Pergamon

0005-1098(95)00071-2

Automatica, Vol. 31, No. 12, pp. 1877–1883, 1995

0005-1098/95 $9.50 + 0.00

Choice of State-space Basis in Combined Deterministic–Stochastic Subspace Identification*

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Choice of basis in subspace identification.

Key Words—System identification; subspace methods; model reduction; state-space models; multivariable systems; state-space methods; linear algebra.

Abstract—This paper describes how the state-space basis of models identified with subspace identification algorithms can be determined. It is shown that this basis is determined by the input spectrum and by user-defined input and output weightings. Through the connections between subspace identification and frequency-weighted balancing, the state-space basis of the subspace-identified models is shown to coincide with a frequency-weighted balanced basis.

1. INTRODUCTION

The identification problem considered in the combined deterministic–stochastic subspace identification papers by Larimore (1990), Van Overschee and De Moor (1994), Verhaegen (1994) and Viberg *et al.* (1993) is the following: let $u_k \in \mathbb{R}^m$, $y_k \in \mathbb{R}^l$ be the observed input and output generated by the unknown system

$$x_{k+1} = Ax_k + Bu_k + w_k \quad y_k = Cx_k + Du_k + v_k, \tag{1}$$

with

$$\mathbf{E}\left[\begin{pmatrix} w_k \\ v_k \end{pmatrix}(w_l^{\mathrm{T}} \quad v_l^{\mathrm{T}})\right] = \begin{pmatrix} Q & S \\ S^{\mathrm{T}} & R \end{pmatrix}\delta_{kl} \geq 0, \tag{2}$$

and $A, Q \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times n}$, $D \in \mathbb{R}^{l\times m}$, $S \in \mathbb{R}^{n\times l}$ and $R \in \mathbb{R}^{l\times l}$ (here $\mathbf{E}$ denotes the expected-value operator and δ_{kl} the Kronecker delta). $v_k \in \mathbb{R}^l$ and $w_k \in \mathbb{R}^n$ are unobserved, zero-mean, white noise vector sequences. $\{A, C\}$ and $\{A, [B \; Q^{1/2}]\}$ are assumed to be observable and controllable respectively. The main identification problem is then stated as follows. Given N input–output measurements generated by the system (1), (2), find A, B, C, D, Q, R and S up to a similarity transformation. Several solutions of this problem have been described by Larimore (1990), Van Overschee and De Moor (1994), Verhaegen (1994) and Viberg *et al.* (1993). Although the solutions look very different at first sight, it was shown by Van Overschee and De Moor (1995) that these algorithms use the same basic subspace, but weighted in a different way. In this paper the effect of these weights will be further explored.

It will be shown that the state-space basis of the identified model corresponds to an input–output frequency-weighted balanced basis as described by Enns (1984). The weights in the frequency domain are a function of the input (u_k) applied to the system and of the weighting matrices W_1 and W_2 introduced by Van Overschee and De Moor (1995). The main theorem introduces specific choices for these weighting matrices, such that the system is identified in a predefined state space basis. As a special case, we will investigate the state-space basis of the N4SID algorithm of Van Overschee and De Moor (1994).

A nice side result is the lower-order identification problem. Since the basis in which the state-space matrices are identified is well defined and is frequency-weighted balanced, it is very easy to truncate the model after identification to a lower-order model. This corresponds exactly to the technique of frequency-weighted model reduction of Enns (1984).

The paper is organized as follows. In Section 2 we introduce some notation and background. Section 3 briefly introduces the concepts of frequency-weighted balancing. In Section 4 the main theorem is presented. Section 5 addresses the problem of reduced-order identification. Finally, Section 6 contains the conclusions.

2. NOTATION AND BACKGROUND

In this section we introduce the notation used throughout the paper: input and output block Hankel matrices, system related matrices and

*Received 18 July 1994; revised 8 December 1994; received in final form 29 March 1995. This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Bo Wahlberg under the direction of editor Torsten Söderström. Corresponding author Dr Peter Van Overschee. Tel. +32 16 32 10 95; Fax +32 16 32 19 86; E-mail peter.vanoverschee@esat.kuleuven.ac.be.

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weighting matrices. We also briefly revise some results on subspace identification.

Throughout this paper, we shall consider the model (1), (2) in its forward innovations form as (Pal, 1982)

$$x_{k+1} = Ax_k + Bu_k + Ee_k, \quad y_k = Cx_k + Du_k + Fe_k,$$
$$\mathbf{E}[e_k e_l^{\mathrm{T}}] = I\delta_{kl}, \qquad (3)$$

with $E \in \mathbb{R}^{n\times l}$, $F \in \mathbb{R}^{l\times l}$, and the innovations $e_k \in \mathbb{R}^l$ uncorrelated with u_k. The transformation from the system (1), (2) to this innovations model can be done through the solution of a Riccati equation (Pal, 1982). The steady-state Kalman gain is given by $K = EF^{-1} \in \mathbb{R}^{n\times l}$. Subspace identification algorithms make extensive use of input and output block Hankel matrices. We define

