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A Singular Value Decomposition for Higher-Order Tensors and Application to Independent Component Analysis*

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Extended Summary

1. Introduction

The purpose of this paper is twofold. First, we want to present a new result in multilinear algebra: a generalization of the Singular Value Decomposition to the case of higher-order tensors. Secondly, we will show how this new tool can be used to perform the Independent Component Analysis in Higher-Order Statistics.

For reasons of clarity, the discussion will be restricted to third-order tensors with real elements. Our results can immediately be generalized to tensors of order higher than three. The generalization to the complex case is straightforward too.

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2. Tensor Singular Value Decomposition

2.1. The TSVD-model

Let Φ be a third-order $(I \times J \times K)$-tensor with real entries, providing a formal way of expressing a multilinear form on $\mathbb{R}^I \times \mathbb{R}^J \times \mathbb{R}^K$.

If P, Q, R denote the dimension of Φ's "column space", "row space" and "tube space", then the decomposition model is given by

$$\Phi_{ijk} = \sum_p^P \sum_q^Q \sum_r^R A_{ip} B_{jq} C_{kr} \Xi_{pqr} \tag{1}$$

in which $A \in \mathbb{R}^{I \times P}$, $B \in \mathbb{R}^{J \times Q}$ and $C \in \mathbb{R}^{K \times R}$ are (column-wise) orthogonal matrices and the "core tensor" $\Xi_{(P \times Q \times R)}$ is "all-orthogonal". The matrices of $\Xi_{(P \times Q \times R)}$ (with one index fixed) are put in order of non-increasing Frobenius norm.

All-orthogonality means that two matrices in Ξ, corresponding to different fixed values of p, are always orthogonal with respect to the standard scalar product, e.g.

$$\sum_q^Q \sum_r^R \Xi_{p_1 qr} \Xi_{p_2 qr} = 0 \text{ if } p_1 \neq p_2 \tag{2}$$

and that a similar orthogonality applies for fixed q and r.

2.2. Relations with second-order SVD

The columns of A, B and C can be considered as "I-mode", "J-mode" and "K-mode" "singular vectors". The core tensor Ξ contains the generalized "singular values".

If we denote the inner product along a certain mode by $\times_{mode}$ (e.g. $\sum_p^P \Xi_{pqr} A_{ip} \triangleq \Xi \times_p A$), we can write down the model equation (1) in a form that provides an easy way to a visual interpretation as the third-order equivalent of the Singular Value Decomposition:

$$\Phi_{(I \times J \times K)} = \Xi_{(P \times Q \times R)} \times_p A_{(I \times P)} \times_q B_{(J \times Q)} \times_r C_{(K \times R)} \tag{3}$$

in which A, B, C are (column-wise) orthogonal and Ξ is all-orthogonal.

Equation (3) is visualized in figure 1. It should be compared to the expression for the SVD of a real $(I \times J)$-matrix F, which is, using the notation defined above:

$$F_{(I \times J)} = \Sigma_{(P \times Q)} \times_p A_{(I \times P)} \times_q B_{(J \times Q)} \quad (\equiv A \cdot \Sigma \cdot B^t) \tag{4}$$

in which A, B are (column-wise) orthogonal and Ξ is diagonal (see also figure 2).

Clearly, the tensor equation (3) is a formal generalization of the matrix equation (4).

