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A Subspace Algorithm for the Identification of Discrete Time Frequency Domain Power Spectra*

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A new subspace algorithm results in the fast and accurate identification of state space models from given power spectra.

Key Words—System identification; subspace methods; power spectra; state-space models; multivariable systems; state-space methods; linear algebra

Abstract—In this paper we present a new subspace algorithm for the identification of multi-input multi-output linear discrete time systems from measured power spectrum data. We show how the state space system matrices can be determined by taking the inverse discrete Fourier transform of the given data and applying the result to a new realization algorithm. Special attention is paid to ensure the positive realness of the identified power spectrum. The computational speed is improved by applying a Lanczos algorithm. The algorithm is illustrated with two practical examples. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Identification of multi-input multi-output (MIMO) systems from a measured power spectrum is still considered to be a challenge. This type of data typically arises when modeling disturbances, in which case the frequency domain power spectrum is often easily obtained. The algorithm described in this paper determines a state space realization of this power spectrum. For disturbance modeling, the spectral factor of this power spectrum can then be used for instance in the design of an optimal disturbance rejection controller (Boyd and Barrat, 1991).

For time domain measurements, a vast number of state-space subspace identification algorithms is available (Larimore, 1990; Van Overschee and De Moor, 1994; Verhaegen, 1994; Viberg, 1995; De Moor and Van Overschee 1995; Van Overschee and De Moor, 1996a). The major advantage of subspace identification algorithms over the classical prediction error methods (Ljung, 1987) is the absence of non-linear parametric optimization problems. Indeed, subspace identification algorithms are non-iterative, and thus never get stuck in local minima or suffer from convergence problems. In short, they always produce a (sub-optimal) result, which is often surprisingly good for practical data.

Subspace identification algorithms for the identification of frequency domain data have already been extensively described in the literature (Liu *et al.*, 1994; McKelvey *et al.*, 1996; Van Overschee and De Moor, 1996b). These techniques are however not directly usable for the identification of frequency domain power spectra.

The problem addressed in this paper is the one of fitting a linear discrete time power spectrum through given measured frequency domain power spectrum samples. A parametric approach to this problem would consist of using a non-linear least squares criterion, that is then optimized using a non-linear search in the parameter space. However, the typical disadvantages of the time domain prediction error methods also carry through to this frequency domain setting i.e. an *a priori* given parametrization is needed (which is especially hard to find for MIMO systems), non-linear parametric optimization is required and convergence problems could occur.

The major contributions of this paper are the following:

- We show how the inverse discrete Fourier transform of the given power spectrum can be expressed as a function of the system matrices.

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A realization theory, similar to the one in Ho and Kalman (1966) and Kung (1978) and based on the results of McKelvey *et al.* (1996) is then devised to obtain the system matrices from this inverse discrete Fourier transform. An important observation is that the computed singular value spectrum determines two times the system order (instead of once the system order for the classical realization algorithms). A detailed step by step description of the algorithm is provided.

- A given set of system matrices does not necessarily describe a valid power spectrum. Indeed, a power spectrum has to be positive real to be physically meaningful. We recognize this problem and solve it through a slight modification of the basic subspace power spectrum identification algorithm. The identified power spectrum is always positive real.
- We illustrate the practical relevance of the problem treated in this paper with two examples which solve an acceleration sensitivity and an acoustic power spectrum modeling problem.

This paper is organized as follows: In Section 2 the discrete time power spectrum subspace identification problem is described and the notation is introduced. In Section 3 we derive the main theorem and show how this leads to a subspace identification algorithm. Section 4 addresses the problem of positivity of the identified power spectrum. The practical examples, illustrative of the algorithm described in this paper, are contained in Section 5.

2. PROBLEM DESCRIPTION AND NOTATION

In this section we define the power spectrum, the spectral factor and the general notation. We also introduce the main identification problem.

Consider the $l \times l$ dimensional square discrete time system:

$$x_{k+1} = Ax_k + Bu_k, \tag{1}$$

$$y_k = Cx_k + Du_k, \tag{2}$$

with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times l}$, $C \in \mathbb{R}^{l \times n}$ and $D \in \mathbb{R}^{l \times l}$ non-singular. The vector sequences $u_k, y_k \in \mathbb{R}^l$ are the input and output sequences, respectively. The system (1)–(2) is assumed to be stable and strictly minimum phase: all eigenvalues of A and $A - BD^{-1}C$ lie strictly inside the unit circle. The matrix pairs $\{A, B\}$ and $\{A, C\}$ are controllable and observable, respectively. The matrix A is assumed to be non-singular. The system (1)–(2) is thus a minimal stochastic system. The transfer function of the system (1)–(2) is denoted by $G(z)$:

$$G(z) \stackrel{\text{def}}{=} D + C(z\, I_n - A)^{-1}B. \tag{3}$$

The power spectrum associated with (1)–(2) is denoted by $S(z) \in \mathbb{C}^{l \times l}$ and is defined as

$$S(z) \stackrel{\text{def}}{=} G(z)\, G^{\mathrm{T}}(z^{-1}). \tag{4}$$

The original system (1)–(2) is called the innovation form, unity variance, minimum phase spectral factor associated with this power spectrum $S(z)$. From (4) we find that for all z on the unit circle ($|z| = 1$) the power spectrum satisfies*

$$S(z) > 0, \quad |z| = 1. \tag{5}$$

This is the positive realness condition which will play an important role in this paper. Indeed, this condition imposes a constraint on the given data samples S_k (each of them has to be a positive definite matrix), as well as on the identified power spectrum $S(z)$ (which has to be a positive real spectrum).