$$U_{\mathrm{p}} \stackrel{\mathrm{def}}{=} \begin{pmatrix} u_0 & u_1 & \cdots & u_{j-1} \\ u_1 & u_2 & \cdots & u_j \\ \vdots & \vdots & & \vdots \\ u_{i-1} & u_i & \cdots & u_{i+j-2} \end{pmatrix},$$

$$U_{\mathrm{f}} \stackrel{\mathrm{def}}{=} \begin{pmatrix} u_i & u_{i+1} & \cdots & u_{i+j-1} \\ u_{i+1} & u_{i+2} & \cdots & u_{i+j} \\ \vdots & \vdots & & \vdots \\ u_{2i-1} & u_{2i} & \cdots & u_{2i+j-2} \end{pmatrix}.$$

Somewhat loosely, we denote U_{p} as the *past* inputs and U_{f} as the *future* inputs. Through a similar definition, Y_{p} and Y_{f} are defined as the past and future outputs respectively, and E_{p} and E_{f} as the past and future innovations respectively. We assume that $j \to \infty$ throughout the paper, and that all sequences are ergodic. The time-averaging operator $\mathbf{E}_j$ is defined as:

$$\mathbf{E}_j[\ \ ] \stackrel{\mathrm{def}}{=} \lim_{j\to\infty} \frac{1}{j}[\ \ ].$$

The covariance matrix of the past inputs R_{p} will play an important role in several derivations: $R_{\mathrm{p}} \stackrel{\mathrm{def}}{=} \mathbf{E}_j[U_{\mathrm{p}}U_{\mathrm{p}}^{\mathrm{T}}] \stackrel{\mathrm{def}}{=} L_{\mathrm{p}} \cdot L_{\mathrm{p}}^{\mathrm{T}}$, where L_{p} is a lower-triangular square root of R_{p} obtained, for example, via a Cholesky decomposition of R_{p}. The extended $(i > n)$ observability matrix Γ_i is defined as

$$\Gamma_i \stackrel{\mathrm{def}}{=} (C^{\mathrm{T}} \quad (CA)^{\mathrm{T}} \quad \ldots \quad (CA^{i-1})^{\mathrm{T}})^{\mathrm{T}}.$$

The extended $(i > n)$ reversed deterministic Δ_i^{d} and stochastic Δ_i^{s} controllability matrices are defined as

$$\Delta_i^{\mathrm{d}} \stackrel{\mathrm{def}}{=} (A^{i-1}B \quad \ldots \quad AB \quad B),$$

$$\Delta_i^{\mathrm{s}} \stackrel{\mathrm{def}}{=} (A^{i-1} \quad \ldots \quad AE \quad E).$$

Furthermore, we define the block Toeplitz matrices containing the Markov parameters of the deterministic and the stochastic system as

$$H_i^{\mathrm{d}} \stackrel{\mathrm{def}}{=} \begin{pmatrix} D & 0 & \cdots & 0 \\ CB & D & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ CA^{i-2}B & CA^{i-3}B & \cdots & D \end{pmatrix},$$

$$H_i^{\mathrm{s}} \stackrel{\mathrm{def}}{=} \begin{pmatrix} F & 0 & \cdots & 0 \\ CE & F & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ CA^{i-2}E & CA^{i-3}E & \cdots & F \end{pmatrix}.$$

The (non-steady-state) Kalman filter state sequence $\tilde{X}_i$ is defined as in Van Overschee and De Moor (1994):

$$\tilde{X}_i \stackrel{\mathrm{def}}{=} (\tilde{x}_i \quad \tilde{x}_{i+1} \quad \tilde{x}_{i+2} \quad \ldots \quad \tilde{x}_{i+j-1}).$$

Each of its columns is the output of a non-steady-state Kalman filter (for more details, see Van Overschee and De Moor, 1994, 1995). The $\mathscr{Z}$ transforms (initial state equal to zero) of u_k and y_k are denoted by $U(z)$ and $Y(z)$ respectively, while the spectral factor of e_k is denoted by $E(z)$. From (3), we then find $Y(z) = G(z)U(z) + H(z)E(z)$, with $G(z) = D + C(zI - A)^{-1}B$ and $H(z) = F + C(zI - A)^{-1}E$. The spectral factor of u_k is denoted by $S_u(z): U(z)U^{\mathrm{T}}(z^{-1}) = S_u(z)S_u^{\mathrm{T}}(z^{-1})$, with all poles of $S_u(z)$ and $S_u^{-1}(z)$ inside the unit circle. We define the input and output weighting matrix functions as

$$W_u(z) \stackrel{\mathrm{def}}{=} D_u + C_u(zI - A_u)^{-1}B_u,$$
$$W_y(z) \stackrel{\mathrm{def}}{=} D_y + C_y(zI - A_u)^{-1}B_y.$$

