normally only the Euclidean or the infinity norm are used ($q = 2$ or ∞). In [3], El Ghaoui also introduces a parameter performance measure for the H_2-norm of linear dynamic systems.

The two margins can be computed in the same way. In [4] a quadratically convergent algorithm is described for computing the margins. However, it can only be used when there two real parameters. In this paper, we propose an iterative way to compute the Euclidean parameter margins without restriction on the number of parameters. The method was introduced in [1]. It was first used to solve Structured Total Least Square problems. The computation of the Euclidean parameter margin can be written as a similar problem. The convergence of the iterative scheme is not guaranteed, but in practice it gives good results.

The structure of the paper is as follow: In section 2, the definition of the q-parameter stability margin is stated again. The definition and calculation of the q-parameter performance margins are summarized in section 3. In section 4, we show how the calculation of the Euclidean parameter margins can be reduced to an iterative scheme. In section 5, these concepts are illustrated with an example. The conclusions are summarized in section 6.

2 The q-parameter stability margin

In this section, the basic definitions are recapitulated for the q-parameter stability margin (q-PSM). This section is based on [4].

Consider a linear, time-invariant closed-loop system of order p

$$\dot{x} = A(a)x$$

where x is the state and $A(a)$ is the $p \times p$ dynamic matrix of the plant. The entries of $A(a)$ are polynomially dependent on the parameter vector a with

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length n. The nominal value of a is assumed to be 0. This is not a restriction.
The q-PSM of $A(a)$ is defined as

$$R_q(A) := \min_{a\in\mathbb{R}^n}\{\|a\|_q \mid A(a) \text{ is unstable}\} \quad (1)$$

In this paper only the Euclidean norm ($q = 2$) will be considered. Problems as (1), e.g. μ-analysis are often solved by looking at the following problem:

$$\min_{a\in\mathbb{R}^n}\{\|a\|_2 \mid \exists\omega : \det(A(a) - j\omega I) = 0\} \quad (2)$$

To avoid the problem of having to perform a frequency sweep, the method proposed in [4] (also in [6]) is to express the critical constraint ($A(a)$ having an eigenvalue $j\omega$) as a determinantal equation

$$\min_{a\in\mathbb{R}^n}\{\|a\|_2 \mid \det[\mathbf{A}(a)] = 0\} \quad (3)$$

$\mathbf{A}(a)$ is real, polynomial matrix of order $\frac{p(p+1)}{2}$, called a Lyapunov matrix. An algorithm to construct $\mathbf{A}(a)$ can be found in [5].
If $\mathbf{A}(a)$ has terms of higher degree in a, it is possible to rewrite (3) as

$$\det\left[A + \sum_{i=1}^{n} a_i A_i\right] = 0 \quad (4)$$

where $A, A_i \ (i = 1 \ldots n)$ are constant augmented matrices.
In [2] it has been shown that (4) can eventually be formulated as:

$$\begin{aligned} R_2(M, r) &= \min_{a\in\mathbb{R}^n}\{\|a\|_2 \mid \det[I + M\Delta(a)] = 0\} \\ \Delta(a) &= \text{block-diag}\left[\begin{array}{ccc} a_1 I_{r_1} & \cdots & a_n I_{r_n}\end{array}\right] \quad (5)\end{aligned}$$

In [4], it is explained how this size of the matrix M can even be reduced further. r is a vector of length n, of which the elements are the sizes of the blocks of $\Delta(a)$.

3 The q-parameter performance margin

This section is based on [3]. It is explained how an H_2 performance measure can be calculated, with respect to real parameter variations.

