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Schelfhout G., De Moor B., ``A relative error model reduction method using balancing", in *Proc. of the 33rd IEEE Conference Decision Control*, Lake Buena Vista, Florida, Dec. 1994, 2734 p., Lirias number: 180279.

A relative error model reduction method using balancing.

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February 23, 1994

Keywords: relative error model reduction, balanced stochastic truncation, frequency weighting

33rd CDC

Abstract

This paper proposes to use Enns's frequency weighted balanced truncation with as a weighting function the inverse of the outer factor of the system to be reduced. This results in a relative error model reduction method very similar to Balanced

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†The following text presents research results obtained within the framework of the Belgian programme on Interuniversity Attraction Poles (IUAP-17 and IUAP-50) initiated by the Belgian State - Prime Minister's Office - Science Policy Programming. The scientific responsibility is assumed by its authors.

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Stochastic Truncation (BST), in the sense that in both methods, there is a spectral factorization of $G(s)G^T(-s)$, involving the solution of a Riccati equation, and a controllability gramian to be computed. It is shown that the computation of the observability gramian of the weighted system in Enns's method can be reduced in this case to the computation of the observability gramian of the weighting function only. Moreover, a Schur decomposition of the Hamiltonian associated with the Riccati equation can be found with little effort, so that the Riccati equation may be solved at a much smaller cost.

1 Background

Let $G(s)$ be a square stable invertible plant of order n, and let $G^{-1}(s)$ be its inverse.
Some notation:
$[A, B, C, D] \triangleq C(sI - A)^{-1}B + D$
Let $[A, B, C, D]$ be a minimal realization of $G(s)$, $A \in \mathbb{R}^{n\times n}, B, C^T \in \mathbb{R}^{n\times m}, D \in \mathbb{R}^{m\times m}$, then $[A_i, B_i, C_i, D_i]$, where $A_i \triangleq A - BD^{-1}C$, $B_i \triangleq BD^{-1}$, $C_i \triangleq -D^{-1}C$, $D_i \triangleq D^{-1}$, is a minimal realization of $G^{-1}(s)$.

We briefly recapitulate Enns's scheme for frequency weighted model reduction: we wish to obtain a system $G_r(s) = [A_r, B_r, C_r, D_r]$ of order at most r such that $\|W_i(s)(G(s) - G_r(s))W_o(s)\|_\infty$ is made small, where $W_i(s) \triangleq [A_i, B_i, C_i, D_i]$ and $W_o(s) \triangleq [A_o, B_o, C_o, D_o]$ are stable and invertible frequency weights. In the unweighted case ($W_i(s) = I, W_o(s) = I$), this is accomplished by balanced truncation [M81] : the observability and controllability grammians Q and P are computed from $A^TQ + QA + C^TC = 0$ and $AP + PA^T + BB^T = 0$, a similarity transform $[T^{-1}AT, T^{-1}B, CT, D]$ is found that balances the grammians (i.e. makes them equal and diagonal : $T^{-1}PT^{-T} = T^TQT = \Sigma = \mathrm{diag}([\sigma_1 \ldots \sigma_n])$), and the reduced model $G_r(s)$ is obtained by truncating the states with the smallest corresponding Hankel singular values σ_i.
The H_∞ norm of the modelling error can be bounded in terms of the Hankel singular values corresponding to the truncated states : $\|G(s) - G_r(s)\|_\infty \leq 2\sum_{i=r+1}^{n} \sigma_i$.