From the Markov parameters of these functions, the weighting matrices W_i^u and W_i^y can be formed:

$$W_i^u \stackrel{\mathrm{def}}{=} \begin{pmatrix} D_u & 0 & \cdots & 0 \\ C_uB_u & D_u & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ C_uA_u^{i-2}B_u & C_uA_u^{i-3}B_u & \cdots & D_u \end{pmatrix},$$

$$W_i^y \stackrel{\mathrm{def}}{=} \begin{pmatrix} D_y & 0 & \cdots & 0 \\ C_yB_y & D_y & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ C_yA_y^{i-2}B_y & C_yA_y^{i-3}B_y & \cdots & D_y \end{pmatrix}.$$

Π_A denotes the operator that projects the row space of a matrix onto the row space of A (which is assumed to be of full row rank): $\Pi_A \stackrel{\mathrm{def}}{=} A^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}A$. The projection of the row space of B onto the row space of A is defined as $B/A \stackrel{\mathrm{def}}{=} B \cdot \Pi_A = BA^{\mathrm{T}}(AA^{\mathrm{T}})^{-1}A$. The orthogonal complement of the row space of A is denoted by $A^{\perp}$.

The unifying theorem of Van Overschee and De Moor (1995) describes how the extended observability matrix Γ_i and the states $\tilde{X}_i$ can be recovered directly from the input–output data u_k and y_k. The theorem introduces two weighting matrices $W_1 \in \mathbb{R}^{li \times li}$ and $W_2 \in \mathbb{R}^{j \times j}$ that will play an important role in this paper. The main results of the theorem (Van Overschee and De Moor, 1995) are that the system order n and the matrices Γ_i and $\tilde{X}_i$ can be determined from an infinite number of input–output data through the non-zero singular values and the left and right singular vectors of the matrix:

$$W_1 \cdot \mathbf{E}_j\left[(Y_f/U_f^{\perp}) \cdot \left(\begin{pmatrix} U_p \\ Y_p \end{pmatrix} \Big/ U_f^{\perp}\right)^{\mathrm{T}}\right]$$

$$\left\{\mathbf{E}_j\left[\left(\begin{pmatrix} U_p \\ Y_p \end{pmatrix} \Big/ U_f^{\perp}\right) \cdot \left(\begin{pmatrix} U_p \\ Y_p \end{pmatrix} \Big/ U_f^{\perp}\right)^{\mathrm{T}}\right]\right\}^{-1} \begin{pmatrix} U_p \\ Y_p \end{pmatrix} \cdot W_2,$$

which has a singular-value decomposition given by

$$U_1 \cdot S_1 \cdot V_1^{\mathrm{T}}. \quad (4)$$

Γ_i and $\tilde{X}_i$ then follow from

$$W_1 \cdot \Gamma_i = U_1 S_1^{1/2}, \quad \tilde{X}_i \cdot W_2 = S_1^{1/2} V_1^{\mathrm{T}}. \quad (5)$$

We refer to Van Overschee and De Moor (1995) for more details. It is known from linear systems theory that Γ_i and $\tilde{X}_i$ are only determined up to within a non-singular similarity transformation $T \in \mathbb{R}^{n \times n}: \Gamma_i \leftarrow \Gamma_i T$ and $\tilde{X}_i \leftarrow T^{-1}\tilde{X}_i$. This implies that the following question makes sense: *In which state space basis are* Γ_i *and* $\tilde{X}_i$ *determined when a subspace method is used to estimate them?* In what follows, we shall show that this basis is a function of the weights W_1 and W_2, and that, by a proper choice of these weights, the basis can be altered in a user-controlled manner. Furthermore, it will be shown that the singular values S_1, (4), used to determine the system order have a clear interpretation from a linear system theoretical point of view.

3. FREQUENCY-WEIGHTED BALANCING

In this section we recapitulate the results of Enns (1984) for frequency-weighted balancing. We also show how the frequency-weighted Gramians introduced by Enns can be calculated from the extended observability and controllability matrices and from the weighting matrices. The notion of 'balanced realization' is well known in system theory (Moore, 1981). Enns has developed a frequency-weighted extension of this result. The idea is that input and output frequency weights can be introduced as to emphasize certain frequency bands in the balancing procedure (Fig. 1). Instead of using the regular controllability and observability Gramians, Enns uses frequency-weighted Gramians.

Definition 1. *Frequency-weighted Gramians.* The solution P_{11} of the Lyapunov equation

$$\begin{pmatrix} A & BC_u \\ 0 & A_u \end{pmatrix}\begin{pmatrix} P_{11} & P_{21}^{\mathrm{T}} \\ P_{21} & P_{22} \end{pmatrix}\begin{pmatrix} A & BC_u \\ 0 & A_u \end{pmatrix}^{\mathrm{T}} + \begin{pmatrix} BD_u & E \\ B_u & 0 \end{pmatrix}\begin{pmatrix} BD_u & E \\ B_u & 0 \end{pmatrix}^{\mathrm{T}} = \begin{pmatrix} P_{11} & P_{21}^{\mathrm{T}} \\ P_{21} & P_{22} \end{pmatrix} \quad (6)$$

