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A multidimensional realization algorithm for parametric uncertainty modelling and multiparameter margin problems

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The parametric uncertainty modelling problem arises in robustness analysis with the structured singular value (μ), or in parametric margin problems. In this paper, it is shown that parametric uncertainty modelling can be cast as a realization problem for a multidimensional system, which is defined in the framework of linear fractional transformations (LFTs). The main result of this paper is an algorithm to solve systematically the multidimensional realization problem.

1. Introduction

The multidimensional realization problem is the problem of realizing a state-space model for a multidimensional system transfer function (Bose 1977). In recent years, however, the same problem appears in robust control. To analyse closed-loop robustness properties, the system transfer function matrix or state-space matrix M, which contains the structured model uncertainties, is represented as a linear fractional transformation (LFT) (Doyle *et al.* 1991)

$$M(\Delta) = F(L, \Delta) \tag{1}$$

where $F(L, \Delta) = L_{22} + L_{21}\Delta(I - L_{11}\Delta)^{-1}L_{12}$, L is the coefficient matrix and is partitioned as

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}$$

and Δ contains the structured model uncertainties. This procedure is called uncertainty modelling (Steinbuch *et al.* 1992). The structured singular value (or μ) analysis method is based on such a LFT framework. When only parametric uncertainties exist, Δ is diagonal with the structure

$$\Delta(p) = \text{blockdiag}\{p_1 I_{r_1}, \ldots, p_q I_{r_q}\}$$

where p is the parameter vector. Consider the case where $p_i \neq 0 \, \forall i$, then (1) can be written as

$$M(p) = L_{22} + L_{21}(\Delta^{-1}(p) - L_{11})^{-1}L_{12} \tag{2}$$

If the entries of $M(p)$ are rational functions, the problem of finding the coefficient matrix L and block structure $bs = [r_1, r_2, \ldots, r_q]$ such that (2) holds is clearly a multidimensional realization problem. This is obvious when each

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parameter p_i is replaced by an integrator or a delay operator, in which case the coefficient matrices L_{11}, L_{12}, L_{21} and L_{22} are state-space matrices A, B, C and D respectively.

A lot of work has been done for one and two-dimensional realization problems in systems theory (Kailath 1980, Kung *et al.* 1977). For the case where the entries of $M(p)$ are linear or affine in the components of parameter vector p, the realization algorithm was given by Doyle *et al.* (1991). In general, however, there is no systematic way to solve this problem when the entries of $M(p)$ are rational functions.

Here, we will describe a systematic multidimensional realization algorithm. The algorithm will not necessarily provide a minimal realization from the system point of view, but it does solve the parametric uncertainty modelling problems and also the multiparameter margin formulation problems (El Ghaoui 1990) (see later).

This paper is organized as follows. Section 2 gives the definition of the multidimensional realization problem. Two examples are given to show how multidimensional realization problems are related to parametric uncertainty modelling and multiparameter margin problems. Section 3, the main part of the paper, describes the algorithm for multidimensional realization, the data structure for rational matrices and also the size reduction. Section 4 gives two examples to show in some detail how the algorithm works. Section 5 gives the conclusions of the paper.

2. Problem definition

We first give a definition of the realization problem, which is based on the concept of the linear fractional transformation (LFT). Then two examples of parametric uncertainty modelling and multiparameter margin formulation based on this definition will be discussed.

2.1. *Definition*

Given a multivariable rational matrix (or a multidimensional system transfer matrix) $M(p)$ with dimension $n \times m$, where p contains q parameters $p_1 \ldots p_q$, the multidimensional realization problem is to find a LFT with $F(L, \Delta(p))$, such that

$$M(p) = F(L, \Delta(p)) \tag{3}$$

where $\Delta(p) := \operatorname{diag}(p_1 I_{r_1}, \ldots, p_q I_{r_q})$, which can also be expressed by a block structure vector $bs = [r_1 r_2 \ldots r_q]$, r_i being the dimension of the identity matrix I_{r_i}, and

$$F(L, \Delta(p)) = L_{22} + L_{21}\Delta(p)(I - L_{11}\Delta(p))^{-1}L_{12}$$

$L \in \mathbb{C}^{(n+r)\times(m+r)}$, $r = \sum_{i=1}^{q} r_i$, is a coefficient matrix and partitioned as

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}$$

The dimensions of L_{11}, L_{12}, L_{21} and L_{22} are $r \times r$, $r \times m$, $n \times r$ and $n \times m$ respectively. If we take $M(p)$ as a system transfer function matrix, then (3) shows that it is equivalent to (or realized by) a system with a feedback matrix

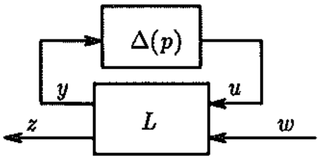

Figure 1. Description of a linear fractional transformation.

$\Delta(p)$ and a coefficient matrix L, see Fig. 1, where w and z are exogenous inputs and outputs, u and y are the feedback inputs and outputs which are connected by the feedback diagonal matrix $\Delta(p)$, and

$$\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix}, \; u = \Delta(p)y$$

Note that the elements of the vector p can be integrator (or delay) operators or uncertainty parameters or both. The realization problem is merely to find the coefficient matrix L and block structure bs such that (3) holds. The following two examples will show in detail how parametric uncertainty modelling and multiparameter margin formulation problems can be phrased as multidimensional realization problems.

In the rest of this paper, for the case where the dimensions of the matrices are not mentioned, we assume they can be inferred from the context.