In Caines (1988) it is shown that the power spectrum $S(z)$ can be split into the sum of two system transfer matrices as follows: With P the solution to the discrete Lyapunov equation:

$$P = APA^{\mathrm{T}} + BB^{\mathrm{T}}, \tag{6}$$

and G and Λ_0 defined as

$$G \stackrel{\text{def}}{=} APC^{\mathrm{T}} + BD^{\mathrm{T}}, \tag{7}$$

$$\Lambda_0 \stackrel{\text{def}}{=} CPC^{\mathrm{T}} + DD^{\mathrm{T}}, \tag{8}$$

the power spectrum $S(z)$ can be decomposed into the sum of two transfer matrices as

$$S(z) = H(z) + H^{\mathrm{T}}(z^{-1}), \tag{9}$$

with

$$H(z) \stackrel{\text{def}}{=} \tfrac{1}{2}\Lambda_0 + C(zI_n - A)^{-1}G. \tag{10}$$

The problem treated in this paper can now be stated as follows:

Given $N + 1$ matrices $S_k \in \mathbb{C}^{l \times l}$ of the power spectrum $S(z)$ evaluated in $N + 1$ equidistant points over the unit circle:

$$S_k = S(e^{j(2\pi k/2N)}), \quad k = 0, \ldots, N. \tag{11}$$

Find:

- The system matrices A, G, C, Λ_0 describing the power spectrum.
- The system matrices A, B, C, D describing the spectral factor (1)–(2)

*$A \geq B$ if $A - B$ is a non-negative definite matrix.

We define the inverse discrete Fourier transform v_k of a given complex signal $V_k = V(e^{j(2\pi k/2N)})$, $k = 0, \ldots, 2N-1$ as:

$$v_k \overset{\text{def}}{=} \frac{1}{2N} \sum_{r=0}^{2N-1} V_r e^{j(2\pi rk/2N)} \tag{12}$$

A^T is the transpose of a matrix A, while A^* denotes the complex conjugate transpose. $\|A\|_F$ is the Frobenius norm of A and $A^\dagger$ the Moore–Penrose pseudo inverse. $\mathscr{R}(A)$ and $\mathscr{I}(A)$ are the real and imaginary parts of a complex matrix A, respectively. $\lambda(A)$ denotes the real eigenvalues of a square Hermitian matrix A.

The permutation matrix $\Pi \in \mathbb{R}^{lN \times lN}$ is given by

$$\Pi \overset{\text{def}}{=} \begin{pmatrix} 0 & 0 & \ldots & I_l \\ \ldots & \ldots & \ldots & \ldots \\ 0 & I_l & \ldots & 0 \\ I_l & 0 & \ldots & 0 \end{pmatrix}$$

The extended observability matrix $\Gamma_q \in \mathbb{R}^{lq \times n}$ $(q \geq n)$ and reversed extended observability matrix $\tilde{\Gamma}_r \in \mathbb{R}^{lr \times n}$ $(r \geq n)$ are defined as

$$\Gamma_q \overset{\text{def}}{=} \begin{pmatrix} C \\ CA \\ \ldots \\ CA^{q-1} \end{pmatrix}, \qquad \tilde{\Gamma}_r \overset{\text{def}}{=} \Pi\Gamma_r = \begin{pmatrix} CA^{r-1} \\ \ldots \\ CA \\ C \end{pmatrix}.$$

The extended controllability matrix $\Delta_r \in \mathbb{R}^{n \times lr}$ and reversed extended controllability matrix $\tilde{\Delta}_q \in \mathbb{R}^{n \times lq}$ are defined as

$$\Delta_r \overset{\text{def}}{=} (G\ AG\ \ldots\ A^{r-1}G),$$

$$\tilde{\Delta}_q \overset{\text{def}}{=} \Delta_q \Pi = (A^{q-1}G \ldots A\ G\ G).$$

Note that since the system (1)–(2) is minimal, the matrices $\Gamma_q, \tilde{\Gamma}_r$ and $\Delta_r, \tilde{\Delta}_q$ are, respectively, of full column and row rank n.

Finally, given n_r real numbers $a_1, \ldots, a_{n_r}$ and $2n_c$ real numbers $b_1, \ldots, b_{n_c}$, $c_1, \ldots, c_{n_c}$ we define the matrix $\Omega(a, b, c)$ as

$$\Omega(a,b,c) \overset{\text{def}}{=} \begin{pmatrix} a_1 & \ldots & 0 & 0 & 0 & \ldots & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & a_{n_r} & 0 & 0 & \ldots & \ldots & 0 & 0 \\ 0 & \ldots & 0 & b_1 & c_1 & \ldots & \ldots & 0 & 0 \\ 0 & \ldots & 0 & -c_1 & b_1 & \ldots & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 0 & 0 & \ldots & \ldots & b_{n_c} & c_{n_c} \\ 0 & \ldots & 0 & 0 & 0 & \ldots & \ldots & -c_{n_c} & b_{n_c} \end{pmatrix}$$

It is trivial to see that the eigenvalues of $\Omega(a, b, c)$ are given by

$$\text{eig}(\Omega(a,b,c)) = a_1, \ldots, a_{n_r}, b_1 \pm jc_1, \ldots, b_{n_c} \pm jc_{n_c}$$

with $j = \sqrt{-1}$.

3. IDENTIFICATION OF POWER SPECTRA

In this section we show how the subspace power spectrum identification problem can be solved. The main theorem states how the system order and the observability and controllability matrices can be determined. The system matrices are then step by step derived from this theorem. The final algorithm is summarized in Fig. 1.

3.1. *Inverse discrete Fourier transform and the main theorem*

Before we start the derivation of the algorithm, we expand the $N+1$ given points S_k to $2N$ points as follows:

$$S_{N+k} = S^*_{N-k}, \quad k = 1, \ldots, N-1. \tag{13}$$

From now on, S_k denotes this signal of length $2N$. The first theorem shows how the inverse discrete Fourier transform s_k of the sequence S_k can be expressed in terms of the system matrices.