3.1 The calculation of the H_2-norm

In this paper, we restrict ourselves to the continuous case. For a stable plant with transfer matrix $G(s)$, the H_2-norm is defined as

$$\|G\|_2 := \left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} \text{Trace}\left[G(j\omega)^* G(j\omega)\right] d\omega\right)^{1/2}$$

For a given state space realization of G

$$G(s) = C(sI - A)^{-1}B$$

The H_2-norm of the transfer function can be computed as

$$\|G\|_2^2 = \text{trace}(LC^tC)$$

where L is the controllability gramian of (A, B). L is the solution of a Lyapunov equation:

$$AL + LA^t + BB^t = 0 \quad (6)$$

Before continuing, a new operator is defined. Given a symmetric $p \times p$ matrix Q, $vec(Q)$ denotes a vector of length $p(p+1)/2$ consisting of the elements of Q on and above the diagonal.
Equation (6) can then be rewritten as

$$\mathbf{A}l = b$$

where $l := vec(L)$, $b = vec(-BB^t)$ and $\mathbf{A}$ is a real matrix of order $n(n+1)/2$, whose elements are linear combinations of the entries of A. $\mathbf{A}$ is called the Lyapunov matrix. An algorithm to construct $\mathbf{A}$ is given in [5]. If the algorithm is used, $vec(.)$ has to store the upper triangle of the matrix column by column:

$$vec\left(\begin{bmatrix} q_{11} & q_{12} & q_{13} & \cdots \\ q_{12} & q_{22} & q_{23} & \cdots \\ q_{13} & q_{23} & q_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}\right) = \begin{bmatrix} q_{11} \\ q_{12} \\ q_{22} \\ q_{13} \\ q_{23} \\ q_{33} \\ \vdots \end{bmatrix}$$

The H_2 norm of a transfer function can thus be written as:

$$\|G\|_2^2 = c\mathbf{A}^{-1}b$$

where $c^t = vec(C^tC .* H)$. '.*' denotes the elementwise product of two matrices. H is defined as

$$H = \begin{bmatrix} 1 & 2 & 2 & 2 & \cdots \\ 2 & 1 & 2 & 2 & \cdots \\ 2 & 2 & 1 & 2 & \cdots \\ 2 & 2 & 2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

3.2 The minimum distance to singularity

If G is perturbed by n real parameters, our performance index becomes

$$J(a) = \|G\|_2^2 = c(a)\mathrm{A}(a)^{-1}b(a)$$

The entries of A, B and C are assumed to be polynomial in the parameters a_i. Given a scalar J_{ref}, a measure for the robustness in H_2 performance is defined as:

$$PPM(J_{ref}) = \min_{a\in\mathbb{R}^n}\{\|a\|_q \mid J(a) \geq J_{ref}\}$$

We look for the minimum Euclidean norm ($q = 2$) of the parameter vector which increases the performance index $J(a)$ up to a certain level above the reference value J_{ref}.
Using the identity

$$\det(I + uv^t) = 1 + v^t u$$

yields

$$J(a) = J_{ref} \Leftrightarrow \det(\mathrm{A} - \frac{1}{J_{ref}}bc) = 0$$

Therefore, the performance margin can be written as:

$$PPM = \min_{a\in\mathbb{R}^n}\left\{\|a\|_2 \mid \det(\mathrm{A}(a) - \frac{1}{J_{ref}}b(a)c(a)) = 0\right\} \quad (7)$$

Expression (7) has the same form as (3). It can be solved in the same way. When $J_{ref} \to \infty$, the parameter stability margin is obtained.

4 The calculation of the Euclidean parameter margin

To solve the Euclidean parameter margin, a new method, described in [1] is used. This method is iterative. It is not guaranteed to converge to the global optimum. In most cases, however, the method works well and fast.
In the previous sections it is shown that the calculation of the Euclidean PSM and PPM, can be transformed to the following problem:

$$\min_{a\in\mathbb{R}^n}\{\|a\|_2 \mid \det[I + M\Delta(a)] = 0\}$$

where $\Delta(a) = \text{block-diag}\begin{bmatrix} a_1 I_{r_1} & \cdots & a_n I_{r_n}\end{bmatrix}$ It is obvious that this problem can be reformulated as

$$\min \|a\|_2^2 \quad \text{subject to} \quad \begin{array}{l} M(a)y = 0 \\ y^t y = 1 \end{array}$$