is called the $W_u(z)$ *weighted observability Gramian*, and is denoted by $P[W_u(z)] \stackrel{\mathrm{def}}{=} P_{11}$. The solution Q_{11} of the Lyapunov equation

$$\begin{pmatrix} A & 0 \\ B_y C & A_y \end{pmatrix}^{\mathrm{T}}\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^{\mathrm{T}} & Q_{22} \end{pmatrix}\begin{pmatrix} A & 0 \\ B_y C & A_y \end{pmatrix} + (D_y C \quad C_y)^{\mathrm{T}}(D_y C \quad C_y) = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^{\mathrm{T}} & Q_{22} \end{pmatrix} \quad (7)$$

is called the $W_y(z)$ *weighted observability Gramian*, and is denoted by $Q[W_y(z)] \stackrel{\mathrm{def}}{=} Q_{11}$. Just as for the classical balancing procedure, a similarity transformation can be found that makes both Gramians diagonal and equal to each other. In that case the system is said to be frequency-weighted balanced.

Definition 2. *Frequency-weighted balancing.* The system (3) is called $[W_u(z), W_y(z)]$ *frequency-weighted balanced* when $P[W_u(z)] = Q[W_y(z)] = \Sigma$, where $\Sigma = \mathrm{diag}[\sigma_1, \sigma_2, \ldots, \sigma_n]$. The diagonal

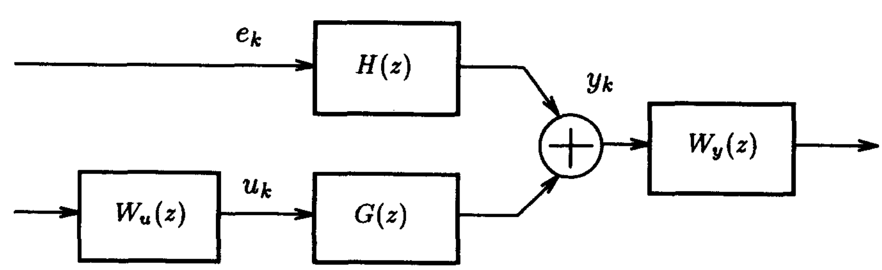

Fig. 1. Cascade system used for the interpretation of fequency-weighted balancing. The weights $W_u(z)$ and $W_y(z)$ are user-defined. Note that the noise input (the input to $H(z)$) has no extra weight, since, from an input–output view, this weight is indistinguishable from $H(z)$.

elements σ_k are called the *frequency-weighted Hankel singular values* and will be denoted by $\sigma_k[W_u(z), W_y(z)]$.

Even though (6) and (7) are easily solvable for $P[W_u(z)]$ and $Q[W_y(z)]$, we present a different way to compute these weighted Gramians. These expressions will enable us to make the connection between subspace identification and frequency weighted balancing.

Lemma 1. With A asymptotically stable, we have

$$P[W_u(z)] = \lim_{i\to\infty} [\Delta_i^d \cdot W_i^u \cdot (W_i^u)^T \cdot (\Delta_i^d)^T + \Delta_i^s \cdot (\Delta_i^s)^T], \quad (8)$$

$$Q[W_y(z)] = \lim_{i\to\infty} [\Gamma_i^T \cdot (W_i^y)^T \cdot (W_i^y) \cdot \Gamma_i]. \quad (9)$$

A proof can be found in the Appendix.

4. SUBSPACE IDENTIFICATION AND FREQUENCY-WEIGHTED BALANCING

In this section we consider the connection between frequency-weighted balancing and subspace identification. We show how the weights W_1 and W_2 influence the Gramians $P[W_u(z)]$ and $Q[W_y(z)]$ corresponding to the state-space basis of Γ_i and $\tilde{X}_i$.

4.1. *Main result*

Theorem 1. *Main theorem.* With A asymptotically stable and $i \to \infty$ (note that first $j \to \infty$ through the operator $\mathbf{E}_j$, after which $i \to \infty$), we have, with

$$W_1 = W_i^y, \quad W_2 = U_p^T R_p^{-1} W_i^u L_p^{-1} U_p + \Pi_{U_p^\perp}, \quad (10)$$

that the $W_u(z)$ weighted controllability Gramian and $W_y(z)$ weighted observability Gramian of the state-space basis corresponding to Γ_i and $\tilde{X}_i$ are given by (with S_1 from (4))

$$P[W_u(z)] = S_1 = Q[W_y(z)]. \quad (11)$$

The proof can be found in the Appendix. The theorem implies that the state-space basis of Γ_i and $\tilde{X}_i$ for the choice of W_1 and W_2 given by (10) is the $[W_u(z), W_y(z)]$ frequency-weighted balanced basis. It also implies that the singular values S_1 are the $[W_u(z), W_y(z)]$ frequency-weighted Hankel singular values.

4.2. *Special cases*

Even though the weighting matrices W_i^u and W_i^y can be chosen arbitrarily, there are some special cases that lead to algorithms published in the literature.

