93-55C

Proceedings of the
American Control Conference
Baltimore, Maryland • June 1994

WP2 - 5:00

DESIGNING REDUCED ORDER OUTPUT FEEDBACK CONTROLLERS USING A POTENTIAL REDUCTION METHOD[1]

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Abstract

Some control problems can be formulated as convex problems involving linear matrix inequalities. Not only controllers for linear time invariant systems can be designed in this way but also controllers for linear systems with time varying uncertainties. It is also possible to design reduced order controllers, but the problem is no longer convex. To design a controller of the lowest possible order that satisfies the constraints, the minimal rank of an affine matrix function has to be found subject to linear matrix inequalities.

In this paper an algorithm is proposed for solving such problems. It is an extension of a potential reduction method for solving convex optimization problems. The problem of finding minimum rank solutions is, however, not convex. Still, with the proposed potential reduction method reduced rank solutions can easily be obtained.

1. Introduction

Fast and reliable algorithms exist to solve convex problems ([2], [3], [11], [12]). This has motivated many researchers to formulate or reformulate problems in control as convex optimization problems. We refer to [2], [3], [4] and the references therein. A lot of analysis and state feedback problems have already been shown to be convex (e.g. [3]).

It has also been shown how certain output feedback controllers can be designed using convex optimization. In [2] the Youla parameterization and a particular approximation of the controller are used. However, the resulting controller is usually of very high order. More recently it has also been shown how these problems can be formulated as convex problems involving linear matrix inequalities. In that framework, it is even possible to design reduced order controllers if low rank solutions of one of the involved matrices can be found, e.g. [1], [4], [8], [10]. These methods are not restricted to linear time invariant systems, but can also be applied to design linear controllers for polytopic linear differential inclusions and norm-bound linear differential inclusions ([4]). An advantage of the problem formulations is that the order of the reduced order controller is not specified in advance: The lower the rank of a certain matrix, the lower the order of the corresponding controller.

In general the minimum rank problems involving linear matrix inequalities that have to be solved, can be written as follows:

$$\min_x \text{Rank } F(x) \geq 0, \quad \text{subject to } C(x) \geq 0,$$

where $F(x)$ and $C(x)$ are affine symmetric matrix functions: $F(x) = F_0 + x_1F_1 + \ldots + x_mF_m$ and $C(x) = C_0 + x_1C_1 + \ldots + x_mC_m$ with $F_i = F_i^t \in \mathbb{R}^{n_f \times n_f}$ and $C_i = C_i^t \in \mathbb{R}^{n_c \times n_c}$, $i = 0 \ldots m$. This problem is not convex. Usually there exist several local and global minima. However, the presented algorithm will quickly find reduced rank solutions that satisfy the constraints.

[1] The following text presents research results obtained within the framework of the Belgian programme on interuniversity attraction poles (IUAP-17,IUAP-50) initiated by the Belgian State - Prime Minister's Office - Science Policy Programming. The scientific responsibility is assumed by its authors.

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In [5] an algorithm to solve these problem was proposed. It was based on the inversion of an analytic center algorithm for convex optimization. In this paper another algorithm is proposed to solve these problems. It is an extension of a potential reduction method for convex optimization. Practical results show that the potential reduction method is better than the algorithm proposed in [5]. It is faster and it is possible to find solutions where the latter algorithm sometimes fails to converge.

The paper is organized as follows. In section 2 the design of a controller that satisfies a given L_2-gain bound is discussed. Two reduced rank solutions have to be solved in this case. How the rank reduction method can be solved using the potential reduction method is described in section 3. Section 4 contains some numerical examples. The conclusions are summarized in section 5.