Theorem 1 (*Inverse discrete Fourier transform*). With $M = (I_n - A^{2N})^{-1}$, the inverse discrete Fourier transform $s_k \in \mathbb{R}^{l \times l}$ of the given power spectrum S_k is given by

$$s_0 = \Lambda_0 + CA^{2N-1}MG + G^T(A^T)^{2N-1}M^TC^T, \tag{14}$$

$$s_k = CA^{k-1}MG + G^T(A^T)^{2N-k-1}M^TC^T, \quad k = 1, \ldots, 2N-1. \tag{15}$$

A proof of this theorem can be found in Appendix A. Note:

- Due to the specific form of the expansion in (13), it is easy to prove that, even though the original power spectrum sequence S_k is complex, the resulting sequence s_k is real.

- For $N \to \infty$, we find with A asymptotically stable that

$$\lim_{N\to\infty} M = I_n.$$

The effect of the matrix M is thus due to the finite number of data.

Theorem 1 is the base for the extraction of the system order and matrices from the given data S_k. In the following theorem the block-Hankel matrix $\mathbb{S} \in \mathbb{R}^{lq \times lr}$ play an important role:

$$\mathbb{S} \overset{\text{def}}{=} \begin{pmatrix} S_1 & S_2 & S_3 & \cdots & S_r \\ S_2 & S_3 & S_4 & \cdots & S_{r+1} \\ S_3 & S_4 & S_5 & \cdots & S_{r+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ S_q & S_{q+1} & S_{q+2} & \cdots & S_{r+q-1} \end{pmatrix}, \quad (16)$$

with the number of block rows $q \geq 2n$ and the number of block columns $r \geq 2n$, and $r + q < 2N$. With this block-Hankel matrix $\mathbb{S}$ and the results of Theorem 1, the main theorem which is proved in Appendix B can now be stated:

Theorem 2 (*Main theorem*). The block-Hankel matrix $\mathbb{S}$ can be decomposed as

$$\mathbb{S} = (\Gamma_q \quad \tilde{\Delta}_q^{\mathrm{T}}) \begin{pmatrix} M & 0 \\ 0 & M^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \Delta_r \\ \tilde{\Gamma}_r^{\mathrm{T}} \end{pmatrix}, \quad (17)$$

which leads to the following results:

- The rank of $\mathbb{S}$ is equal to $2n$ (two times the system order).
- The column space of $\mathbb{S}$ can be expressed in terms of the system matrices as

$$\text{column space } \mathbb{S} = \text{column space } (\Gamma_q \ \tilde{\Delta}_q^{\mathrm{T}}).$$

- The row space of $\mathbb{S}$ can be expressed in terms of the system matrices as

$$\text{row space } \mathbb{S} = \text{row space} \begin{pmatrix} \Delta_r \\ \tilde{\Gamma}_r^{\mathrm{T}} \end{pmatrix}.$$

This theorem is what would be qualified as a typical subspace theorem: it defines a matrix, the rank of which will determine the system order n, and the column and row space of which will generate estimates of the system matrices, respectively. Other typical subspace algorithms of this type are summarized in Van Overschee and De Moor (1996a). This theorem serves as a starting point of the step by step extraction of the system order and matrices. These steps are described in the following subsections. The final algorithm is summarized in Fig. 1.

Power Spectrum Subspace Identification Algorithm:

- Expand the $N+1$ given points S_k to a signal of length $2N$: $S_{N+k} = S_{N-k}^*$ for $k = 1, \ldots, N-1$.
- Compute the $2N$ points inverse discrete Fourier transform s_k of the signal S_k.
- Form the block-Hankel matrix $\mathbb{S}$ as in equation (16).
- Compute the Singular Value Decomposition:

$$\mathbb{S} = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix}.$$

- The number of singular values S_1 different from zero is equal to two times the system order.
- Determine $\mathcal{U}$ and $\mathcal{V}$ as: $\mathcal{U} = U_1.S_1^{1/2}$ and $\mathcal{V} = S_1^{1/2}.V_1^T$.
- Determine the eigenvalue spectrum of $\underline{\mathcal{U}}^{\dagger}.\overline{\mathcal{U}}$ or find the symmetric spectrum from the characteristic equation (23). The stable eigenvalues of this spectrum are denoted by $\alpha_1, \ldots, \alpha_{n_r}$ and $\beta_1 \pm j\gamma_1, \ldots, \beta_{n_c} \pm \gamma_{n_c}$. Determine the system matrix A as $A = \Omega(\alpha, \beta, \gamma)$.
- Determine the matrix $M = (I_n - A^{2N})^{-1}$. Determine the real matrix $T^0 = \begin{pmatrix} T_1^0 & T_2^0 \end{pmatrix}$ from the eigenvectors of $\underline{\mathcal{U}}^{\dagger}.\overline{\mathcal{U}}$.
- Solve the following linear equation for the elements δ_k, λ_k and ρ_k:

$$\mathcal{U}^{\dagger}.\Pi.\mathcal{V}^T = T_1^0.M.\Omega^l(\delta, \lambda, \rho)(T_2^0)^T + T_2^0.\Omega(\delta, \lambda, \rho).M^T.(T_1^0)^T.$$

- Determine the system matrices C and G^T as the $l \times n$ top left respectively bottom right sub-matrices of:

$$\left(\mathcal{U}.T_1^0 \middle| \mathcal{U}.T_2^0.\Omega(\delta, \lambda, \rho)\right).$$

- Determine $\Lambda_0 = s_0 - CA^{2N-1}MG - G^T(A^T)^{2N-1}M^TC^T$.
- Find P through the solution of the Riccati equation (31)-(32) and determine B and D as:

$$B = (G - APC^T).(\Lambda_0 - CPC^T)^{-1/2}, \quad D = (\Lambda_0 - CPC^T)^{1/2}.$$

Fig. 1. Subspace algorithm for the identification of a given discrete time frequency domain power spectrum.