2.2. *Example 1: state-space parametric uncertainty modelling*

Consider a vector $p = [p_1 \;\; \dots \;\; p_q] \in \mathbb{R}^q$, containing q scalar uncertainty parameters. Let the model of the perturbed system be given as a state-space realization in which the entries of the state-space matrices are multidimensional rational functions of the parameter vector p

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A(p) & B(p) \\ C(p) & D(p) \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \tag{4}$$

To analyse the robustness of such systems by the μ analysis method (Doyle *et al.* 1991, Steinbuch *et al.* 1992) one should first solve the parametric uncertainty modelling problem. That is, the system of (4) must be presented by a LFT. To do that, define the matrix

$$M(p) = \begin{bmatrix} A(p) & B(p) \\ C(p) & D(p) \end{bmatrix}$$

The nominal part of the state-space model is given by $M_{\text{nom}} := M(0)$. The uncertainty part of the state-space model is defined as $M_\Delta(p)$ with entries $[M_\Delta]_{ij}(p) = M_{ij}(p) - [M_{\text{nom}}]_{ij}$. By such uncertainty extraction, the perturbed state-space model (4) can be written as

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = M_{\text{nom}} \begin{bmatrix} x \\ u \end{bmatrix} + M_\Delta(p) \begin{bmatrix} x \\ u \end{bmatrix} \tag{5}$$

Define a new input u_Δ and a new output y_Δ, and let the output y_Δ be fed back to the input u_Δ through a diagonal perturbation $\Delta(p) = \text{blockdiag}\,(p_1 I_{r_1}, \dots,$

$p_q I_{r_q}$). Furthermore, construct matrices B_Δ, C_Δ and D_Δ, such that the following equations hold

$$\begin{bmatrix} \dot{x} \\ y \\ y_\Delta \end{bmatrix} = \begin{bmatrix} M_{\text{nom}} & B_\Delta \\ C_\Delta & D_\Delta \end{bmatrix} \begin{bmatrix} x \\ u \\ u_\Delta \end{bmatrix} \tag{6}$$

B_Δ, C_Δ and D_Δ contain information about how the uncertainties affect the nominal model. Obviously it must be possible to reduce (6) to (4). Therefore eliminating u_Δ and y_Δ in (6), we have

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = M_{\text{nom}} \begin{bmatrix} x \\ u \end{bmatrix} + B_\Delta \Delta(p)[I - D_\Delta \Delta(p)]^{-1} C_\Delta \begin{bmatrix} x \\ u \end{bmatrix} \tag{7}$$

Equation (7) is equivalent to (4), if the following equation holds

$$M(p) = M_{\text{nom}} + B_\Delta \Delta(p)[I - D_\Delta \Delta(p)]^{-1} C_\Delta \tag{8}$$

If we let $L_{22} = M_{\text{nom}}$, $L_{21} = B_\Delta$, $L_{12} = C_\Delta$ and $L_{11} = D_\Delta$, Equation (8), compared with (3), shows that the state-space parametric uncertainty modelling problem coincides with a multidimensional realization problem. □

2.3. *Example 2: formulation of the multiparameter margin problem*

Consider a linear, time-invariant closed-loop system of order n

$$\dot{x} = F(p)x$$

where x is the state and $F(p)$ is an $n \times n$ rational matrix of a parameter vector p of length q, whose nominal value can always be reset to zero. The k-parameter margin of $F(p)$ is

$$R_k(F) := \min_{p \in \mathbb{R}^q} \{\|p\|_k | F(p) \text{ is unstable}\} \tag{9}$$

where $\|p\|_k$ is the k-norm of the vector p. Equation (9) can be converted to the following equation in the assumption that the system is stable when $p = 0$ (Tesi and Vicino 1990)

$$R_k(F) = \min_{p \in \mathbb{R}^q} \left\{ \|p\|_k \middle| \det\left[I + \sum_{i=1}^{l} f_i(p) B_i \right] = 0 \right\} \tag{10}$$

where $f_i(p)$ is a rational function of the vector p. The problem now is how to write (10) as

$$R_k(M_b, bs) = \min_{p \in \mathbb{R}^q} \{\|p\|_k | \det[I + M_b \Delta(p)] = 0\} \tag{11}$$

$$\Delta(p) = \text{blockdiag}(p_1 I_{r_1}, \ldots, p_q I_{r_q})$$

and with M_b a constant matrix, bs and I_{r_i} as defined before. For the case where the $f_i(p)$ are linear in p, the algorithm for constructing M_b and bs in (11) from (10) is given by El Ghaoui (1990). He also gave an example to show how to construct M_b and bs when $f_i(p)$ are polynomials in the elements of vector p.

However, the problem of constructing M_b and bs in (11) from (10) is actually nothing more than a multidimensional realization problem, which can be completely solved by the algorithm to be described below. To see this, assume that the following realization problem be solved

$$\sum_{i=1}^{l} f_i(p) B_i = L_{22} + L_{21}\Delta(p)[I - L_{11}\Delta(p)]^{-1}L_{12} \tag{12}$$

Then, the determinant in (10) becomes

$$\det\{I + L_{22} + L_{21}\Delta(p)[I - L_{11}\Delta(p)]^{-1}L_{22}\} = 0 \tag{13}$$

If $\det(I + L_{22}) \neq 0$ from what follows, it will be clear that $L_{22} = 0$ and $\det[I - L_{11}\Delta(p)] \neq 0$ (which must be checked after the parameter margin is obtained), and by manipulating the matrices in (13) and using the equality: $\det(I + AB) = \det(I + BA)$, Equation (13) is equivalent to

$$\det\{I + (-L_{11} + L_{12}[I + L_{22}]^{-1}L_{21})\Delta(p)\} = 0$$

The problem is clearly solved if we let

$$M_b = -L_{11} + L_{12}(I + L_{22})^{-1}L_{21} \tag{14}$$

□

2.4. *Preliminary condition for matrix M(p)*

In the rest of the paper, we assume that each entry $[M(p)]_{ij}$ of the rational matrix $M(p)$ satisfies the following condition

$$[M(p)]_{ij} = c_{ij} + f_{ij}(p) \tag{15}$$

with c_{ij} a constant and $f_{ij}(p)$ of the form

$$\frac{k(p)}{1 + l(p)}, \; k(0) = l(0) = 0$$

This assumption is based on the following facts. For the elements in p which are integrator operators (s^{-1}) or delay operators (z^{-1}), (15) means that the corresponding system is physically causal (the transfer function is proper). For the elements in p which are parametric uncertainties, (15) shows that the uncertainty part of $M(p)$ diminishes when the uncertainty parameters are equal to zero.