2. L_2-gain control

In this section the formulas to design a linear time invariant controller that guarantees a bound on the L_2 gain of a NLDI are summarized ([4]). Consider the following NLDI:

$$\begin{array}{rcl} \dot{x} & = & Ax + B_w w + B_u u + B_p p \\ z & = & C_z x + D_{zw} w + D_{zu} u + D_{zp} p \\ y & = & C_y x + D_{yw} w + D_{yu} u + D_{yp} p \\ q & = & C_q x + D_{qw} w + D_{qu} u + D_{qp} p, \\ & & p = \Delta(t) q, \qquad \|\Delta(t)\| \leq 1, \end{array} \tag{1}$$

where $\|.\|$ is the induced 2-norm for matrices. It is assumed that $D_{yu} = 0$, without loss of generality. The L_2 gain of the system is defined as:

$$\sup_{\|w\|_2 < 1} \|z\|_2 = \sup_{\|w\|_2 \neq 0} \frac{\|z\|_2}{\|w\|_2},$$

where $\|.\|_2$ is the L_2 norm of the signal:

$$\|w\|_2^2 = \int_0^{+\infty} w(t)^t w(t)\, dt.$$

For notational reasons, D_{zw}, D_{zu}, D_{zp}, D_{yw}, D_{yp} and D_{qp} are assumed to be zero. However, the formulas can easily be extended to include the non-zero cases. The stated formulas are the ones needed for the example of section 4.

From arguments in [4] the following theorem can be proved:

Theorem 1 *A controller of order n_c which guarantees that the L_2-gain of the NLDI (1) is lower than γ, exists if matrices $P > 0$ and $Q > 0$, and positive scalars λ, μ and σ can be found such that*

$$\lambda = \mu^{-1}, \tag{2}$$

$$\begin{pmatrix} P & I \\ I & Q \end{pmatrix} \geq 0, \tag{3}$$

$$\begin{pmatrix} QA^t + AQ + \gamma^{-2} B_w B_w^t + \mu B_p B_p^t & QC_z^t & QC_q^t + \gamma^{-2} B_w D_{qw}^t \\ C_z Q & -I & 0 \\ C_q Q + \gamma^{-2} D_{qw} B_w^t & 0 & -\mu I + \gamma^{-2} D_{qw} D_{qw}^t \end{pmatrix} < \sigma \begin{pmatrix} B_u \\ 0 \\ D_{qu} \end{pmatrix} \begin{pmatrix} B_u^t & 0 & D_{qu}^t \end{pmatrix}, \tag{4}$$

$$\begin{pmatrix} A^t P + PA + C_z^t C_z + \lambda C_q^t C_q & PB_w + \lambda C_q^t D_{qw} & PB_p \\ B_w^t P + \lambda D_{qw}^t C_q & -\gamma^2 I + \lambda D_{qw}^t D_{qw} & 0 \\ B_p^t P & 0 & -\lambda I \end{pmatrix} < \sigma \begin{pmatrix} C_y^t \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} C_y & 0 & 0 \end{pmatrix}, \tag{5}$$

and

$$Rank \begin{pmatrix} P & I \\ I & Q \end{pmatrix} \leq n + n_c. \qquad \square$$

σ can be removed from (4) and (5), by multiplication of the null spaces of the matrices at the right hand side ([4]).

To find a controller that satisfies the bound, is a bit more involved than finding a stabilizing controller, as a λ and μ have to be found such that $\lambda = \mu^{-1}$. To find a $\lambda = \mu^{-1}$, El Ghaoui ([9]) proposed to apply the following procedure: First find P, Q, λ and μ that satisfy (3), (4) and (5) and

$$\begin{pmatrix} \lambda & 1 \\ 1 & \mu \end{pmatrix} \geq 0. \tag{6}$$

Then apply a rank minimization on (6) subject to (3), (4) and (5). When a rank 1 matrix is found, $\lambda = \mu^{-1}$ holds. If a rank 1 solution for (6) exists and the other constraints are also satisfied, a full order controller that satisfies the given specifications exists. If no rank 1 solution can be found, it doesn't imply that there doesn't exist a controller such that the L_2-gain is smaller than γ. The formulas are only sufficient for NLDIs and not necessary ([4]).