N4SID. This stands for 'numerical algorithms for subspace state-space system identification' (Viberg *et al.* 1993; Van Overschee and De Moor, 1994). With the results of Van Overschee and De Moor (1995), it is easy to show that N4SID delivers the following choice of weighting matrices in Theorem 1: $W_i^u = L_p$ and $W_i^y = I$.

It is easy to verify that (for $i \to \infty$) the lower-triangular matrix L_p corresponds to the Toeplitz matrix generated by the Markov parameters of the spectral factor $S_u(z)$ of u_k. This implies that (for N4SID) the input weight $W_u(z)$ in the frequency-weighted balancing procedure corresponds to the special factor $S_u(z)$.

Balanced realization. With the weighting matrices $W_i^u = I$ and $W_i^y = I$, we find $P[I(z)] = S_1 = Q[I(z)]$. Now it is easy to verify that $P[I(z)]$ and $Q[I(z)]$ are equal to the unweighted controllability respectively observability Gramian. This implies that the basis of Γ_i and $\tilde{X}_i$ is the classical balanced basis as described by Moore (1981). A similar result for pure deterministic systems was obtained by Moonen and Ramos (1992).

5. CONSEQUENCES FOR REDUCED-ORDER IDENTIFICATION

In this section we apply the results of the main theorem to the identification of lower-order systems. The connections with frequency-weighted model reduction are exploited.

As has been proved in this paper, subspace identification of a model of order n (the exact state-space order) leads to a state-space system that is $[W_u(z), W_y(z)]$ frequency-weighted balanced. This nth-order model can then be easily reduced to a model of lower order r by truncating it as follows:

$$A = \begin{matrix} & r & n-r \\ r & \left(A_{11}\right. & \left.A_{12}\right) \\ n-r & \left(A_{21}\right. & \left.A_{22}\right) \end{matrix}, \quad B = \begin{matrix} & m \\ r & \left(B_1\right) \\ n-r & \left(B_2\right) \end{matrix},$$

$$C = l\begin{matrix} r & n-r \\ (C_1 & C_2) \end{matrix}, \quad E = \begin{matrix} & l \\ r & \left(E_1\right) \\ n-r & \left(E_2\right) \end{matrix}.$$

The reduced-order model is described by the matrices A_{11}, B_1, C_1, D, E_1 and F. The reduced transfer functions are denoted by $\hat{G}(z) = D + C_1(zI - A_{11})^{-1}B_1$ and $\hat{H}(z) = F + C_1(zI - A_{11})^{-1}E_1$. Enns (1984) made the following conjecture: when truncating a $[W_u(z), W_y(z)]$ frequency-weighted system, the infinity norm of the difference between the original and the reduced system can be upper-bounded by the neglected weighted Hankel singular values. In the framework of this paper (see also Fig. 1), this conjecture becomes

$$\| W_y(z)[G(z) - \hat{G}(z)]W_u(z) \mid W_y(z)[H(z) - \hat{H}(z)]\|_\infty \leq 2 \sum_{k=r+1}^{n} \sigma_k[W_u(z), W_y(z)](1+\alpha), \quad \alpha > 0. \quad (12)$$

The conjecture consists of the fact that α is 'small'. Let us ponder a bit about this conjecture. We have tried to find a simple expression for α, but (just like Enns) did not succeed (as far as we know, no one else has either, even though the problem has been open since 1984). Even though the result is ambiguous (α has never been proved to be bounded, let alone to be small), the 'heuristic' model reduction technique seems to work very well in practice (Anderson and Moore, 1989; Wortelboer and Bosgra, 1992). It turns out that, even though it is not a real upper bound, twice the sum of the neglected singular values ($\alpha = 0$) gives a good indication about the size of the error. More importantly, (12) states that the fit of the truncated lower-order model will be good where $W_u(z)$ and $W_y(z)$ are large. This implies that by a proper choice of $W_u(z)$ and $W_y(z)$ the distribution of the error in the frequency domain can be shaped. We find for the special cases the following.

For N4SID,

$$\|[G(z) - \hat{G}(z)]S_u(z) \mid [H(z) - \hat{H}(z)]\|_\infty \leq 2 \sum_{k=r+1}^{n} \sigma_k(1 + \alpha).$$

We can conclude that the error of the model will be small where the frequency content of the input is large. This is a very intuitive result: a lot of input energy in a certain frequency band leads to an accurate model in that band. Note also that the error on the noise model can not be shaped by the user.

For a balanced basis,

$$\|[G(z) - \hat{G}(z)] \mid [H(z) - \hat{H}(z)]\|_\infty \leq 2 \sum_{k=r+1}^{n} \sigma_k.$$

Note that in this case setting $\alpha = 0$ is justified, since for (unweighted) balanced model reduction twice the sum of the Hankel singular values is actually a proved upper bound for the truncation error (see Al-Saggaf and Franklin, 1987).

6. CONCLUSIONS

We have shown that the state-space basis of the subpsace-identified models corresponds to a frequency-weighted balanced basis. The frequency weights are determined by the input spectrum and by user-defined input and output weighting functions.