3. Algorithm

The main idea of the algorithm for the multidimensional realization is to realize the affine part of a rational matrix at first, then the rational function part in the second step. The two realizations are finally 'cascaded' via the Redheffer Star Product.

It is obvious that $M(p)$ can be written as an affine combination in rational functions

$$M(p) = M_0 + \sum_{i=1}^{l} f_i(p) M_i \tag{16}$$

where

$M_0 \in \mathbb{C}^{n\times m}$ constant matrix with entries $[M]_{ij} = c_{ij}$ (see (15))
$f_i(p)$, $i = 1, \ldots, l$ multivariable rational functions, each of $f_i(p)$ is just one of $f_{ij}(p)$ in (15), l ($\leq nm$) is the number of the rational functions $f_{ij}(p)$ which are different
$M_i \in \mathbb{C}^{n\times m}$, $i = 1, \ldots, l$ constant matrices, which tell how the $f_i(p)$ enter matrix $M(p)$

The first step of the algorithm is to find a coefficient matrix L_a and a block structure bs_a such that

$$M(p) = L_{a22} + L_{a21}\Delta(f)[I - L_{a11}\Delta(f)]^{-1}L_{a12} \tag{17}$$

where

$$L_a = \begin{bmatrix} L_{a11} & L_{a12} \\ L_{a21} & L_{a22} \end{bmatrix},$$

$$\Delta(f) = \operatorname{diag}[f_1(p)I_{r_{a1}}, \ldots, f_l(p)I_{r_{al}}], \quad \text{and} \quad bs_a = [r_{a1}, \ldots, r_{al}]$$

The second step is to find a coefficient matrix L_b and a block structure bs such that

$$\Delta(f) = L_{b22} + L_{b21}\Delta(p)[I - L_{b11}\Delta(p)]^{-1}L_{b12} \tag{18}$$

where

$$L_b = \begin{bmatrix} L_{b11} & L_{b12} \\ L_{b21} & L_{b22} \end{bmatrix}, \quad \Delta(p) = \operatorname{diag}(p_1 I_{r_1}, \ldots, p_q I_{r_q}) \text{ and } bs = [r_1, \ldots, r_q]$$

Based on the coefficient matrices L_a and L_b, the coefficient matrix L, the final realization of (3), can be easily constructed by the Redheffer Star-Product (Doyle *et al.* 1991): $L = S(L_b, L_a)$, where $S(L_b, L_a)$ is the star product of L_b and L_a defined as

$$S(L_b, L_a) := \begin{bmatrix} L_{b11} + L_{b12}L_{a11}(I - L_{b22}L_{a11})^{-1}L_{b21} & L_{b21}(I - L_{a11}L_{b22})^{-1}L_{a12} \\ L_{a21}(I - L_{b22}L_{a11})^{-1}L_{b21} & L_{a22} + L_{a21}L_{b22}(I - L_{a11}L_{b22})^{-1}L_{a12} \end{bmatrix} \tag{19}$$

The corresponding block structure vector bs for L is the same as that obtained in the second step. This two-step procedure is depicted in Fig. 2.

3.1. *Realization: L_a and bs_a*

The first realization step is just to realize the affine part of $M(p)$, taking $f_i(p)$ as independent variables or parameters. Because the entries of $M(p) = M(f(p))$ are linear in the vector $f(p) = [f_1(p), \ldots, f_l(p)]$, we can use the same method as that used in the case where the entries in $M(p)$ are linear in the vector p. For this linear case, the algorithm is based on Safonov and Athans' Internal Feedback Loop (IFL) parameter representation (Safonov and Athans 1977, El Ghaoui 1990). To do that, first, take the LR decomposition for each $n \times m$ matrix of M_i, $i = 1, \ldots, l$, from its singular value decomposition

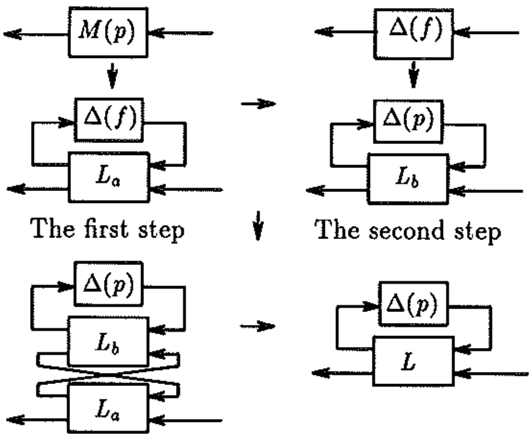

Figure 2. Description of the two-step realization algorithm. First $M(p)$ is realized as a LFT with a diagonal rational $\Delta(f)$. $\Delta(f)$ is then realized as a LFT with a block diagonal $\Delta(p)$. The two realizations are then 'cascaded' via a Redheffer Star product.

$$M_i = U_i \Sigma_i V_i^t \tag{20}$$

where U_i and V_i are $n \times r_{ai}$ and $r_{ai} \times m$ matrices respectively, $r_{ai} = \text{rank}(M_i)$. Σ_i is a $r_{ai} \times r_{ai}$ positive definite diagonal matrix. Then let

$$L_i = U_i \Sigma_i^{1/2}, \quad R_i = \Sigma_i^{1/2} V^t \tag{21}$$

such that $M_i = L_i R_i$ or $f_i(p) M_i = L_i [f_i(p) I_{r_{ai}}] R_i$. Finally construct

$$U = [L_1, \ldots, L_l] \quad \text{and} \quad V = [R_1, \ldots, R_l] \tag{22}$$

Now (3) becomes

$$M(p) = M_0 + \sum_{i=1}^{l} L_i [f_i(p) I_{r_{ai}}] R_i = M_0 + U\Delta(f) V^t$$

where $\Delta(f)$ is defined in (17). Comparing this equation with (17), it is easy to find that

$$L_{a11} = 0, \qquad L_{a12} = V^t \tag{23}$$
$$L_{a21} = U, \qquad L_{a22} = M_0$$

The corresponding block structure is $bs_a = [r_{a1}, \ldots, r_{al}]$ with r_{ai} found as above. Note that L in (19) can be simplified significantly as a result of $L_{a11} = 0$

$$L = \begin{bmatrix} L_{b11} & L_{b12} L_{a12} \\ L_{a21} L_{b21} & L_{a22} + L_{a21} L_{b22} L_{a12} \end{bmatrix} \tag{24}$$

3.2. *Realization: L_b and bs*

The second step is to realize the rational function part. This step is not so trivial as the first one. The algorithm for this step consists of three substeps.