When λ and μ are fixed, the order of the controller can also be reduced by minimizing the rank of (3) subject to (4) and (5). When P, Q and the rank of (3) are known, the order of the controller and the corresponding matrix $\tilde{P}$ can be found.

Finally the controller is designed, solving a convex feasibility problem for a feasible K. If solutions λ, μ, P and Q are found we know from the theory that a solution K has to exist. Usually a whole set of reduced order controllers exists that satisfy the constraints. Extra conditions can also be taken into account.

3. The potential reduction method

In this section it is shown how a potential reduction method can be used to solve reduced rank linear matrix inequality (LMI) problems. The algorithm is an extension of a potential reduction method for optimizing a linear objective over a convex set. The potential reduction method is an interior point method for convex optimization.

Consider again the general minimum rank problem as described in the introduction. The desired solutions x are situated at the boundary of the set of feasible solutions satisfying the constraints $F(x) \geq 0$ and $C(x) \geq 0$. This set is convex ([3],[11]).

The algorithm works as follows. The following objective function is constructed:

$$\phi(x) = q \log\det F(x) - \log\det \begin{pmatrix} C(x) & 0 \\ 0 & F(x) \end{pmatrix}.$$

The first term is a concave term that becomes $-\infty$ when $F(x)$ is singular. The second term is a convex term that becomes $+\infty$ at the boundary of the feasible domain. It can be interpreted as a barrier function. To find a reduced rank solution of $F(x)$, $\phi(x)$ has to be minimized.

It might look strange that $F(x)$ is included in the convex term that becomes $+\infty$ when $F(x)$ is singular. This does not matter if q is taken high enough as $\phi(x)$ is equivalent to $(q-1)\log\det F(x) - \log\det C(x)$. It makes the implementation easier however. It is shown that q has to satisfy the following constraint:

$$q > 1 + \frac{n_c}{n_f}.$$

$\phi(x)$ is not a convex function, thus the theory of Nesterov and Nemirovsky ([11]) cannot be applied directly. However, if the concave term is linearized the theory can be applied. The linearized function in x_k is

$$\psi(x) = q g_F^t x - \log\det \begin{pmatrix} C(x) & 0 \\ 0 & F(x) \end{pmatrix} + c,$$

where g_F is the gradient of $\log\det F(x)$: $g_F = \begin{pmatrix} g_1 & \dots & g_m \end{pmatrix}^t$ and $g_i = \text{trace} F(x)^{-1} F_i$, $i = 1 \dots m$. c is a constant. The concave term does not matter for the minimization. The first term of $\phi(x)$ is concave. This implies that the value of $\phi(x)$ will always be lower than that of $\psi(x)$. Thus if the linearized function is minimized, the corresponding value of the objective function will be even lower.

The Newton direction of $\psi(x)$ is $-H(x)^{-1}g(x)$, where $H(x)$ is the Hessian and $g(x)$ is the gradient. It can then easily be computed as the solution of a least squares problem:

$$-H^{-1}g = \underset{v}{\text{argmin}} \left\| I_c - \sum_{i=1}^{m} v_i Z_c^{-1} C_i Z_c^{-t} \right\|_F + \left\| I_f(1-q) - \sum_{i=1}^{m} v_i Z_f^{-1} F_i Z_f^{-t} \right\|_F,$$

where Z_c and Z_f are the Cholesky factors of $C(x)$ and $F(x)$: $C(x) = Z_c Z_c^t$ and $F(x) = Z_f Z_f^t$.