Acknowledgements—Peter Van Overschee is a research assistant and Bart De Moor a senior research associate of the Belgian National Fund for Scientific Research. This paper presents research results of the Belgian Programme on Interuniversity Poles of Attraction (IUAP 17 and 50), initiated by the Belgian State, Prime Minister's office for Science, Technology and Culture. The scientific responsibility rests with its authors.

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APPENDIX—PROOFS

Proof of Lemma 1

We first prove (8). We consider the different sub-blocks of the weighted controllability Lyapunov equation (6):

$$P_{11} = AP_{11}A^{\mathrm{T}} + EE^{\mathrm{T}} + AP_{21}^{\mathrm{T}}C_u^{\mathrm{T}}B^{\mathrm{T}} + BC_uP_{21}A^{\mathrm{T}} + B(C_uP_{22}C_u^{\mathrm{T}} + D_uD_u^{\mathrm{T}})B^{\mathrm{T}}, \quad (A.1)$$

$$P_{21} = A_uP_{21}A^{\mathrm{T}} + (A_uP_{22}C_u^{\mathrm{T}} + B_uD_u^{\mathrm{T}})B^{\mathrm{T}}, \quad (A.2)$$

$$P_{22} = A_uP_{22}A_u^{\mathrm{T}} + B_uB_u^{\mathrm{T}}. \quad (A.3)$$

We first prove that, with $\Delta_i^u = (A_u^{i-1}B_u \ \ldots \ A_uB_u \ B_u)$, we have

$$P_{22} = \lim_{i\to\infty} [\Delta_i^u \cdot (\Delta_i^u)^{\mathrm{T}}], \quad (A.4)$$

$$P_{21} = \lim_{i\to\infty} [\Delta_i^u \cdot (W_i^u)^{\mathrm{T}} \cdot (\Delta_i^{\mathrm{d}})^{\mathrm{T}}]. \quad (A.5)$$

Proof of (A.4).

$$\Delta_i^u \cdot (\Delta_i^u)^{\mathrm{T}} = (A_u \cdot \Delta_{i-1}^u \mid B_u)\begin{pmatrix} (\Delta_{i-1}^u)^{\mathrm{T}}A_u^{\mathrm{T}} \\ \hline B_u^{\mathrm{T}} \end{pmatrix} = A_u[\Delta_{i-1}^u \cdot (\Delta_{i-1}^u)^{\mathrm{T}}]A_u^{\mathrm{T}} + B_u \cdot B_u^{\mathrm{T}}. \quad (A.6)$$

For stable A_u ($\lim_{i\to\infty} A_u^{i-1} = 0$), we also have

$$\lim_{i\to\infty}[\Delta_i^u \cdot (\Delta_i^u)^T]$$
$$= \lim_{i\to\infty}[\Delta_{i-1}^u \cdot (\Delta_{i-1}^u)^T + \underbrace{A_u^{i-1}}_{\to 0} B_u B_u^T \underbrace{(A_u^{i-1})^T}_{\to 0}]$$
$$= \lim_{i\to\infty}[\Delta_{i-1}^u \cdot (\Delta_{i-1}^u)^T]. \quad (A.7)$$

Taking the limit as $i \to \infty$ on both sides of (A.6), we find with (A.7) that $\lim_{i\to\infty}[(\Delta_i^u(\Delta_i^u)^T]$ is the solution of the same Lyapunov equation (A.3) as P_{22}, and thus prove (A.4). □

Proof of (A.5).

$$\Delta_i^u \cdot (W_i^u)^T \cdot (\Delta_i^d)^T$$
$$= (A_u\Delta_{i-1}^u \mid B_u)\begin{pmatrix} (W_{i-1}^u)^T & (\Delta_{i-1}^u)^T C_u^T \\ 0 & D_u^T \end{pmatrix} \times \begin{pmatrix} (\Delta_{i-1}^d)^T A^T \\ B^T \end{pmatrix}$$
$$= A_u \cdot [\Delta_{i-1}^u (W_{i-1}^u)^T \cdot (\Delta_{i-1}^d)^T] \cdot A^T + \{A_u[\Delta_{i-1}^u \cdot (\Delta_{i-1}^u)^T]C_u^T + B_u D_u^T\}B^T. \quad (A.8)$$

For stable A_u and A, we have

$$\lim_{i\to\infty}[\Delta_i^u \cdot (W_i^u)^T \cdot (\Delta_i^d)^T] = \lim_{i\to\infty}[\Delta_{i-1}^u \cdot (W_{i-1}^u)^T \cdot (\Delta_{i-1}^d)^T].$$

Taking the limit as $i \to \infty$ on both sides of (A.8), we thus find, with (A.4), that

$$\lim_{i\to\infty}[\Delta_i^u \cdot (W_i^u)^T \cdot (\Delta_i^d)^T] = A_u \cdot \lim_{i\to\infty}[\Delta_i^u \cdot (W_i^u)^T \cdot (\Delta_i^d)^T] \cdot A^T + (A_u P_{22} C_u^T + B_u D_u^T)B^T,$$

which proves that $\lim_{i\to\infty}[\Delta_i^u \cdot (W_i^u)^T \cdot (\Delta_i^d)^T]$ is the solution of the same equation as P_{21}, (A.2), and thus proves (A.5). □