3.2. *Singular value decomposition and determination of the system order*

As described in Theorem 2, the matrix $\mathbb{S}$ is rank deficient and can be factored into system related matrices. This factorization is achieved with the Singular Value Decomposition (**SVD**) of $\mathbb{S}$:

$$\mathbb{S} = (U_1 \ U_2) \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^{\mathrm{T}} \\ V_2^{\mathrm{T}} \end{pmatrix}, \quad (18)$$

where $S_1 \in \mathbb{R}^{2n \times 2n}$, $U_1 \in \mathbb{R}^{ql \times 2n}$ and $V_1 \in \mathbb{R}^{rl \times 2n}$. Two times the system order ($2n$) can thus immediately be determined from the number of singular values different from zero. The singular vectors of the **SVD** (18) also lead to the definition of the important matrices $\mathcal{U}$ and $\mathcal{V}$:

$$\mathcal{U} \overset{\text{def}}{=} U_1 S_1^{1/2}, \qquad \mathcal{V} \overset{\text{def}}{=} S_1^{1/2} V_1^{\mathrm{T}}.$$

We find from the **SVD** (18) and Theorem 2 that (with $T \in \mathbb{R}^{2n \times 2n}$ a non-singular matrix):

$$\mathcal{U} = (\Gamma_q \tilde{\Delta}_q^{\mathrm{T}}) T^{-1}, \quad (19)$$

$$\mathcal{V} = T \begin{pmatrix} M & 0 \\ 0 & M^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \Delta_r \\ \tilde{\Gamma}_r^{\mathrm{T}} \end{pmatrix}. \quad (20)$$

3.3. *Determination of A*

In classical realization theory of Ho and Kalman (1966) and Kung (1978), for instance, the range of

the extended observability matrix Γ_q is determined from the **SVD** of the impulse response block-Hankel matrix. The system matrix A is then determined through the shift structure property of this extended observability matrix Γ_q, which essentially states that

$$\overline{\Gamma}_q = \underline{\Gamma}_q A,$$

where $\overline{X}$ and $\underline{X}$ denote the matrix X with the l first and last rows deleted, respectively.

From (19) we know that for the power spectrum identification of this paper, the range of the block-Hankel matrix determines not the range of Γ_q, but the range of $(\Gamma_q \tilde{\Delta}_q^{\mathrm{T}})$. We thus resort to an alternative shift structure formulation:

$$\overline{(\Gamma_q \tilde{\Delta}_q^{\mathrm{T}})} = \underline{(\Gamma_q \tilde{\Delta}_q^{\mathrm{T}})} \begin{pmatrix} A & 0 \\ 0 & (A^{\mathrm{T}})^{-1} \end{pmatrix}.$$

Using (19) this last equation can be converted to

$$\underline{\mathscr{U}}^{\dagger}\overline{\mathscr{U}} = T \begin{pmatrix} A & 0 \\ 0 & (A^{\mathrm{T}})^{-1} \end{pmatrix} T^{-1}. \tag{21}$$

The left-hand side of this equation can be computed from the singular value decomposition of $\mathbb{S}$. Clearly, the spectrum of this matrix is equal to the combined spectrum of A and $(A^{\mathrm{T}})^{-1}$. Since we know that A is stable, the stable eigenvalues of $\underline{\mathscr{U}}^{\dagger}\overline{\mathscr{U}}$ determine the eigenvalue spectrum of A.

The matrix A can thus be constructed as follows. Let $\underline{\mathscr{U}}^{\dagger}\overline{\mathscr{U}}$ have n_r real stable eigenvalues $\alpha_1, \ldots, \alpha_{n_r}$ and $n_c = \frac{1}{2}(n - n_r)$ pairs of complex conjugate stable eigenvalues $\beta_1 \pm j\gamma_1, \ldots, \beta_{n_c} \pm j\gamma_{n_c}$. The matrix A can then be set equal to

$$A = \Omega(\alpha, \beta, \gamma). \tag{22}$$

Note that, for simplicity, we have assumed that A has distinct eigenvalues. If this is not the case, the definition of $\Omega(\alpha, \beta, \gamma)$ can be extended to also include block-Jordan forms for repeated eigenvalues. This would only slightly complicate the discussion to follow.

In practice, when the data are perturbed by noise, the simple solution (21) is not guaranteed to have n stable and n unstable eigenvalues. An alternate solution, which avoids this problem, is to fit the null space of $\mathscr{U}$ as in Viberg *et al.* (1997). Indeed, in Viberg *et al.* (1997) it is shown how a basis for the null space of the extended observability matrix can be parametrized as a function of the coefficients of the characteristic polynomial. Similarly, we can parameterize the null space of $\mathscr{U}$ with the coefficients of the symmetric characteristic polynomial:

$$z^{2n} + a_1 z^{2n-1} + a_2 z^{2n-2} + \cdots + a_{n-1} z^{n+1} + a_n z^n + a_{n-1} z^{n-1} + \ldots + a_2 z^2 + a_1 z + 1, \tag{23}$$

which for each root r_k also has a root $1/r_k$. In this way the solution will always have n stable and n unstable eigenvalues. A drawback of this method is that it is not as numerically reliable as (21).