where $M(a) = M_0 + a_1M_1 + a_2M_2 + \ldots + a_nM_n$ and $M_0 = I$.
This problem can easily be solved using Lagrange multipliers:

$$\mathcal{L} = \|a\|_2^2 + 2l^tM(a)y + \lambda(y^ty - 1)$$

where l is a vector of Lagrange multipliers and λ is a scalar Lagrange multiplier. Differentiation with respect to the different unknowns yields necessary conditions for the solution

$$\frac{\partial\mathcal{L}}{\partial a_k} : a_k + l^tM_ky = 0 \quad (8)$$

$$\frac{\partial\mathcal{L}}{\partial y} : M(a)^tl + y\lambda = 0 \quad (9)$$

$$\frac{\partial\mathcal{L}}{\partial \lambda} : y^ty = 1 \quad (10)$$

$$\frac{\partial\mathcal{L}}{\partial l} : M(a)y = 0 \quad (11)$$

After some manipulations the following 'non-linear generalized singular value problem' comes up:

$$Iy = D_yx\sigma \quad (12)$$

$$Ix = D_xy\sigma \quad (13)$$

where

$$D_x = \sum_{i=1}^{n}(M_i^tx)(M_i^tx)^t \quad (14)$$

$$D_y = \sum_{i=1}^{n'}(M_iy)(M_iy)^t \quad (15)$$

and $y^ty = 1$, $x^tx = 1$ and $l = x\sigma$.
The 'non-linear generalized singular value problem', is solved using an inverse iteration algorithm ([1]). Until now, convergence is not guaranteed. However, in practice, most problems converge fast and accurately.

5 Example

In this section we will show an example. The example has only 2 parameters to make graphical representation easier. However, as explained in the previous section, the method used is not restricted to

2 parameters. The example of [3] is used, slightly modified. Consider the following state feedback problem

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 7+a_1 & -8+a_2 \\ 6 & -7 \end{bmatrix} w \\ & + \begin{bmatrix} 7+a_1 & -8+a_2 \\ 6 & -7 \end{bmatrix} u \\ z &= \begin{bmatrix} x \\ \sqrt{10}u \end{bmatrix} \\ y &= x \end{aligned}$$

The LQR controller is:

$$u = -Kx = -\begin{pmatrix} 0.1413 & 0.07358 \\ -0.1604 & -0.0872 \end{pmatrix} x$$

In figure 1, the H_2 norm of the closed loop transfer function is shown as function of the parameters a_1 and a_2. The levels shown are $J_{ref} = \gamma J(0)$, for $\gamma = 1, 3, 10, \infty$. The level $J_{ref} = \infty$ is also the border between the stable and the unstable region. The Euclidean parameter performance margin for $J_{ref} = 3J(0)$ is 14.6019. In the parameter space, the region bounded by the Euclidean PPM is represented by the dotted circle. For all the points within that circle, it is guaranteed that the corresponding closed loop has a H_2-norm smaller than 3 times the nominal one. The circle touches the level $J_{ref} = 3J(0)$ at $(a_1, a_2) = (-14.4567\ , -2.0543)$.
For comparison the ∞-PPM for $J_{ref} = 3J(0)$ is also computed. The computation is based on [3]. The parameter region bounded by this margin is a square with edges of length 2×10.9439. The square touches the level $J_{ref} = 3J(0)$ at its upper left hand corner. Remark: The Euclidean parameter margin (circle) is not necessarily more conservative than the ∞-parameter margin (box). This can clearly be observed on the plots. Which region is more conservative only depends on the location of the constraint. In figure 2, the logarithm of the singular values of $M(a)$ are plotted, as the iteration proceeds. We started from random vectors x and y (both with norm 1). The stop criterion used was: $\|M(a)y\|_2 < 1e-8$. It can be clearly observed from the plot that the smallest singular value decreases until the matrix is singular enough. It is not sure that the method will converge to the global optimum or that it will converge at all. Therefore, it is safe to start form different random vectors. E.g. in the above example the iterations sometimes converged to a point with Euclidean norm 14.7342. This is larger than the performance margin mentioned above. The smallest value is taken as the parameter performance margin.