Proof of (8).

$$\lim_{i\to\infty}[\Delta_i^d \cdot W_i^u \cdot (W_i^u)^T \cdot (\Delta_i^d)^T + \Delta_i^s \cdot (\Delta_i^s)^T]$$
$$= \lim_{i\to\infty}\Bigg[(A\Delta_{i-1}^d \mid B)\begin{pmatrix} W_{i-1}^u & 0 \\ C_u\Delta_{i-1}^u & D_u \end{pmatrix} \times \begin{pmatrix} (W_{i-1}^u)^T & (\Delta_{i-1}^u)^T C_u^T \\ 0 & D_u^T \end{pmatrix}\begin{pmatrix} (\Delta_{i-1}^d)^T A^T \\ B^T \end{pmatrix} + (A\Delta_{i-1}^s \mid E)\begin{pmatrix} (\Delta_{i-1}^s)^T A^T \\ E^T \end{pmatrix}\Bigg]$$
$$= A \cdot \lim_{i\to\infty}[\Delta_{i-1}^d \cdot W_{i-1}^u \cdot (W_{i-1}^u)^T \cdot (\Delta_{i-1}^d)^T + \Delta_{i-1}^s \cdot (\Delta_{i-1}^s)^T]A^T + EE^T$$
$$+ A \cdot \lim_{i\to\infty}[\Delta_{i-1}^d \cdot W_{i-1}^u \cdot (\Delta_{i-1}^u)^T]C_u^T B^T$$
$$+ BC_u \cdot \lim_{i\to\infty}[\Delta_{i-1}^u \cdot (W_{i-1}^u)^T \cdot (\Delta_{i-1}^d)^T]A^T$$
$$+ B\Big\{C_u \cdot \lim_{i\to\infty}[\Delta_{i-1}^u \cdot (\Delta_{i-1}^u)^T]C_u^T + D_u D_u^T\Big\}B^T$$
$$= A \cdot \lim_{i\to\infty}[\Delta_{i-1}^d \cdot W_{i-1}^u \cdot (W_{i-1}^u)^T \cdot (\Delta_{i-1}^d)^T + \Delta_{i-1}^s \cdot (\Delta_{i-1}^s)^T]A^T + EE^T + AP_{21}^T C_u^T B^T + BC_u P_{21} A^T + B(C_u P_{22} C_u^T + D_u D_u^T)B^T. \quad (A.9)$$

Through similar reasoning as before, it is easy to prove that, with A and A_u stable,

$$\lim_{i\to\infty}[\Delta_i^d \cdot W_i^u \cdot (W_i^u)^T \cdot (\Delta_i^d)^T + \Delta_i^s \cdot (\Delta_i^s)^T] = \lim_{i\to\infty}[\Delta_{i-1}^d \cdot W_{i-1}^u \cdot (W_{i-1}^u)^T \cdot (\Delta_{i-1}^d)^T + \Delta_{i-1}^s \cdot (\Delta_{i-1}^s)^T],$$

which proves with (A.9) that

$$\lim_{i\to\infty}[\Delta_i^d \cdot W_i^u \cdot (W_i^u)^T \cdot (\Delta_i^d)^T + \Delta_i^s \cdot (\Delta_i^s)^T]$$

is the solution of the same equation as P_{11}, (A.1), and thus proves (8). The proof of (9) is analogous to the proof of (8). □

Proof of Theorem 1

From Van Overschee and De Moor (1994), we have

$$\tilde{X}_i = ((A^i - Q_i\Gamma_i)S(R^{-1})_{1|mi} + \Delta_i^d - Q_iH_i^d \mid Q_i) \times \begin{pmatrix} U_p \\ Y_p \end{pmatrix}. \quad (A.10)$$

For the meaning of the symbols, we refer to Van Overschee and De Moor (1994). The only properties we use here are (from Van Overschee and De Moor (1994), for stable A)

$$\lim_{i\to\infty}(A^i - Q_i\Gamma_i) = 0 \quad \lim_{i\to\infty}(Q_i\Gamma_i) = 0,$$
$$\lim_{i\to\infty}[Q_iH_i^s \cdot (H_i^s)^T Q_i^T] = \lim_{i\to\infty}[\Delta_i^s \cdot (\Delta_i^s)^T]. \quad (A.11)$$

Using $Y_p = \Gamma_i X_p + H_i^d U_p + H_i^s E_p$, where X_p are the past states of the forward innovation model (3) (see Van Overschee and De Moor, 1994), we can rewrite (A.10) as

$$\tilde{X}_i = M_i + \Delta_i^d U_p + Q_i H_i^s E_p, \quad (A.12)$$

with

$$M_i \overset{\text{def}}{=} (A^i - Q_i\Gamma_i)S(R^{-1})_{1|mi}U_p + Q_i\Gamma_i X_p.$$