Before going into detail, we define a permutation matrix which is related to another integer matrix and will be frequently used later. Let B be an $h \times q$ matrix with integer entries b_{ij}, and

$$s_{ij} = \sum_{k=1}^{(i-1)} \sum_{l=1}^{q} b_{kl} + \sum_{l=1}^{j} b_{il},\ i = 1, \ldots, h;\ j = 1, \ldots, q$$

$$s_{i0} = s_{(i-1)q},\ i = 2, \ldots, h;\ s_{10} = 0$$

and $s_{hq} \times (s_{ij} - s_{i(j-1)})$ matrices

$$E_{ij} = [e_{s_{i(j-1)}+1}, \ldots, e_{s_{ij}}], \quad i = 1, \ldots, h;\ j = 1, \ldots, q$$

where e_i is a column vector with zero entries except that the ith entry is 1. Now define a permutation matrix P_B as follows

$$P_B = [E_{11} \ldots E_{h1} \ldots\ldots E_{1q} \ldots E_{hq}] \tag{25}$$

Obviously, the matrix P_B depends on the matrix B, the rows of which are the block structure vectors (see later). The following example is given to show clearly how the permutation matrix P_B is formed.

Example 3.1: Let a 2×3 matrix B be given as

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Then from the definition above

$$s_{10} = 0 \quad s_{11} = 1 \quad s_{12} = 3 \quad s_{13} = 6$$

$$s_{20} = 6 \quad s_{21} = 10 \quad s_{22} = 15 \quad s_{23} = 21$$

and

$$E_{11} = [e_{s_{10}+1}, \ldots, e_{s_{11}}] = e_1$$

$$E_{12} = [e_{s_{11}+1}, \ldots, e_{s_{12}}] = [e_2\ e_3]$$

$$E_{13} = [e_{s_{12}+1}, \ldots, e_{s_{13}}] = [e_4\ e_5\ e_6]$$

$$E_{21} = [e_{s_{20}+1}, \ldots, e_{s_{21}}] = [e_7\ e_8\ e_9\ e_{10}]$$

$$E_{22} = [e_{s_{21}+1}, \ldots, e_{s_{12}}] = [e_{11}\ e_{12}\ e_{13}\ e_{14}\ e_{15}]$$

$$E_{23} = [e_{s_{22}+1}, \ldots, e_{s_{23}}] = [e_{16}\ e_{17}\ e_{18}\ e_{19}\ e_{20}\ e_{21}]$$

Finally, we can form the corresponding permutation matrix as

$$P_B = [e_1\ e_7\ e_8\ e_9\ e_{10}\ e_2\ e_3\ e_{11}\ e_{12}\ e_{13}\ e_{14}\ e_{15}\ e_4\ e_5\ e_6\ e_{16}\ e_{17}\ e_{18}\ e_{19}\ e_{20}\ e_{21}]$$

□

3.2.1. *L_b and bs for $\Delta(f)$.* Assume that the rational functions $f_i(p)$, $i = 1, \ldots l$, have their LFTs with corresponding coefficient matrices L_{f_i} and block structure vectors bs_{f_i} respectively. Assume also that the block structure vector bs_a is already known from the last section. Then we can form the LFT for $\Delta(f)$ by lumping all LFTs for the $f_i(p)$, each repeated $bs_a(i)$ times, see Fig. 3. First, we write the aggregate system consisting of the appended LFTs as

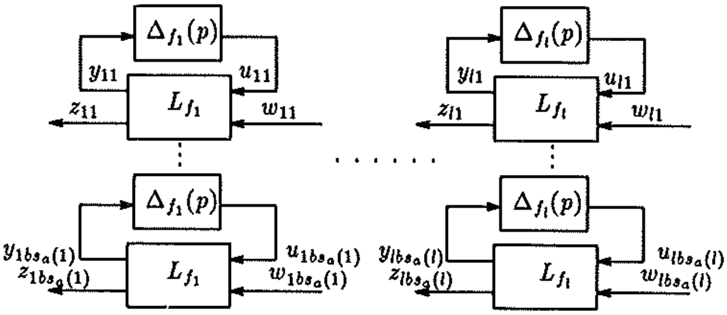

Figure 3. The LFT for $\Delta(f)$ by lumping all LFTs for rational functions (f_i).

$$\begin{bmatrix} y_{11} \\ z_{11} \\ \vdots \\ y_{1bs_a}(1) \\ z_{1bs_a}(1) \\ \vdots \\ \vdots \\ y_{l1} \\ z_{l1} \\ \vdots \\ y_{lbs_a(l)} \\ z_{lbs_a(l)} \end{bmatrix} = L'_b \begin{bmatrix} u_{11} \\ w_{11} \\ \vdots \\ u_{1bs_a}(1) \\ w_{1bs_a}(1) \\ \vdots \\ \vdots \\ u_{l1} \\ w_{l1} \\ \vdots \\ u_{lbs_a(l)} \\ w_{lbs_a(l)} \end{bmatrix}$$

$$u_{ij} = \Delta_{fi}(p) y_{ij}, \quad i = 1, \ldots, l; \quad j = 1, \ldots, bs_a(i)$$

where $\Delta_{fi}(p)$ has the block structure bs_{f_i} and

$$L'_b = \text{blockdiag}\,(I_{bs_a(i)} \otimes L_{f_1}, \ldots, I_{bs_a(l)} \otimes L_{f_l}) \tag{26}$$

where $\otimes$ denotes the Kronecker Product. However, L'_b is not equal to L_b, as its rows (and columns) corresponding to the repeated feedback inputs (outputs) and exogenous inputs (outputs) are mixed together. But L_b can be obtained by a permutation as follows. Define an $l \times (q+1)$ integer matrix

$$B = \begin{bmatrix} b_1 \\ \vdots \\ b_l \end{bmatrix} \text{ with } b_i = \left.\begin{bmatrix} bs_{f_i} & 1 \\ \vdots & \vdots \\ bs_{f_i} & 1 \end{bmatrix}\right\} bs_a(i) \text{ rows} \tag{27}$$