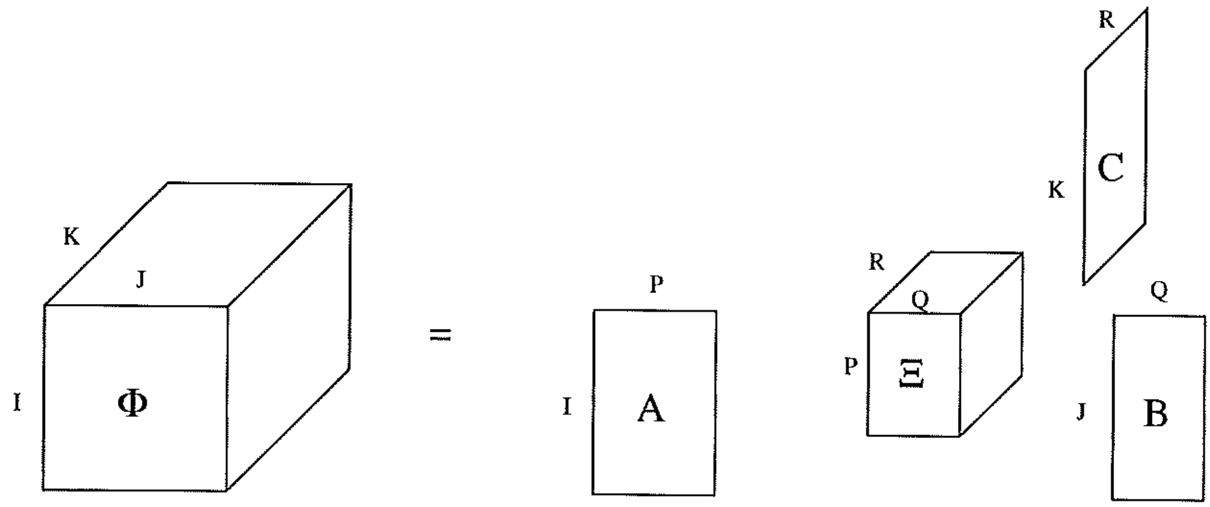

Figure 1: Third-order Singular Value Decomposition

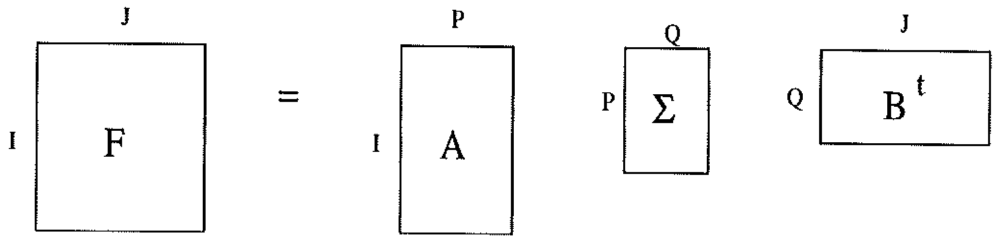

Figure 2: Second-order Singular Value Decomposition

Moreover, it can be proved that the Tensor Singular Value Decomposition of a second-order tensor boils down to its matrix SVD (under the condition that all singular values are different) [1].

Many properties, like the link with the Eigenvalue Decomposition or the aspects of unicity, have already been generalized. They all show a strong analogy between the matrix and the tensor case [2].

2.3. Calculation

The matrices A, B and C can be calculated as the right singular matrices of the $(JI \times K)$, $(KJ \times I)$ and $(IK \times J)$ matrix unfoldings of Φ (Figure 3).

In general, the Tensor SVD of an Nth-order tensor leads to N matrix Singular Value Decompositions.

The core tensor follows from the equation:

$$\Xi_{(P \times Q \times R)} = \Phi_{(I \times J \times K)} \times_i A_{(I \times P)} \times_j B_{(J \times Q)} \times_k C_{(K \times R)} \quad (5)$$

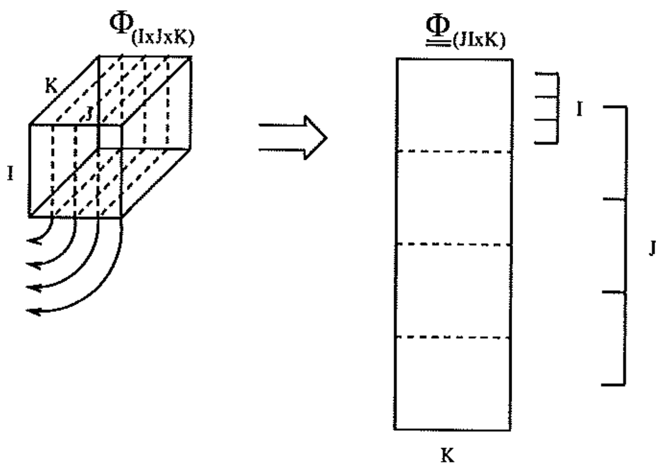

Figure 3: Unfolding of the $(I \times J \times K)$ tensor Φ to the $(JI \times K)$ matrix $\underline{\underline{\Phi}}$

3. Application to Independent Component Analysis

3.1. Problem description

Consider the following linear transfer of a K-dimensional zero-mean stochastic "source vector" X to a zero-mean stochastic "output vector" Y:

$$Y = MX \tag{6}$$

The matrix M is assumed to be square (for convenience) and regular; the components of X are statistically independent.

The goal of Independent Component Analysis is the determination of M, given only realizations of Y. This is done by minimizing the statistical dependence of the components of the corresponding source vector estimate [3].

3.2. Solution

Generally, the problem is solved by factorisation of the transfer matrix:

$$M = TQ \tag{7}$$

in which T is regular and Q is orthogonal.

Second-order statistical independence of the source components can be realized by the determination of T from a congruence transformation of the output covariance C_2^Y:

$$C_2^Y = TT^t \tag{8}$$

One alternative is the computation of the Eigenvalue Decomposition of C_2^Y (which is equivalent to a matrix SVD since C_2^Y is symmetric):

$$C_2^Y = E\Sigma^2 E^t = (E\Sigma)(E\Sigma)^t \tag{9}$$

The resulting degree of freedom, the orthogonal factor Q, is recovered from the higher-order statistics of Y.

As a new result, we are able to prove that Q is equal to the singular matrix of the symmetric tensor $\mathbf{C}_3$, defined as:

$$\mathbf{C}_3 \triangleq \mathbf{C}_3^Y \times_1 T^{-1} \times_2 T^{-1} \times_3 T^{-1} \tag{10}$$

in which $\mathbf{C}_3^Y$ denotes the third-order cumulant of Y.

Besides the proof, we will give a numerical example in the final paper.

3.3. Discussion

In contrast to the approaches described in literature up till now, the new method guarantees the determination of the global optimum of the non-linear higher-order part of the Independent Component Analysis.

From a computational point of view, the new approach is based on the SVD of matrices: this means that robust algorithms can be used.

Finally, we want to stress the conceptual importance of the new approach. It reveals an important symmetry when considering the problems of Principal Component Analysis (second-order) and Independent Component Analysis (higher-order). In "classical" *second*-order statistics, the problem of interest is to remove the correlation from data measured after linear transfer of independent source signals. The key tool to realize this, comes from "classical" *linear* algebra: it is the matrix SVD. More recently, researchers also aimed at the removal of higher-order dependence, which is a problem of *higher*-order statistics. We proved that one can resort to a tool from *multilinear* algebra, which is precisely the generalization of the SVD for higher-order tensors.

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