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A Multidimensional Realization Algorithm for Parametric Uncertainty Modeling Problems*

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Abstract

The parametric uncertainty modeling problem arises in robustness analysis, and it can be cast as a realization problem for a multidimensional system, which is defined in the framework of Linear Fractional transformations (LFTs). The main result of this paper is an algorithm to systematically solve the multidimensional realization problem.

1 Introduction

The multidimensional realization problem is the problem of realizing a state space model for a multidimensional system transfer function [1]. In recent years, however, the same problem appears in robust control[2, 3]. A lot of work has been done for one and two dimensional realization problems in systems theory, see [1] and the references there in. In general, however, there is no systematic way to solve this problem when the dimension is large than 2. Here we will describe a systematic multidimensional realization algorithm.

2 Problem definition

Given a multivariable rational matrix $M(p)$ with dimension $n \times m$, where p contains q parameters $p_1 \cdots p_q$, the multidimensional realization problem is to find a LFT with a block structure vector $bs = [r_1\ r_2\ \cdots\ r_q]$ with r_i being integer and a coefficient matrix $L \in \mathbb{C}^{(n+r)\times(m+r)}$, $r = \sum_{i=1}^{q} r_i$, partitioned as:

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}$$

such that

$$M(p) = F(L, \Delta(p)) \tag{1}$$

where $\Delta(p) := \mathrm{diag}(p_1 I_{r_1}, \cdots, p_q I_{r_q})$, and $F(L,\Delta(p)) = L_{22} + L_{21}\Delta(p)(I - L_{11}\Delta(p))^{-1}L_{12}$. The dimensions of L_{11}, L_{12}, L_{21} and L_{22} are $r \times r$, $r \times m$, $n \times r$ and $n \times m$ respectively.

In robust control, the state space parametric uncertainty modeling problem [2, 3]is as follows: Given a system matrix

$$M(p) = \begin{bmatrix} A(p) & B(p) \\ C(p) & D(p) \end{bmatrix}$$

with the entries being multidimensional functions of the uncertainty parameter vector p, find constant matrices M_{nom}, B_Δ, C_Δ and D_Δ such that the following equation holds:

$$M(p) = M_{nom} + B_\Delta \Delta(p)[I - \Delta(p) D_\Delta]^{-1} C_\Delta \tag{2}$$

If we let $L_{22} = M_{nom}$, $L_{21} = B_\Delta$, $L_{12} = C_\Delta$ and $L_{11} = D_\Delta$, Eq.(2), compared with Eq.(1), shows that the state space parametric uncertainty modeling problem coincides with a multidimensional realization problem.

In the rest of the paper, we assume that each entry $[M(p)]_{ij}$ of the rational matrix $M(p)$ satisfies the following preliminary condition:

$$[M(p)]_{ij} = c_{ij} + f_{ij}(p) \tag{3}$$

with c_{ij} a constant and $f_{ij}(p)$ of the form:

$$\frac{k(p)}{1 + l(p)},\ k(0) = l(0) = 0$$

3 Algorithm

The main idea of the algorithm for the multidimensional realization is to realize the affine part of a rational matrix at first, then the rational function part in the second step. The two realizations are finally "cascaded" via the Redheffer Star Product.

It is obvious that $M(p)$ can be written as an affine combination in rational functions:

$$M(p) = M_0 + \sum_{i=1}^{l} f_i(p) M_i$$

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where $M_0 \in \mathbb{C}^{n\times m}$: constant matrix with entries $[M]_{ij} = c_{ij}$ (see Eq.(3)); $f_i(p)$, $i = 1, \cdots, l$: multivariable rational functions (see Eq.(3)) and $M_i \in \mathbb{C}^{n\times m}$, $i = 1, \cdots, l$: constant matrices, which tell how the $f_i(p)$ enter the matrix $M(p)$.