The Newton algorithm is:

$$x_{k+1} = x_k - \alpha H^{-1} g.$$

The damping factor α still has to be determined. To ensure that x_{k+1} is again inside the feasible set if x_k is in the feasible set, the Nesterov-Nemirovsky damping factor is taken. This damping factor depends on the so-called Newton decrement $\delta(x)$ ([3],[11]):

$$\delta(x) = \|H(x)^{-1/2} g(x)\|.$$

The Nesterov-Nemirovsky damping factor is then:

$$\begin{cases} \alpha = \frac{1}{1+\delta(x)} & \text{if } \delta(x) > .25, \\ \alpha = 1 & \text{if } \delta(x) \leq .25. \end{cases}$$

If v is known, $\delta(x)$ can easily be computed as:

$$\delta(x) = \left\| \sum_{i=1}^{m} v_i Z_c^{-1} C_i Z_c^{-t} \right\|_F + \left\| \sum_{i=1}^{m} v_i Z_f^{-1} F_i Z_f^{-t} \right\|_F.$$

Starting from an initial feasible point, the algorithm will move towards a minimum. Each of the updates will be inside the feasible set. To stop the following criteria can be used. As x remains feasible, the eigenvalues of $F(x)$ are always positive. An eigenvalue is considered to be singular if it is lower than a certain threshold. It can be shown that when x is near the boundary $F(x) \geq 0$,

$$\left(\frac{\| \sum_{i=1}^{m} v_i Z_f^{-1} F_i Z_f^{-t} \|_F}{q-1} \right)^2 \tag{7}$$

is an estimate of the number of decreasing eigenvalues. When (7) is rounded towards the nearest integer it indicates the number of decreasing eigenvalues of $F(x)$. Let n_d be that number. The stopping criterion can then be formulated as follows: Sort the eigenvalues in ascending order and check if the eigenvalue indicated by n_d is smaller than the threshold. If so, stop, else continue. The rank of the resulting $F(x)$ will be $n_f - n_d$.

However, the algorithm can also get stuck in a finite local minimum. It is known that if $\delta(x)$ is lower than .25, the algorithm will converge quadratically towards a minimum. If $\delta(x)$ becomes small, the algorithm gets stuck in a finite local minimum. No reduced rank solution is found in this case.

More details of the algorithm can be found in [6] and [7].

4. Examples

In this section the above theory is illustrated with an example. Reduced order controllers are designed for a pendulum with a varying length, shown in figure 1. This system can be described as a NLDI:

$$\begin{aligned}
\begin{pmatrix}\dot\theta_1\\ \dot\theta_2\end{pmatrix} &= \begin{pmatrix}0 & 1\\ -\frac{g}{l_0} & 0\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} + \begin{pmatrix}0\\ \frac{1}{ml_0}\end{pmatrix} w \\
&\quad + \begin{pmatrix}0\\ \frac{1}{ml_0}\end{pmatrix} u + \begin{pmatrix}0\\ \delta l\end{pmatrix} p, \\
z &= \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}, \\
y &= \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}, \\
q &= \begin{pmatrix}\frac{g}{l_0^2} & 0\end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} - \frac{1}{ml_0^2} w - \frac{1}{ml_0^2} u, \\
& p = \Delta(t) q,
\end{aligned}$$

with $\|\Delta(t)\| < 1$, where $\Delta(t)$ is an unknown bounded time varying uncertainty, and where l_0 is the nominal length, δl is the maximal deviation from this nominal length. θ_1 is the angle of the pendulum. θ_2 is the angular velocity. m is the mass of the load.

A disturbing input w is assumed, parallel to the control input u. The controlled output is the angle θ. The following values are used:

$$\begin{aligned} l_0 &= 1 \quad (m), & \delta l &= .1 \quad (m), \\ m &= .5 \quad (kg), & g &= 9.81 \quad (m/s^2). \end{aligned} \tag{8}$$

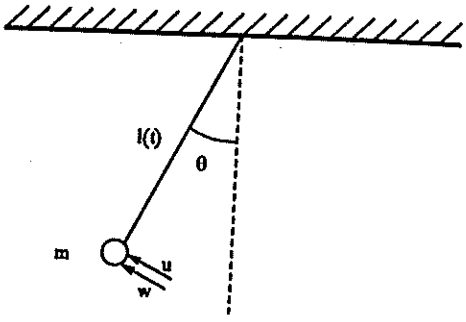

Figure 1: The pendulum with varying length setup.