3.4. *Determination of C, G and* Λ_0

Once A is determined from (22), we have to determine the matrix T appearing in (19)–(20).* The first equation which T has to satisfy follows from (21):

$$T^{-1}\underline{\mathscr{U}}^{\dagger}\overline{\mathscr{U}}T = \begin{pmatrix} A & 0 \\ 0 & (A^{\mathrm{T}})^{-1} \end{pmatrix}. \tag{24}$$

One matrix T^0 satisfying this equation can be easily found from the eigenvalue decomposition of $\underline{\mathscr{U}}^{\dagger}\overline{\mathscr{U}}$. A real matrix T^0 is constructed as follows:

- For every real eigenvalue and associated real eigenvector v_k of $\underline{\mathscr{U}}^{\dagger}\overline{\mathscr{U}}$, the corresponding column in T^0 simply equals v_k.
- For every complex conjugate eigenvalue pair and associated complex conjugate eigenvectors v_k and $\tilde{v}_k$ of $\underline{\mathscr{U}}^{\dagger}\overline{\mathscr{U}}$, the corresponding columns in T^0 equal $\mathscr{R}(v_k)$ and $\mathscr{I}(v_k)$.

The matrix T^0, however, is not unique in the sense that any matrix $T = T^0 F$ with:

$$F \overset{\text{def}}{=} \begin{pmatrix} \Omega(\kappa, \mu, \nu) & 0 \\ 0 & \Omega(\delta, \lambda, \rho) \end{pmatrix}, \tag{25}$$

also satisfies equation (24). Here we have introduced $2(n_r + 2n_c) = 2n$ real numbers $(\kappa_k, \lambda_k, \rho_k)$ and (δ_k, μ_k, ν_k). However, the number of degrees of freedom in the model, after fixing A (22) is only n: any similarity transformation transforming A into itself is of the form $\Omega(a, b, c)$ and has only n free parameters. We thus need a second equation to eliminate n degrees of freedom from T.

This second equation, which determines the coefficients of F follows from the combination of equations (19) and (20). With $T = (T_1 \; T_2)$ and $T^0 = (T_1^0 \; T_2^0)$ where $T_1, T_2, T_1^0, T_2^0 \in \mathbb{R}^{2n \times n}$, we find (see Appendix C):

$$\mathscr{U}^{\dagger}\Pi\mathscr{V}^{\mathrm{T}} = T_1 M T_2^{\mathrm{T}} + T_2 M^{\mathrm{T}} T_1^{\mathrm{T}}. \tag{26}$$

Using $T = T^0 F$ and equation (25), this leads to

$$\mathscr{U}^{\dagger}\Pi\mathscr{V}^{\mathrm{T}} = T_1^0 \Omega(\kappa, \mu, \nu) M \Omega^{\mathrm{T}}(\delta, \lambda, \rho)(T_2^0)^{\mathrm{T}} + T_2^0 \Omega(\delta, \lambda, \rho) M^{\mathrm{T}} \Omega^{T}(\kappa, \mu, \nu)(T_1^0)^{\mathrm{T}}. \tag{27}$$

*Note that in the classical realization theory of Ho and Kalman (1966) and Kung (1978) the matrix T does not have to be determined. This is because, in that case, C and G^{T} are determined as the first l rows and columns of the extended observability and controllability matrices, respectively. The reason why we have to determine T for this identification problem is that C and G appear both in (19) and (20), and thus cannot be determined independently from each other.

We can now fix n degrees of freedom in equation (27) by setting $\Omega(\kappa, \mu, \nu) = I_n$:

$$\mathscr{U}^{\dagger}\Pi\mathscr{V}^{\mathrm{T}} = T_1^0 M\Omega^{\mathrm{T}}(\delta, \lambda, \rho)(T_2^0)^{\mathrm{T}} + T_2^0\Omega(\delta, \lambda, \rho)M^{\mathrm{T}}(T_1^0)^{\mathrm{T}}. \quad (28)$$

This equation is linear in the parameters δ_k, λ_k and ρ_k, which implies that, even though a little tricky, it is straightforward to solve for the unknowns.* Equation (19) now becomes

$$(\Gamma_q \Delta_q^{\mathrm{T}}) = \mathscr{U}T = \mathscr{U}T^0F = \mathscr{U}T^0\begin{pmatrix} I_n & 0 \\ 0 & \Omega(\delta, \lambda, \rho) \end{pmatrix} = (\mathscr{U}T_1^0 | \mathscr{U}T_2^0\Omega(\delta, \lambda, \rho)). \quad (29)$$

The system matrices C and G^{T} can be determined as the $l \times n$ upper left sub-matrix and lower right sub-matrix of (29), respectively.

Once A, G and C are determined, Λ_0 easily follows from (14) as (with $M = (I_n - A^{2N})^{-1}$):

$$\Lambda_0 = s_0 - CA^{2N-1}MG - G^{\mathrm{T}}(A^{\mathrm{T}})^{2N-1}M^{\mathrm{T}}C^{\mathrm{T}}. \quad (30)$$

3.5. *Determination of B and D*

The second part of the identification problem consists of the determination of the system matrices B and D of the spectral factor (1)–(2). As explained in Caines (1988) and Van Overschee and De Moor (1996a) this can be done by solving the following Riccati equation for P:

$$P = APA^{\mathrm{T}} + (G - APC^{\mathrm{T}})(\Lambda_0 - CPC^{\mathrm{T}})^{-1} \times (G - APC^{\mathrm{T}})^{\mathrm{T}}. \quad (31)$$

The positive definite solution of the Riccati equation (31) can be found from the generalized eigenvalue problem:

$$\begin{pmatrix} A^{\mathrm{T}} - C^{\mathrm{T}}\Lambda_0^{-1}G^{\mathrm{T}} & 0 \\ -G\Lambda_0^{-1}G^{\mathrm{T}} & I_n \end{pmatrix}\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} I_n & -C^{\mathrm{T}}\Lambda_0^{-1}C \\ 0 & A - G\Lambda_0^{-1}C \end{pmatrix}\begin{pmatrix} W_1 \\ W_2 \end{pmatrix}\Lambda, \quad (32)$$

as $P = W_2W_1^{-1}$. Λ contains the n stable (i.e. inside the unit circle) eigenvalues of the generalized eigenvalue pencil. The matrices B and D can then be put equal to:

$$B = (G - APC^{\mathrm{T}})(\Lambda_0 - CPC^{\mathrm{T}})^{-1/2}, \quad (33)$$

$$D = (\Lambda_0 - CPC^{\mathrm{T}})^{1/2}. \quad (34)$$

The fact that B and D are computed through the solution of the Riccati equation (31) guarantees that the resulting system is minimum phase, i.e. that the eigenvalues of $(A - BD^{-1}C)$ are all stable.

