Katholieke Universiteit Leuven

Departement Elektrotechniek

ESAT-SISTA/TR 1993-66

A Singular Value Decomposition for Higher-Order Tensors [1]

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September 1993

Published in:
Proceedings of the ATHOS Workshop
on
System Identification and High-Order Statistics
Nice, France, September 20-21, 1993.

[1]This report is available by anonymous ftp from *gate.esat.kuleuven.ac.be* in the directory *pub/SISTA/delathau/reports/athos-nice93.ps.Z*

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Abstract

Due to the scientific boom in higher-order signal processing, the interest in algebraic manipulations of tensors is rapidly increasing. We studied a model that can be interpreted as the tensorial equivalent of the Singular Value Decomposition. In the paper we mainly focus on the algebraic properties of this model.

A Singular Value Decomposition for Higher-Order Tensors

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Due to the scientific boom in higher-order signal processing, the interest in algebraic manipulations of tensors is rapidly increasing.

By researchers in mathematical psychology, a model was proposed which is generally suitable as a tensorial decomposition. When putting the right constraints, this model turns out to be the tensorial equivalent of the Singular Value Decomposition.

In higher-order statistics, the "Tensor SVD" can e.g. be used to perform the Independent Components Analysis. In this paper we mainly focus on the algebraic properties of the model.

Let Φ be a third-order $(I \times J \times K)$ tensor with real entries, providing a formal way of expressing a multilinear form on $\Re^I \times \Re^J \times \Re^K$. Our results can immediately be generalized to tensors of order higher than three. The generalization to the complex case is straightforward too.

If P, Q, R denote the dimension of Φ's "column space", "row space" and "tube space", then the decomposition model is given by

$$\Phi_{ijk} = \sum_p^P \sum_q^Q \sum_r^R A_{ip} B_{jq} C_{kr} \Xi_{pqr} \quad (1)$$

in which $A \in \Re^{I \times P}$, $B \in \Re^{J \times Q}$ and $C \in \Re^{K \times R}$ are (column-wise) orthogonal matrices and the "core tensor" $\Xi_{(P \times Q \times R)}$ is "all-orthogonal". All-orthogonality means that two submatrices in Ξ, corresponding to different fixed values of p (or q, or r), are always orthogonal with respect to the inner product. The submatrices of Ξ are put in order of descending energy.

*Supported by the Belgian I.W.O.N.L.
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Second Order	Third Order
$F = A \cdot \Sigma \cdot B^t$	$\Phi = \Xi \times_p A \times_q B \times_r C$
$P_I = F \cdot F^t$ contains on position (i, i') the inner product of rows i and i' in F	$P_I = \underline{\underline{\Phi}}_{(I\times KJ)} \cdot \underline{\underline{\Phi}}^t_{(I\times KJ)}$ contains on position (i, i') the inner product of horizontal planes i and i' in Φ
$P_I = A \cdot D_A \cdot A^t$ with A containing the left singular vectors of F	$P_I = A \cdot D_A \cdot A^t$ with A containing the I-mode singular vectors of Φ
The inner product of rows p and p' in Σ is the (p, p')th element of D_A. Different rows of Σ are orthogonal.	The inner product of horizontal planes p and p' in Ξ is the (p, p')th element of D_A. Different horizontal planes of Ξ are orthogonal.
$\|\|F\|\| = \|\|\Sigma\|\|$ (Frobenius norms)	$\|\|\Phi\|\| = \|\|\Xi\|\|$ (Frobenius norms)

Table 1: Comparison between second and third order Singular Value Decomposition

There are several ways to write down the model equations. One equivalent is to consider Φ as a sum of rank-1 tensors:

$$\Phi = \sum_p^P \sum_q^Q \sum_r^R \Xi_{pqr} \, \underline{A_p} \circ \underline{B_q} \circ \underline{C_r} \tag{2}$$

in which $\underline{A_p}$, $\underline{B_q}$, $\underline{C_r}$ are the columns of A, B, C and $\circ$ denotes the vectorial outer product.

If we denote the inner product along a certain mode by $\times_{mode}$, we can as well put:

$$\Phi_{(I\times J\times K)} = \Xi_{(P\times Q\times R)} \times_p A_{(I\times P)} \times_q B_{(J\times Q)} \times_r C_{(K\times R)} \tag{3}$$

A pure matrix equation is obtained by unfolding Φ and Ξ to $(JI \times K)$ and $(QP \times R)$ matrices $\underline{\underline{\Phi}}$ and $\underline{\underline{\Xi}}$, with J and Q slowlier varying than I resp. P:

$$\underline{\underline{\Phi}} = (B \otimes A) \cdot \underline{\underline{\Xi}} \cdot C^t \tag{4}$$

in which $\otimes$ denotes the Kronecker product.

In the second-order case this model boils down to the well-known Singular Value Decomposition (under the condition that all singular values are different). Generalizations of some second-order properties are listed in Table 1.

The generalized SVD provides the means for calculation of a generalization of the best rank-k approximation of matrices. We study the problem in which a given tensor Φ is approximated in least-squares sense by an other tensor $\hat{\Phi}$, satisfying the generalized SVD model for fixed P, Q, R. In other words, the column-wise orthonormal matrices $A_{(I\times P)}$, $B_{(J\times Q)}$, $C_{(K\times R)}$ and the core tensor Ξ have to be determined, such that

$$\hat{\Phi}_{(I\times J\times K)} = \Xi_{(P\times Q\times R)} \times_p A \times_q B \times_r C \tag{5}$$

minimizes for a given $(I \times J \times K)$ tensor Φ the residual sum of squares

$$\sum_i \sum_j \sum_k (\Phi_{ijk} - \hat{\Phi}_{ijk})^2 \tag{6}$$

The best estimation can be computed by means of an alternating least squares algorithm. The initial value of this iteration process is obtained by truncation of the generalized SVD model.