Then

$$L_b = P_B^t L'_b P_B \tag{28}$$

P_B is defined in (25) with B in (27). Note that permuting L'_b by P_B is nothing more than putting together rows (columns) which correspond to the repeated feedback inputs (outputs) or exogenous inputs (outputs). It is easy to construct the block structure vector *bs*

$$bs = \sum_{i=1}^{l} bs_a(i) bs_{f_i} \tag{29}$$

Note that from the preliminary condition on the matrix $M(p)$, $f_i(0) = 0$, so $\Delta(f)_{p=0} = 0$, this leads to $L_{b22} = 0$, see (18). Then the coefficient matrix L can be further simplified from (24) to

$$L = \begin{bmatrix} L_{b11} & L_{b12} L_{a12} \\ L_{a21} L_{b21} & L_{a22} \end{bmatrix} \tag{30}$$

3.2.2. *L_f and bs_f for one rational function.* The coefficient matrix L_f and block structure vector bs_f of the LFT for one rational function $f(p)$ are constructed here in the assumption that all LFTs of monomials in its numerator and denominator polynomials $k(p)$ and $l(p)$ are known. First, the coefficient matrix L_k and block structure vector bs_k for $k(p)$ will be constructed. Let the monomials of $k(p)$ be $m_j(p)$, $j = 1, \ldots s$, s is the number of the monomials in $k(p)$, that is

$$k(p) = \sum_{j=1}^{s} m_j(p)$$

and L_{m_j} and bs_{m_j}, $j = 1, \ldots s$ be the coefficient matrices and the block structure vectors of LFTs for the corresponding monomials $m_j(p)$. So what we try to do now is to find a LFT with L_k and bs_k for the linear combination of all LFTs for $m_j(p)$, see Fig. 4. It is easy to verify

$$\begin{bmatrix} y_1 \\ \vdots \\ y_s \\ z \end{bmatrix} = L'_k \begin{bmatrix} u_1 \\ \vdots \\ u_s \\ w \end{bmatrix}$$

$$y_j = \Delta_j(p) u_j, \quad i = 1, \ldots, s$$

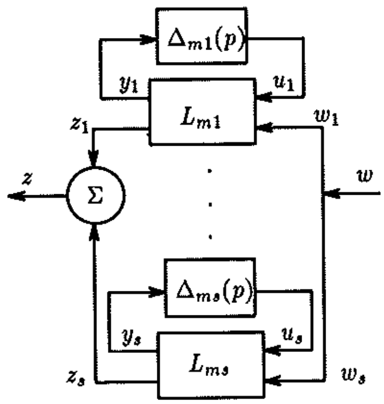

Figure 4. A polynomial LFT as a linear combination of monomial LFTs.

where

$$L_k' = \begin{bmatrix} (L_{m_1})_{11} & 0 & \cdots & \cdots & 0 & (L_{m_1})_{12} \\ 0 & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & 0 & \\ 0 & & \cdots & 0 & (L_{m_s})_{11} & (L_{m_s})_{12} \\ (L_{m_1})_{21} & & \cdots & & (L_{m_s})_{21} & \sum_{j=1}^{s}(L_{m_j})_{22} \end{bmatrix} \tag{31}$$

and $\Delta_j(p)$ has the block structure vector bs_{m_j}. The next step is to permute L_k' such that the repeated feedback inputs (outputs) are arranged together. Define the $s \times (q+1)$ integer matrix

$$B = (b_1^t, \ldots, b_s^t)^t \quad \text{with} \quad b_j = [bs_{m_j}\ 0],\ j = 1, \ldots, (s-1) \quad \text{and}$$
$$b_s = [bs_{m_s}\ 1] \tag{32}$$

Using the permutation matrix P_B defined in (25) with B in (32), we have

$$L_k = P_B^t L_k' P_B \tag{33}$$

It is trivial to get the block structure vector bs_k

$$bs_k = \sum_{j=1}^{s} bs_{m_j} \tag{34}$$

We can use the same method to construct the coefficient matrix L_l and the block structure vector bs_l of the LFT for the polynomial $l(p)$.

Now the LFT of L_f and bs_f can be constructed as a linear combination of LFTs of L_k, bs_k, L_l and bs_l, see Fig. 5, for $f(p)$ is the product of two rational functions

$$f(p) = k(p)\left[\frac{1}{1 + l(p)}\right]$$

and $1/(1 + l(p))$ is just a closed loop function with the feedback gain $-l(p)$ and forward gain 1. It is easy to verify that the coefficient matrix of the LFT for $1/(1 + l(p))$ is

$$L_l^f = \begin{bmatrix} L_{l11} - L_{l12}EL_{l21} & L_{l12}E \\ -EL_{l21} & E \end{bmatrix}, \quad E = (I + L_{l22})^{-1} \tag{35}$$

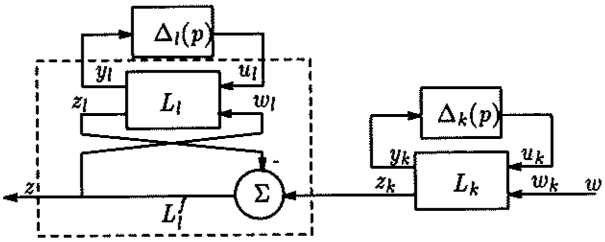

Figure 5. A rational function LFT is a linear combination of two polynomial LFTs.