This concludes the description of the steps of the power spectrum identification problem. The final algorithm is summarized in Fig. 1.

3.6. *Consistency*

We now discuss briefly how the algorithm behaves in the presence of noise. Assume that the given data $\hat{S}_k$ are corrupted as

$$\hat{S}_k = S(e^{j(2\pi k/2N)}) + n_k,$$

where n_k is a zero mean complex random variable with covariance†

$$\mathbf{E}\left[\begin{pmatrix} \mathscr{R}(n_k) \\ \mathscr{I}(n_k) \end{pmatrix}(\mathscr{R}(n_s^{\mathrm{T}}), \mathscr{I}(n_s^{\mathrm{T}}))\right] = \begin{pmatrix} \frac{1}{2}R_k & 0 \\ 0 & \frac{1}{2}R_k \end{pmatrix}\delta_{ks}.$$

We thus assume that the perturbations between different frequency points are independent and that the real and imaginary parts at a fixed frequency point are also independent. Furthermore, we assume the covariance R_k to be uniformly bounded.

Following the derivation in McKelvey *et al.* (1996) and with

$$\hat{\mathbb{S}} = \mathbb{S} + \Delta\mathbb{S}$$

we then find that for the number of data N going to infinity ($N \to \infty$), the Frobenius norm of the perturbation goes to zero $\|\Delta\mathbb{S}\|_{\mathrm{F}} \to 0$. This is intuitively clear from the averaging effect when taking the inverse discrete Fourier transform which will zero out the noise contributions. This observation has as a direct consequence (McKelvey *et al.*, 1996) that the algorithms presented in this paper are strongly consistent.

4. ENSURING THE POSITIVE REALNESS OF THE POWER SPECTRUM

In this Section, we investigate the consequences of the fact that every physically meaningful power spectrum should be positive real (5). *In practice, when the data are corrupted by noise, this property is not guaranteed by the algorithm of Fig.* 1 *and should thus sometimes be forced afterwards. This section describes two optimal ways to do so.*

As already stated in equation (5), each of the given data matrices S_k has to be positive definite. When this is the case, and when the data S_k are noise free data generated by a linear system of

* Note that when the data are perturbed by noise, the equation is not exactly solvable and should thus be solved in a least squares sense.

† $\mathbf{E}$ denotes the expected value operator and δ_{ks} the Kronecker delta.

a finite order, the identified power spectrum will also be positive real and satisfy (5).

However, problems arise when the given data were *not* generated by a finite dimensional linear system or when the finite data sample is noise corrupted, which is the case for all practical problems. In this case, there is no guarantee that the identified power spectrum, which is determined by A, G, C and Λ_0 is positive real and thus satisfies equation (5) for all points on the unit circle. One important implication of this is that when the identified sequence is not positive real, the Riccati equation (31) has no positive definite solution and the spectral factor cannot be computed. See also Van Overschee and De Moor (1996a) for more details on positive real spectra and sequences.

In this section we present two possible solutions to this problem. Both of these solutions start from given matrices A and C (determined from the algorithm in Fig. 1). The solutions then state how G and Λ_0 are determined through the solution of an optimization problem which guarantees a positive real identified power spectrum.

4.1. *Linear matrix inequalities*

Given the system matrices A and C from the algorithm in Fig. 1, a positive real identified power spectrum can be guaranteed by solving the following constrained optimization problem (see also Van Overschee and De Moor, 1996a):

Given the known transfer matrix

$$L(z) = (C(zI_n - A)^{-1}|I_l). \tag{35}$$

Solve

$$\min_{Q,S,R} \sum_{k=0}^{2N-1} \|S_k - L(e^{j(2\pi k/2N)})\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \times L^T(e^{-j(2\pi k/2N)})\|_F^2, \tag{36}$$

Constrained to

$$\begin{pmatrix} Q & S \\ S^T & R \end{pmatrix} \geq 0. \tag{37}$$

The system matrices G and Λ_0 can then be found by solving the set of equations:

$$P = APA^T + Q, \qquad G = APC^T + S,$$
$$\Lambda_0 = CPC^T + R.$$

The constraint (37) guarantees that the resulting identified quadruple A, G, C, Λ_0 leads to a positive real power spectrum (Van Overschee and De Moor, 1996a), which in turn implies that the Riccati equation (31) has a positive definite solution and that the spectral factor can be computed.

The optimization problem (36)–(37) can be converted to a linear matrix inequality (Boyd *et al.*, 1994) (LMI), which means that it has a unique solution. The LMI software obtainable from anonymous ftp (Boyd, 1995) is a good tool to find a numerical solution. The drawback of this approach however is that the software in Boyd (1995) is not really suited (yet) for solving large problems. That is why we present a second approach.

4.2. *Non-linear least squares*

Given the system matrices A and C from the algorithm in Fig. 1, a positive real identified power spectrum can be guaranteed by solving the following unconstrained, non-linear least squares optimization problem (with $L(z)$ defined in (35)):

$$\min_{B,D} \sum_{k=1}^{2N-1} \|S_k - L(e^{j(2\pi k/2N)})\begin{pmatrix} B \\ D \end{pmatrix}(B^T D^T) \times L^T(e^{-j(2\pi k/2N)})\|_F^2, \tag{38}$$

which can be solved by a non-linear least squares solver. The optimization problem (38) guarantees a positive real power spectrum. To insure a minimum phase model, the equations (6)–(8) can be solved for G and Λ_0, after which a new B and D (guaranteeing a minimum phase model) can be computed through the solution of the Riccati equation (31) and equations (33) and (34).