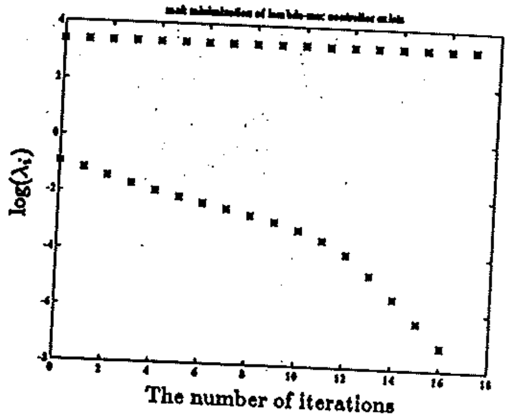

Figure 2: The eigenvalues of (6) as a function of the number of iterations. One of the eigenvalues goes to zero. This implies that a solution $\lambda = \mu^{-1}$ exists.

We want to design a controller such that the L_2-gain of the closed loop is less than 1. First a feasible solution has to be found satisfying (3)-(6). Once a feasible point is found, a λ and μ have to be computed such that the rank of (6) is 1, subject to (3), (4) and (5). The evolution of the eigenvalues of (6) is shown in figure 2. One eigenvalue goes to 0. Thus a solution $\lambda = \mu^{-1}$ exists, such that the other inequalities are satisfied: $\lambda = 4.7870e-04$, $\mu = 2.0890e+03$. With λ and μ fixed, the rank of (3) can be reduced subject to (4) and (5). The evolution of the eigenvalues of (3) is shown in figure 3. The final rank of (3) is 3, implying a first order controller. The state space realization of the controller is

$$\left(\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right) = \left(\begin{array}{c|c} -190.3 & 1097 \\ \hline 109.5 & -633.1 \end{array}\right).$$

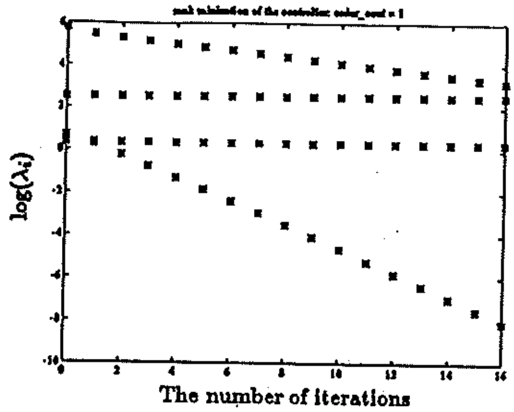

Figure 3: The eigenvalues of (3) as a function of the number of iterations. One eigenvalue decreases to zero. This implies that the rank of the matrix is only three. The order of the controller will be one lower than the order of the plant.

The actual L_2-gain is 0.1594, which is much lower than the given bound.

5. Conclusions

In this paper a potential reduction method is proposed to solve minimum rank problems involving linear matrix inequalities. This problem shows up in the design of reduced order controllers for linear time-invariant problems. The minimum rank procedure is not restricted to the problems mentioned in this paper. A lot of other problems in controller design can be solved in this way. Other interesting problems like the Frisch scheme in identification can also be solved in this way.

The proposed method works better than another method based on the inversion of the analytic path. The potential method works faster and has the advantage that all the constraints are always taken into account. The step never has to be recomputed.

Although it is not guaranteed that a minimum rank solution is obtained, often it is reached. In almost all cases at least a reduced rank solution is found, if there exists one that satisfies all the constraints.

Acknowledgments

The authors would like to thank Laurent El Ghaoui, Lieven Vandenberghe and Eric Feron for their suggestions and helpful remarks.

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