Enns [E84a, E84b, AM89] proposed the following extension for frequency weighted balanced truncation :

Compute the controllability grammian $P \triangleq \begin{bmatrix} P_g & P_{gw} \\ P_{gw}^T & P_w \end{bmatrix}$ of $(sI-A)^{-1}BW_i(s)$ from

$$\begin{bmatrix} A & BC_i \\ 0 & A_i \end{bmatrix}\begin{bmatrix} P_g & P_{gw} \\ P_{gw}^T & P_w \end{bmatrix}+\begin{bmatrix} P_g & P_{gw} \\ P_{gw}^T & P_w \end{bmatrix}\begin{bmatrix} A & BC_i \\ 0 & A_i \end{bmatrix}^T+\begin{bmatrix} BD_iD_i^TB^T & BD_iB_i^T \\ B_iD_i^TB^T & B_iB_i^T \end{bmatrix}=0$$

and the observability grammian $Q \triangleq \begin{bmatrix} Q_g & Q_{gw} \\ Q_{gw}^T & Q_w \end{bmatrix}$ of $W_o(s)C(sI-A)^{-1}$ from

$$\begin{bmatrix} A & 0 \\ B_oC & A_o \end{bmatrix}^T\begin{bmatrix} Q_g & Q_{gw} \\ Q_{gw}^T & Q_w \end{bmatrix}+\begin{bmatrix} Q_g & Q_{gw} \\ Q_{gw}^T & Q_w \end{bmatrix}\begin{bmatrix} A & 0 \\ B_oC & A_o \end{bmatrix}+\begin{bmatrix} C^TD_o^TD_oC & C^TD_o^TC_o \\ C_o^TD_oC & C_o^TC_o \end{bmatrix}=0.$$

Balancing P and Q makes no sense, since they need not even be of the same dimension. However, balancing P_g and Q_g and truncation of the states corresponding to the smallest diagonal elements σ_i of the balanced matrices yields a good reduced model in the frequency weighted norm $\|W_o(s)(.)W_i(s)\|_\infty$.

Enns showed that the reduced model is generically stable when $W_o(s)=I$ or $W_i(s)=I$ (Zhou et al. [ZZL93] proved that *'generically'* may be replaced with *'always'* in the discrete time case). Unfortunately, an error bound as in the unweighted case is not available.

2 Relative error model reduction

Zhou et al. [ZZL93] noted (for the discrete-time case in fact) that Enns's method with the inverse system of $G(s)$ as a weighting function is equivalent to Balanced Stochastic Truncation (BST) if $G(s)$ is minimum phase, and show that the relevant gramians may be computed by solving two n-by-n Lyapunov equations, and an error bound was given that is equivalent to the error bound proposed by Green [G88] for BST. We take the opportunity to point out that this bound can be replaced by a tighter one equivalent to the one proposed by Wang et al. [WS90], i.e. $\|G^{-1}(G-G_r)\|_\infty \leq 2\sum_{i=r+1}^n \sigma_i(\sqrt{1+\sigma_i^2}+\sigma_i)$, with σ_i defined as in [ZZL93].

We propose to generalize Zhou's approach by weighting with the inverse of the outer spectral factor $W(s)$ of the system $G(s)$ at hand, i.e. $W(s)$ is stable and minimum phase s.t. $W(s)W^T(-s)=G(s)G^T(-s)$. We then take $W_o(s)\triangleq(W(s))^{-1}=C_i(sI-A_{wi})^{-1}B_{wi}+D_i$, where $A_{wi}\triangleq A_i+XC_i^TC_i$, $B_{wi}\triangleq B_i+XC_i^TD_i$, $C_{wi}\triangleq$, $D_{wi}\triangleq$, and X is the solution of the algebraic Riccati equation

$$A_iX + XA_i^T + XC_i^TC_iX = 0 \tag{1}$$

such that $A_{wi} = A_i + XC_i^TC_i$ has all its eigenvalues in the left half plane.

Using $W_o(s)$ as a weighting function on the output side, it is easily verified that P_g is the solution of $AP_g + P_gA^T + BB^T = 0$ and $Q_g = -Q_{gw} = Q_w$ where Q_w is the solution of $A_{wi}^TQ_w + Q_wA_{wi} + C_i^TC_i = 0$.