Our experience with this method is that it converges well, when good initial guesses for the system matrices B and D are provided. This is the topic of the next subsection.

4.3. *About initial guesses*

To solve the non-linear optimization problem (38) we need an initial guess for the matrices B and D. However, when the power spectrum associated with the identified quadruple $\{A, G, C, \Lambda_0\}$ is not positive, the matrices B and D cannot be computed since the Riccati equation (31) has no positive definite solution. In this subsection we describe how the identified matrix Λ_0 can be perturbed to a matrix $\tilde{\Lambda}_0$ so that the power spectrum associated with the resulting quadruple $\{A, G, C, \tilde{\Lambda}_0\}$ is positive real. Assume the perturbed $\tilde{\Lambda}_0$ is of the form:

$$\tilde{\Lambda}_0 = \Lambda_0 + \tau I_l,$$

where $\tau > 0$. Taking τ large will trivially ensure positive realness of the power spectrum. However, we would like to keep τ as small as possible. This problem can be posed as an LMI as follows (see also constraint (37)):

$$\begin{aligned} \min \quad & \tau \\ \text{subject to} \quad & \tau > 0 \\ & P > 0, \\ & \begin{pmatrix} P - APA^T & G - APC^T \\ G^T - CPA^T & \Lambda_0 - CPC^T + \tau I_l \end{pmatrix} > 0 \end{aligned} \tag{39}$$

Power Spectrum Subspace Identification Algorithm:

- Repeat all but the last step of the algorithm in Figure 1.
- When the power spectrum is not positive, solve (4.3) for τ and replace Λ_0 with $\Lambda_0 + \tau . I_l$.
- Find P through the solution of the Riccati equation (31)-(32) and determine initial B^0 and D^0 matrices as:

$$B^0 = (G - APC^T).(\Lambda_0 - CPC^T)^{-1/2} \quad , \quad D^0 = (\Lambda_0 - CPC^T)^{1/2} .$$

- Solve the non-linear least squares optimization problem with B^0 and D^0 as start-up values for the non-linear optimization procedure:

$$\min_{B,D} \sum_{k=1}^{2N-1} \| S_k - L(e^{j.\frac{\pi k}{2N}}). \begin{pmatrix} B \\ D \end{pmatrix} . \begin{pmatrix} B^T & D^T \end{pmatrix} . L^T(e^{-j.\frac{\pi k}{2N}}) \|_F^2 .$$

- To ensure a minimum phase model, solve (6)-(8) for G and Λ_0. Recompute B and D through the solution of the Riccati equation (31) and equations (33) and (34).

Fig. 2. Subspace algorithm for the identification of a given discrete time frequency domain power spectrum. This algorithm ensures a positive real power spectrum and finds the optimal matrices B and D.

and P symmetric. This LMI can be easily solved with the software of Boyd (1995). Note that this LMI even lends itself to introduce general perturbations on Λ: $\tilde{\Lambda}_0 = \Lambda_0 + T$, where the Frobenius norm of T could be minimized with respect to the above constraints. We will however not pursue this in this paper.

The Riccati equation associated with $\{A, G, C, \tilde{\Lambda}_0\}$ can now be solved. This leads to matrices B^0 and D^0 which can serve as initial guesses for the optimization problem of Subsection 4.2.

The steps of the resulting algorithm are summarized in Fig. 2, which is the final power spectrum identification algorithm of this paper.

5. EXAMPLES

In this section we consider two practical power spectrum identification problems which illustrate the capabilities of the algorithm described in this paper.

5.1. *Modeling human sensitivity for car comfort analysis*

In a first example we model the human sensitivity for accelerations, to predict the comfort of a car as experienced by the driver. The measured data consists of a quantitative sensitivity index as a function of the frequency the car is excited with (the data here are for vertical accelerations). A high index implies high sensitivity at that frequency and vice-versa. Naturally, this data does not contain any phase information and is thus suitable for the algorithm described in this paper. Fig. 3 (a) shows the measured power spectrum (the sensitivity index as a function of the frequency). The spectrum consists of 512 measured amplitude samples. We have applied the algorithm of Fig. 2 to these data (with $r = q = N = 512$). Figure 3(b) shows the singular value spectrum which determines (two times) the system order. We choose a system order equal to 2 and 7. The identified power spectra were both