and the corresponding block structure vector is just bs_l. For the series connection of the two LFTs, we have

$$\begin{bmatrix} y_l \\ y_k \\ z \end{bmatrix} = L'_f \begin{bmatrix} u_l \\ u_k \\ w \end{bmatrix}$$

$$u_l = \Delta_l(p)y_l, \; u_k = \Delta_k(p)y_k$$

where

$$L'_f = \begin{bmatrix} L^f_{l11} & L^f_{l12}L_{k21} & L^f_{l12}L_{k22} \\ 0 & L_{k11} & L_{k12} \\ L^f_{l21} & L^f_{l22}L_{k21} & L^f_{l22}L_{k22} \end{bmatrix} \tag{36}$$

Again, permutation is needed to arrange the repeated feedback inputs (outputs) together. This time, the integer matrix B is defined as

$$B = \begin{bmatrix} bs_l & 0 \\ bs_k & 1 \end{bmatrix} \tag{37}$$

Then

$$L_f = P'_B L'_f P_B \tag{38}$$

The block structure vector bs_f is

$$bs_f = bs_l + bs_k \tag{39}$$

3.2.3. *L_m and bs_m for one monomial.* Let a monomial have the form: $ap_1^{k_1}, \ldots p_q^{k_q}$, where a is the coefficient of the monomial and k_i $(i = 1, \ldots, q)$ is the degree of the monomial with respect to p_i. The total degree of the monomial is

$$k = \sum_{i=1}^{q} k_i$$

This monomial, as a transfer function, is depicted as a series of blocks in Fig. 6. By pulling all variables or parameters out (see the dashed line), it is trivial to construct the $(k+1) \times (k+1)$ coefficient matrix L_m and the block structure vector bs_m of the LFT for the monomial as

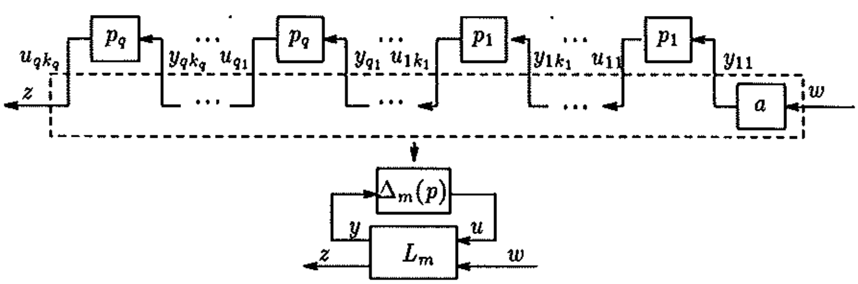

Figure 6. A monomial LFT as a linear combination of its variables.

$$L_m = \begin{bmatrix} 0 & \ddots & \cdots & 0 & a \\ 1 & \ddots & & & 0 \\ 0 & \ddots & \ddots & \vdots & \\ & & \ddots & & \vdots \\ \vdots & & \ddots & 0 & \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \tag{40}$$

and

$$bs_m = [k_1 \ldots k_q] \tag{41}$$

The coefficient a and the degrees k_i of a monomial can be directly obtained from the input data. The second realization step for L_b and bs is completed here, for we can construct L_m and bs_m for each monomial from the input data $(a, k_i, \forall i)$ first, then L_f and bs_f for each rational function from the corresponding L_m and bs_m, and finally L_b and bs_b for $\Delta(f)$ from all L_f and bs_f.

3.3. *Data structure and program code*

In order to make a computer program for the realization problem with the above algorithm, the data structure for representation of the multivariable rational matrix $M(p)$ has to be decided first. Here, we give as an example the data structure with two matrices. The first one is the $n \times lm$ matrix M defined as (see § 3)

$$M = [M_0 \quad M_1 \quad \ldots \quad M_l]$$

The matrix M tells how each rational function $f_i(p)$ enters the matrix $M(p)$. The second one, which tells the structures of all rational functions, is defined as

$$N = [n_1^t, \quad \ldots, \quad n_h^t]^t$$

with each row including all data for one monomial

$$n_r = [i, j, a_{ik}, bs_{mik}], \quad r = 1, \ldots, h$$

where

- i integer, denotes that the monomial is in the ith rational function $f_i(p)$
- j integer, denotes if the monomial is numerator $(j = 0)$ or denominator $(j = 1)$
- k integer, denotes that the monomial is in the kth position
- a_{ik} $\in \mathbb{C}$, the coefficient of the monomial
- bs_{mik} integer vector, the elements of it denote the degrees of corresponding variables. It is equal to the block structure vector for the monomial.
- h integer, total number of the monomials

An example is given next to show how the input matrices are formed.

Example 3.2: Assume that a 2×2 rational matrix $M(p)$ is

$$M(p) = \begin{bmatrix} 1 + \dfrac{p_2}{1 + 2p_1} & \dfrac{5p_1p_2}{1 + 6p_1p_2} \\[2ex] \dfrac{5p_1p_2}{1 + 6p_1p_2} & 1 + \dfrac{3p_1}{1 + 4p_2} \end{bmatrix}$$

the input matrix $M = [M_0\ M_1\ M_2\ M_3]$, where

$$M_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and the corresponding rational functions are

$$f_1(p) = \frac{p_2}{1 + 2p_1}, \ f_2(p) = \frac{3p_1}{1 + 4p_2} \text{ and } f_3 = \frac{5p_1p_2}{1 + 6p_1p_2}$$

Therefore, the input matrix N is

$$N = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 & 0 \\ 2 & 0 & 3 & 1 & 0 \\ 2 & 1 & 4 & 0 & 1 \\ 3 & 0 & 5 & 1 & 1 \\ 3 & 1 & 6 & 1 & 1 \end{bmatrix}$$

With the input data M and N defined above, we can now summarize the algorithm (the code is available in MATLAB as a function file) as

Step 1. Input data, M and N.
Step 2. Compute L_a and bs_a from M, using (20)–(23).
Step 3. Compute L_b and bs_b from N and bs_a.
Step 3.1. Compute L_m and bs_m from N using (40) and (41).
Step 3.2. Compute L_k, L_l, bs_k and bs_l from L_m and bs_m using (31)–(34).
Step 3.3. Compute L_f and bs_f from L_k, bs_k and L_l, bs_l using (35)–(39).
Step 3.4. Compute L_b and bs from $L_f(s)$, $bs_f(s)$ and bs_a using (26)–(29).
Step 4. Compute L from (30).