The first step of the algorithm is to find a coefficient matrix L_a and a block structure bs_a such that:

$$M(p) = L_{a22} + L_{a21}\Delta(f)[I - L_{a11}\Delta(f)]^{-1}L_{a12}$$

where

$$L_a = \begin{bmatrix} L_{a11} & L_{a12} \\ L_{a21} & L_{a22} \end{bmatrix},\ bs_a = [r_{a1}, \cdots, r_{al}]$$

$$\Delta(f) = \text{diag}[f_1(p)I_{r_{a1}}, \cdots, f_l(p)I_{r_{al}}]$$

It can be proved (see [4]) that $L_{a11} = 0$, $L_{a12} = V^T$, $L_{a21} = U$ and $L_{a22} = M_0$ where U and V are from the LR decompositions of M_i: $U = [U_1\Sigma_1^{\frac{1}{2}}, \cdots, U_1\Sigma_l^{\frac{1}{2}}]$ and $V = [\Sigma_1^{\frac{1}{2}}V_1^T, \cdots, \Sigma_l^{\frac{1}{2}}V_l^T]$, U_i, V_i and Σ_i are from singular value decomposition: $M_i = U_i\Sigma_iV_i^T$, and $r_{ai} = \text{Rank}(\Sigma_i)$.

The second step is to find a coefficient matrix L_b and a block structure bs_b such that:

$$\Delta(f) = L_{b22} + L_{b21}\Delta(p)[I - L_{b11}\Delta(p)]^{-1}L_{b12}$$

where

$$L_b = \begin{bmatrix} L_{b11} & L_{b12} \\ L_{b21} & L_{b22} \end{bmatrix},\ bs_b = [r_1, \cdots, r_q]$$

$$\Delta(p) = \text{diag}(p_1I_{r_1}, \cdots, p_qI_{r_q})$$

This step is to realize the rational function part and is not so trivial as the first one. The algorithm for this step consists of two parts.

1. One rational function: For each monomial $ap_1^{k_1} \cdots p_q^{k_q}$, where a is the coefficient of the monomial and k_i is the degree of the monomial with respect to p_i, we can construct a LFT with a coefficient matrix L_m and the structure vector bs_m with

$$L_m = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 & a \\ 1 & \ddots & & & \vdots & 0 \\ 0 & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & 0 \end{bmatrix} \text{ and } bs_m = [k_1 \cdots k_q]$$

It is now just a routine to get the coefficient matrix L_f and block structure vector bs_f of the LFT for one rational function $f(p)$ from the linear interconnection of the LFTs for all monomials of $f(p)$ (see [4] for details).

2. L_b and bs_b for $\Delta(f)$: Now we can form a LFT for $\Delta(f)$ by lumping all LFTs for the $f_i(p)$, each repeated $bs_a(i)$ times:

$$L_b = P_b^t L_b' P_b, \quad bs_b = \sum_{i=1}^{l} bs_a(i)bs_{f_i}$$

where $L_b' = \text{blockdiag}(I_{bs_a(1)} \otimes L_{f_1}, \cdots I_{bs_a(l)} \otimes L_{f_l})$, $\otimes$ denoting the Kronecker Product, P_b is a permutation matrix. Note that from the preliminary condition on the matrix $M(p)$, $f_i(0) = 0$, so $\Delta(f)_{p=0} = 0$, this leads to L_{b22}=0.

Finally the coefficient matrix L can be easily constructed by the Redheffer Star-Product [2]:

$$L = \begin{bmatrix} L_{b11} & L_{b12}L_{a12} \\ L_{a21}L_{b21} & L_{a22} \end{bmatrix}$$

with $bs = bs_b$.

4 Conclusion

An algorithm to systematically solve the multidimensional realization problem was given, which is the main contribution of this paper. The trick we used is to realize the affine part of a rational matrix at first, then the polynomial part in second step. The algorithm does in general not give the minimal realization, that is, the size of the result $\Delta(p)$ is not minimal. But the minimal realization problem for multidimensional systems ($j > 1$) is known as a very difficult problem in system theory even for 2D systems. The work for reducing the size is still under investigation. Here we give some preliminary results on the issue. Suppose that for a LFT, we can find a symmetric matrix P (or Q) which commutes with $\Delta(p)$ such that

$$PL_{11} = L_{11}^TP \text{ and } PL_{12}L_{12}^TP = L_{21}^TL_{21} \qquad (4)$$

$$(\text{or } L_{11}Q = QL_{11}^T \text{ and } L_{12}L_{12}^T = QL_{21}^TL_{21}Q)$$

Then it can be easily proved that the size of the LFT can be reduced to the rank of P (or Q). So the problem now is to find a symmetric matrix P (or Q) with a minimal rank such that Eq. 4 holds. However the solution of this minimal rank problem is still not solved yet.

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