Note that the Riccati equation $A_iX + XA_i^T + XC_i^TC_iX = 0$ may be solved more cheaply than a general algebraic Riccati equation. One may take advantage of the fact that there is a trivial solution $X = 0$; this solution is the stabilizing one if $G(s)$ is minimum phase, and one then finds $Q_g = Q_w = Q_i$ (cf. [ZZL93]). Moreover, if the solution Q_i of $A_i^TQ_i + Q_iA_i + C_i^TC_i = 0$, is invertible (which is generically true), then Q_i^{-1} is one solution to the Riccati equation, for which $A_{wi} = -Q_i^{-1}A_i^TQ_i$, so this is the stabilizing one if $G^{-1}(s)$ is antistable; in the latter case, $Q_g = Q_w = -Q_i$, since $(-Q_iA_iQ_i^{-1})(-Q_i)+(-Q_i)(-Q_i^{-1}A_i^TQ_i)+C_i^TC_i = Q_iA_i + A_i^TQ_i + C_i^TC_i = 0$.

When $G^{-1}(s)$ is neither stable nor antistable, a Schur decomposition of the Hamiltonian of the Riccati equation may be written down directly:

$$\begin{bmatrix} A_i & 0 \\ -C_i^TC_i & -A_i^T \end{bmatrix} \begin{bmatrix} 0 & U \\ UJ & 0 \end{bmatrix} = \begin{bmatrix} 0 & U \\ UJ & 0 \end{bmatrix} \begin{bmatrix} -JS^TJ & -JU^HC^TCU \\ 0 & S \end{bmatrix}$$

after computing a Schur decomposition of $A_i = USU^H$, with all right half plane eigenvalues in the upper left block (J is the permutation matrix of the same size as A_i with ones on the antidiagonal and zeroes elsewhere). Using an orthogonal transformation $V \triangleq \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$, one can reorder the $2n \times 2n$ Schur matrix such that all right half plane eigenvalues are in the upper left block in maximally $O(12np(n+p))$ flops, including the flops needed for reordering the Schur matrix S, where p is the number of nonminimum phase zeroes of $G(s)$. This compares favorably to the approximately $O(200n^3)$ flops normally needed for a Schur decomposition of size $2n \times 2n$. Therefore

$$\begin{bmatrix} A_i & 0 \\ -C_i^TC_i & -A_i^T \end{bmatrix} \begin{bmatrix} 0 & U \\ UJ & 0 \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} = \begin{bmatrix} 0 & U \\ UJ & 0 \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} S_+ & S_{12} \\ 0 & S_- \end{bmatrix}$$

where S_+ (S_-)is upper triangular with only RHP (resp. LHP) eigenvalues. The stabilizing solution to the Riccati equation (1) is then $X = UV_{21}V_{11}^{-1}JU^H$.

3 Model reduction procedure

1. Solve $AP_g + P_gA^T + BB^T = 0$ for P_g
2. Compute $X = UV_{21}V_{11}^{-1}JU^H$ as shown above.
3. Solve $A_{wi}^TQ_g + Q_gA_{wi} + C_i^TC_i = 0$ for Q_g, where $C_i \triangleq -D^{-1}C$ and $A_{wi} \triangleq A + BC_i + XC_i^TC_i$.

   We use Safonov's Schur method [SC88] to do the balanced truncation to a system of order r:

4. Compute matrices V_r and V_l, whose r columns form bases for the right and left eigenspaces of P_gQ_g associated with the r biggest eigenvalues.
5. Compute a singular value decomposition $U_e\Sigma_eV_e^T = V_l^TV_r$
6. Defining $L \triangleq \Sigma_e^{-\frac{1}{2}}U_e^TV_l^T$ and $T \triangleq V_rV_e\Sigma_e^{-\frac{1}{2}}$, $A_r \triangleq LAT$, $B_r \triangleq LB$, $C_r \triangleq CT$, the reduced order model is $G_r(s) \triangleq [A_r, B_r, C_r, D]$.

There is of course also a dual input-side weighted equivalent, where Q_g is computed from $A^TQ_g + Q_gA + C^TC = 0$, X from $A_i^TX + XA_i + XB_iB_i^TX = 0$, and P_g from $A_{wi}P_g + P_gA_{wi}^T + B_iB_i^T = 0$, where this time $B_i \triangleq BD^{-1}$ and $A_{wi} \triangleq A - B_iC + B_iB_i^TX$.