6 Conclusions

In this paper we showed how the Euclidean SPM and PPM can be computed by solving a 'non-linear generalized singular value' problem. This can be solved iteratively. This method can be used for any number of real parameters. This is an advantage over the method proposed in [4], that is restricted to 2 parameters. The iterative method can also be extended to compute complex parameter margins.

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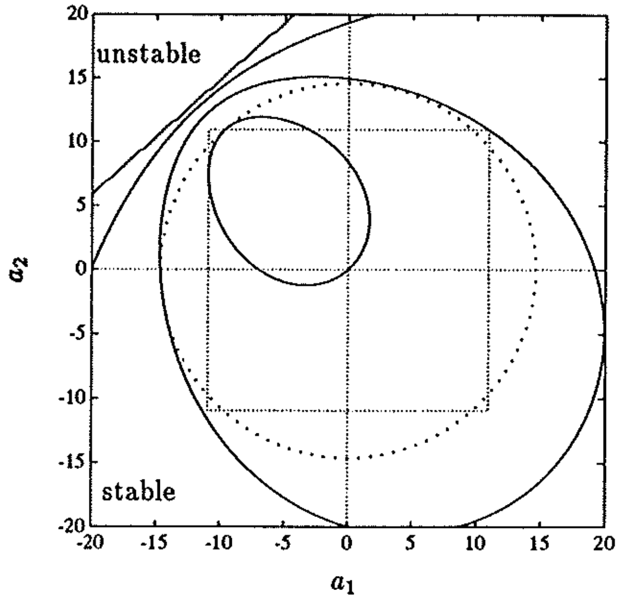

Figure 1: The H_2-norm of the closed loop system for varying a_1 and a_2. The levels $J_{ref} = \gamma J(0)$, for $\gamma = 1, 3, 10, \infty$ are shown. The dotted circle and box are the regions bounded by the Euclidean PPM and the ∞-PPM for the level $J_{ref} = 3J(0)$. These regions touch the level $J_{ref} = 3J(0)$.

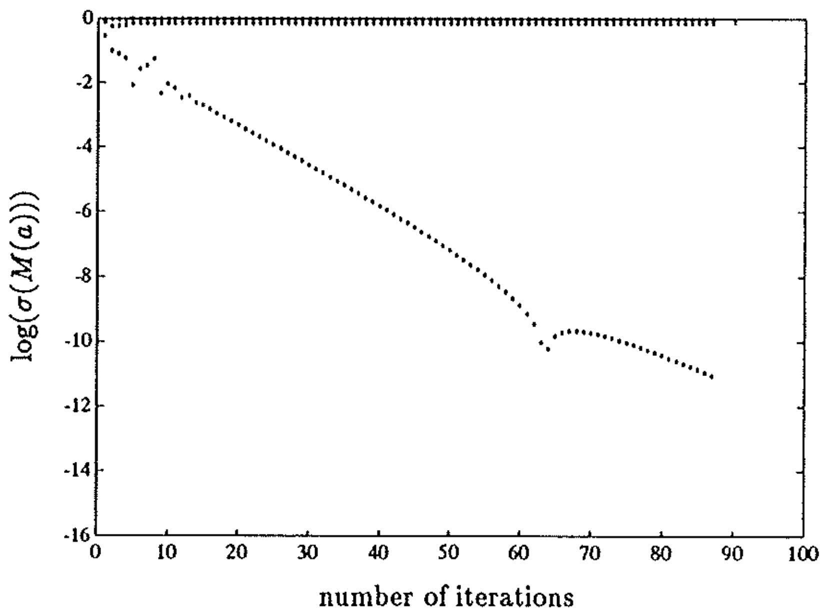

Figure 2: The logarithm of the singular values of the matrix $M(a)$ as a function of the number of iterations. The smallest singular value becomes small, implying that $M(a)$ becomes singular.