3.4. *Size reduction*

As mentioned already, the algorithm does, in general, not give the minimal realization, that is, the size of the result $\Delta(p)$ is not minimal. If we take each rational function entry in $M(p)$ as a $f_i(p)$, then the rank r_{ai} of M_i is 1 and the algorithm will give a block structure vector

$$bs = \sum_{i=1}^{h} n_i(4{:}\ q + 3)$$

and the size of $\Delta(p)$ is

$$\sum_{i=1}^{q} bs(i)$$

Clearly this size is larger then the minimal one in the case $j = 1$. But the minimal realization problem for multidimensional systems ($j > 1$) is known as a very difficult problem in system theory even for two-dimensional systems (Kung *et al*. 1977).

However, there are some methods which can lead to a reduction (not minimum) of the dimension of $\Delta(p)$. Here we modify the algorithm to reduce the dimension of $\Delta(p)$, when solving the parametric modelling problem in § 2.3. To solve the realization problem of (12), let the program input matrices be

$$M_0 = 0,\ M_i = B_i$$

After finishing Step 1 and Step 2 of the algorithm, we obtained L_{a12}, L_{a21} and bs_a and have $L_{a11} = 0$ and $L_{a22} = M_0 = 0$. Before going to Step 3, we can reduce the dimension of $\Delta(f)$. As $L_{a11} = 0$ and $L_{a22} = M_0 = 0$, (13) is equivalent to $\det[I + L_{a21}\Delta(f)L_{a12}] = 0$, which is further equivalent to

$$\det[I + M_a\Delta(f)] = 0,\ M_a = L_{a12}L_{a21} \tag{42}$$

Now the reduction algorithm based on Singular Value Decomposition (see Algorithm 3.1 of Fan and Tits 1986 for detail) can be used to reduce the size of $\Delta(f)$ such that (42) is equivalent to

$$\det[I + M_{ar}\Delta_r(f)] = 0$$

where $\Delta_r(f)$ has a reduced block structure vector bs_{ar}, and each 'block row' and 'block column' of M_{ar} (partitioned conformally to bs_{ar}) is of full rank. After the size of $\Delta(p)$ is reduced, we can go to Step 3 with new L_a and bs_a

$$L_{a11} = 0,\ L_{a12} = I,\ L_{a21} = M_{ar},\ L_{a22} = 0 \text{ and } bs_a = bs_{ar}$$

Then from Steps 3 and 4, we obtain L and bs. By constructing M_b with (14), we have

$$\det[I + M_b\Delta(p)] = 0 \tag{43}$$

Again, the same reduction procedure can be used to reduce the size of $\Delta(p)$. By this double reduction, the final size of $\Delta(p)$ is reduced significantly as demonstrated in the example in the next section. Note that such a reduction procedure may not be used for the general realization problem as $\sum_{i=1}^{l} f_i(p)B_i \neq M_b\Delta(p)$.

4. Examples

In this section, we give two examples to show how the algorithm and the size reduction works.

Example 3.2 (continued): The first example here is to solve a general realization problem with the rational matrix $M(p)$ the same as in Example 3.2. Using the input matrices M and N obtained in Example 3.2, we can obtain, from the program, the following LFT coefficient matrices and block structure vectors

$$L_a = \left[\begin{array}{cccc|cc} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad \text{and} \quad bs_a = [1 \quad 1 \quad 2]$$

for the affine realization part and

$$L_b = \left[\begin{array}{cccccccccccc|cccc}
-2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -6 & 0 & 0 & 0 & 0
\end{array}\right]$$

$$bs = [6 \quad 6]$$

for the rational realization part. After the Redheffer Star Product of L_a and L_b, the final realization has the coefficient matrix as depicted in Fig. 7 with the block structure vector $bs = [6\ 6]$. □

Example 4.1: The second example is, from El Ghaoui (1990), to solve the realization problem for the formulation of the multiparameter margin, see § 2.3.

Given the open loop transfer function for a three mass/spring system as

$$\frac{(p_1 + 1)(p_2 + 1)/6}{s^2[s^4 + (\frac{5}{6}p_1 + \frac{3}{2}p_2 + \frac{7}{3}) + (p_1 + 1)(p_2 + 1)]}$$

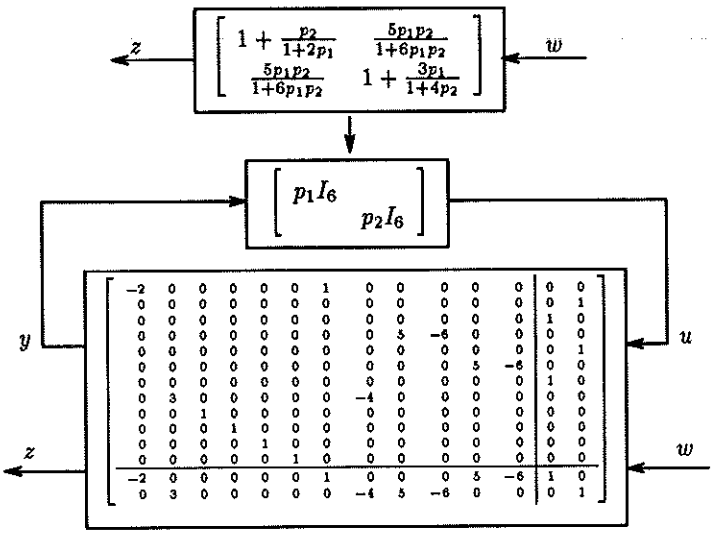

Figure 7. The LFT for the realization problem in Example 3.2.