Since the input u_k and the noise e_k are uncorrelated, we also know that (see Van Overschee and De moor, 1994)

$$\mathbf{E}_j[U_p \cdot W_2 \cdot W_2^T \cdot E_p^T] = 0,$$

$$\mathbf{E}_j[M_i \cdot W_2 \cdot W_2^T \cdot E_p^T] = 0,$$

$$\mathbf{E}_j[E_p \cdot W_2 \cdot W_2^T \cdot E_p^T] = \mathbf{E}_j[E_P \cdot E_p^T] = I.$$

Equation (A.12) combined with the expression for W_2 in (10) leads to

$$\begin{aligned}
&\mathbf{E}_j[\tilde{X}_i \cdot W_2 \cdot W_2^T \cdot \tilde{X}_i^T] \\
&= \mathbf{E}_j[(M_i + \Delta_i^d U_p + Q_i H_i^s E_p) \cdot W_2 \cdot W_2^T \\
&\quad \times (M_i + \Delta_i^d U_p + Q_i H_i^s E_p)^T] \\
&= \Delta_i^d \mathbf{E}_j[U_p \cdot W_2 \cdot W_2^T \cdot U_p^T](\Delta_i^d)^T \\
&\quad + Q_i H_i^s \mathbf{E}_j[E_p \cdot W_2 \cdot W_2^T \cdot E_p^T](H_i^s)^T Q_i^T \\
&\quad + \underbrace{\mathbf{E}_j[(M_i + \Delta_i^d U_p) \cdot W_2 \cdot W_2^T \cdot E_p^T](H_i^s)^T Q_i^T}_{=0,\ (A.13)} \\
&\quad + \underbrace{Q_i H_i^s \mathbf{E}_j[E_p \cdot W_2 \cdot W_2^T \cdot (M_i + \Delta_i^d U_p)^T]}_{=0,\ (A.13)} \\
&\quad + \underbrace{\mathbf{E}_j[M_i \cdot W_2 \cdot W_2^T \cdot M_i^T + M_i \cdot W_2 \cdot W_2^T \cdot U_p^T(\Delta_i^d)^T + \Delta_i^d U_p \cdot W_2 \cdot W_2^T \cdot M_i^T]}_{\overset{\text{def}}{=} C_i} \\
&= \Delta_i^d \underbrace{\mathbf{E}_j[U_p U_p^T]}_{=R_p} R_p^{-1} W_i^u L_p^{-1} \underbrace{\mathbf{E}_j[U_p U_p^T]}_{=R_p} \\
&\quad \times L_p^{-T}(W_i^u)^T R_p^{-1} \underbrace{\mathbf{E}_j[U_p U_p^T]}_{=R_p}(\Delta_i^d)^T \\
&\quad + Q_i H_i^s \underbrace{\mathbf{E}_j[E_p \cdot W_2 \cdot W_2^T \cdot E_p^T]}_{=I,\ (A.13)}(H_i^s)^T Q_i^T + C_i \\
&= \Delta_i^d W_i^u (W_i^u)^T (\Delta_i^d)^T + Q_i H_i^s (H_i^s)^T Q_i^T + C_i.
\end{aligned} \tag{A.14}$$

Using (A.11), it is easy to show that $\lim_{i\to\infty} C_i = 0$. Taking the limit as $i \to \infty$ on both sides of (A.14) thus gives (using A.11))

$$\lim_{i\to\infty} \mathbf{E}_j[\tilde{X}_i \cdot W_2 \cdot W_2^T \cdot \tilde{X}_i^T] = \lim_{i\to\infty} [\Delta_i^d W_i^u (W_i^u)^T (\Delta_i^d)^T + \Delta_i^s (\Delta_i^s)^T]. \tag{A.15}$$

On the other hand, we know from (5) that

$$\lim_{i\to\infty} \mathbf{E}_j[\tilde{X}_i \cdot W_2 \cdot W_2^T \cdot \tilde{X}_i^T] = \lim_{i\to\infty} (S_1^{1/2} V_1^T V_1 S_1^{1/2}) = S_1. \tag{A.16}$$

Finally, from (A.15) and (A.16), we find

$$\lim_{i\to\infty} [\Delta_i^d \cdot W_i^u \cdot (W_i^u)^T \cdot (\Delta_i^d)^T + \Delta_i^s \cdot (\Delta_i^s)^T] = S_1.$$

From Lemma 1, we know that the left-hand side of this expression is equal to $P[W_u(z)]$. This leads to $P[W_u(z)] = S$, which is exactly the first part of (11).

From (5), we know that $\Gamma_i^T W_1^T W_1 \Gamma_i = S_1^{1/2} U_1^T U_1 S_1^{1/2} = S_1$. Combining this with Lemma 1, (9), and with $W_1 = W_i^y$, (10), we find

$$Q[W_y(z)] = \lim_{i\to\infty} [\Gamma_i^T \cdot (W_i^y)^T \cdot W_i^y \cdot \Gamma_i] = \lim_{i\to\infty} (\Gamma_i^T W_1^T W_1 \Gamma_i) = S_1,$$

which is the second part of (11).