A bound on the error has not been proved so far, but we formulate the following

Conjecture 1 *If σ_i^2 are the eigenvalues of P_gQ_g, with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$, and if the number of nonminimum phase zeroes p is not greater than $n - r$, then*

$$\|G(s)^{-1}(G(s) - G_r(s))\|_\infty \leq 2\sum_{i=r+1}^{n} \sigma_i(\sqrt{1+\sigma_i^2} + \sigma_i)$$

4 Example

Take the following example :

$$G(s) \triangleq \frac{4z^8 + 6z^7 + 2z^6 - 6z^5 + 4z^4 - 5z^3 - 4z^2 - 1z - 2}{3z^8 + 15z^7 + 37z^6 + 65z^5 + 78z^4 + 59z^3 + 27z^2 + 8z + 1}$$

This system is stable and has 3 nonminimum phase zeroes. It is reduced to order $r = 4$ using both BST and our method. The magnitude Bode plots for the original system (full

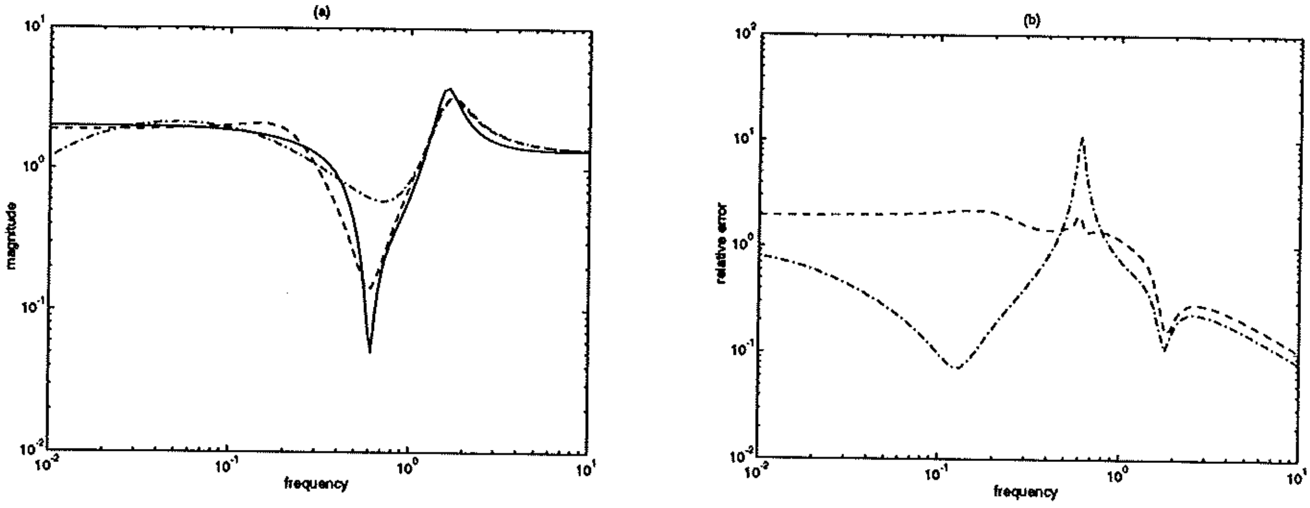

Figure 1: Full line : original system, dashed line : our method, dash-dotted line : BST .
(a) magnitude Bode plots (b) error magnitude

line), BST (dash-dotted line) and our method (dashed line) are given in Fig. 1(a). The relative error magnitude vs. frequency is for BST (dash-dotted line) and our method (dashed line) are given in Fig. 1(b).

Our conjectured bound yields $\|G(s)^{-1}(G(s) - G_r(s))\|_\infty \leq 4.15$. The actual error is $\|G(s)^{-1}(G(s)-G_r(s))\|_\infty = 2.17$, whereas the error for BST is $\|G(s)^{-1}(G(s)-G_{BST}(s))\|_\infty = 11.87$.

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