and the compensator's transfer function as

$$\frac{0.7466s^3 - 0.1132s^2 + 0.5385s + 0.0416}{s^3 + 0.8233s^2 + 1.2384s + 0.3330}$$

the closed loop system matrix $F(p)$ has the following form

$$F(p) = A_0 + p_1p_2A_1 + p_1A_2 + p_2A_3$$

where

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -0.124 & 0 & -1 & 0 & -2.33 & 0 & 0.121 & 0.0643 & 0.0345 \\ 1 & 0 & 0 & 0 & 0 & 0 & -0.823 & -0.124 & -0.333 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -0.124 & 0 & -1 & 0 & 0 & 0 & 0.121 & 0.0643 & 0.0345 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -0.124 & 0 & -1 & 0 & -0.833 & 0 & 0.121 & 0.0643 & 0.0345 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -0.124 & 0 & -1 & 0 & -1.5 & 0 & 0.121 & 0.0643 & 0.0345 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In order to solve the parameter margin problem of (9), we convert (9) to (10) by the algorithm of Tesi and Vicino (1990), with

$$f_1(p) = p_1 p_2, \quad f_2(p) = p_1, \quad f_3(p) = p_2$$

and obtain 45×45 matrices B_1, B_2 and B_3 (which are obviously too large to write here). Now start our algorithm with inputs

$$M = [0_{45\times45} \quad B_1 \quad B_2 \quad B_3] \quad \text{and} \quad N = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 0 \\ 3 & 0 & 1 & 0 & 1 \end{bmatrix}$$

After Step 2, we can form M_a from L_a by (42) with M_a, a 27×27 matrix and $bs_a = [9 \ \ 9 \ \ 9]$. The first reduction gives M_{ar}, a 18×18 matrix and $bs_{ar} = [6 \ \ 6 \ \ 6]$. The coefficient matrix L_b and the block structure bs are obtained from Step 3 with L_b, a 42×42 matrix and $bs = [12 \ \ 12]$. The matrix M_b in (43) is constructed from L given by Step 4 with M_b, a 24×24 matrix (conformal to $bs = [12 \ \ 12]$). Finally, by the second reduction, M_b and bs are

$$M_b = \left[\begin{array}{rrrrrrrrrrrr}
1·9929 & -0·2391 & 0·0747 & 0·1245 & -0·0838 & -0·1007 & 3·7298 & -1·6699 & -0·8111 & -0·3755 & 0·0991 & 0·5748 \\
-9·5041 & 1·1314 & 0·1896 & 0·4209 & 0·1793 & 0·0069 & -27·5936 & 2·1338 & -1·9431 & -0·4148 & 0·7559 & -0·1398 \\
1·7670 & 0·4742 & 0·9628 & -0·1175 & 0·0902 & -0·0457 & 6·1184 & 1·9612 & 0·5615 & -1·7542 & -0·1803 & -0·0402 \\
15·3116 & 1·7784 & -0·3696 & 0·0746 & 0·2095 & -0·1161 & 48·4273 & 5·5715 & 0·3985 & 0·2696 & 0·6219 & 0·1086 \\
10·1655 & -2·6222 & 0·0265 & 0·0984 & 0·1719 & -0·0678 & 24·2354 & -9·4436 & 0·7867 & -0·1839 & 0·4058 & -0·0602 \\
18·9315 & 0·5319 & 0·2194 & 0·0654 & -0·2201 & 0·1460 & 53·4500 & -0·9555 & -1·8426 & -0·0551 & -0·4325 & -0·1086 \\
-0·0671 & -0·0419 & 0·0054 & 0·0259 & -0·0112 & -0·0106 & 1·5411 & -0·0981 & -0·0474 & -0·0314 & 0·0058 & 0·0619 \\
-1·1655 & -0·1205 & -0·2577 & 0·1151 & -0·0387 & 0·0133 & -1·9742 & 1·6895 & 0·1627 & 0·3244 & 0·0620 & -0·2106 \\
0·0472 & -0·1152 & -0·6470 & 0·1433 & -0·0927 & -0·0002 & 1·2129 & -0·3618 & 0·9928 & 0·7967 & -0·1273 & -0·2007 \\
-0·3916 & -0·0731 & -0·2299 & 0·0749 & -0·0355 & 0·0026 & -0·3014 & 0·1342 & 0·0450 & 1·2813 & -0·0066 & -0·1051 \\
1·2426 & -0·0007 & -0·4926 & 0·0432 & -0·0653 & -0·0105 & 3·2295 & -1·0823 & -0·1579 & 0·6067 & 0·7888 & -0·0390 \\
-1·4522 & -0·1311 & -0·7770 & 0·1830 & -0·0823 & 0·0467 & -3·4990 & 1·0193 & 0·3525 & 1·0321 & -0·0049 & 0·4138
\end{array}\right]$$

$$bs = [6 \ \ 6]$$

But in El Ghaoui (1990), $bs = [7 \ \ 7]$. The corresponding euclidean parameter margin obtained by the structured total least square (STLS) method (DeMoor 1993) is 0·50171 at point $p_1 = -0·49953$ and $p_2 = -0·046697$. (In El Ghaoui 1990, the parameter margin is 0·5036 at point $p_1 = -0·498$ and $p_2 = -0·0938$.)

5. Conclusions

First, it was shown that the parametric uncertainty modelling can be cast as the realization of a multidimensional system, which is defined in the framework of linear fractional transformations (LFTs). An algorithm to solve the multidimensional realization problem systematically was given, which is the main contribution of this paper. The trick we used is to realize the affine part of a rational matrix at first, then the polynomial part second. The disadvantage of the algorithm is clearly the large size of the resulting matrix. Consequently, a size reduction is introduced in each realization step for the problem of the formulation of the multiparameter margin. However, such a problem-dependent size reduction may not be generally used for realization problems. Further research work to investigate the minimality issue is definitely required.

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