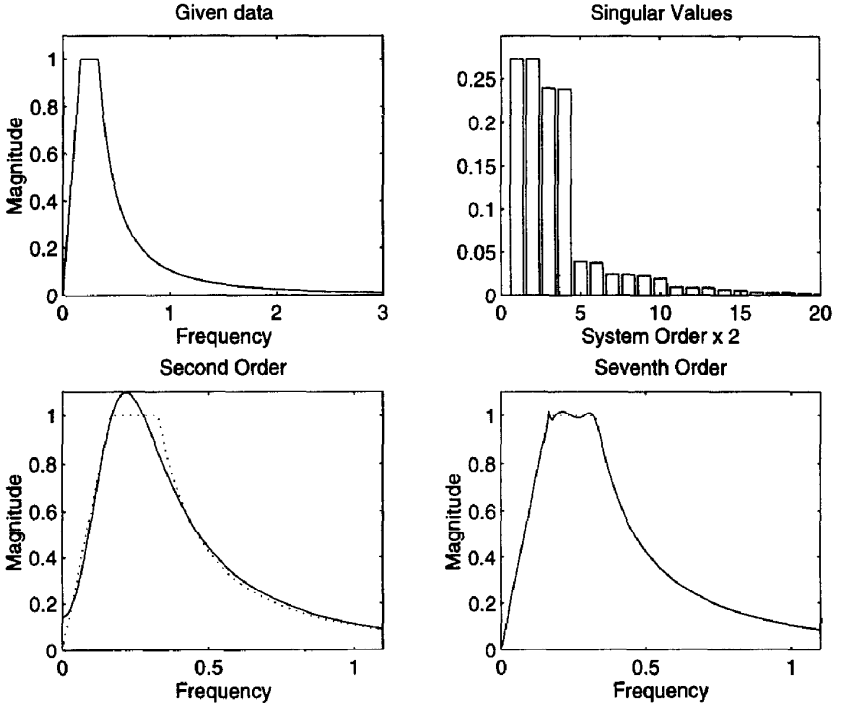

Fig. 3. (a) Given road disturbance power spectrum. (b) Singular value spectrum which determines two times the system order. We selected order 2 and 7 (which corresponds to the bar at 4 and 14 in the plot). (c) and (d) Second and seventh order fit (full line) and original data (dotted line). Note that we cut the frequency axis to zoom in on the more relevant part of the power spectrum. The seventh order model fits the flat part of the spectrum very well.

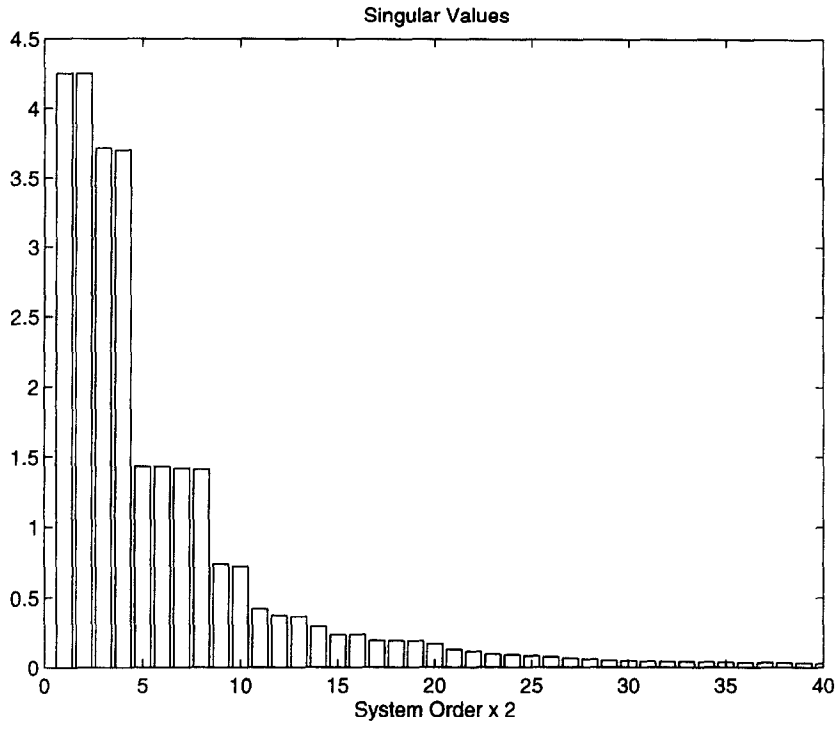

Fig. 4. Singular values associated with the radiation efficiency model. A fifth order system (bar 10) gave a reasonably good result, but a tenth order system (bar 20) resulted in the best fit of the numerical integral solution (equation (41)).

positive real, so no extra correction was needed. The resulting fits are shown in Figs 3(c) and (d).

The computationally most demanding step in the algorithm of Fig. 2 is the computation of the Singular Value Decomposition of $\mathbb{S} \in \mathbb{R}^{lN \times lN}$ (a 512×512 matrix for this example). Most of the computed singular values and vectors are however never used, since we only need the smaller matrices $U_1 \in \mathbb{R}^{lN \times n}$, $S_1 \in \mathbb{R}^{n \times n}$ and $V_1 \in \mathbb{R}^{lN \times n}$ in the algorithm. Instead of using the direct algorithm for the SVD as in Golub and Van Loan (1989), it is much more efficient to use a Lanczos algorithm which only computes the most significant singular values S_1 and vectors U_1 and V_1. For this example, only 20 singular values and vectors were computed with the software package ARPACK (Lehoucq *et al.*, 1995). With the Lanczos algorithm, the attained computational speed-up factor in the SVD step was approximately 35 (compared to the full SVD of Golub and Van Loan (1989)).

5.2. *Modeling of an acoustic power spectrum*

Efficient active reduction of the noise radiated by a structure requires an accurate model of the radiation efficiency which relates the dynamic response of the structure to the total acoustic energy radiated* (Baumann *et al.*, 1991). It can be shown that, when the velocity distribution of the structure can be decomposed as an expansion of P mode shapes, this total acoustic energy E equals:

$$E = \int_0^\infty \Psi^*(j\omega) M(j\omega) \Psi(j\omega)\, d\omega, \qquad (40)$$

with the matrix $M(j\omega)$ *containing the radiation efficiencies defined as*

$$M(j\omega) = \int_0^{2\infty} \int_0^\infty H^*(j\omega) H(j\omega) \sin(\theta)\, d\theta\, d\phi, \qquad (41)$$

where

$$\Psi(j\omega) = (\psi_1(j\omega)\ \psi_2(j\omega)\ \ldots\ \psi_P(j\omega))\mathrm{T},$$

$$H(j\omega) = (h_1(j\omega)\ h_2(j\omega)\ \ldots\ h_P(j\omega)).$$

$\psi_i(j\omega)$ denotes the modal velocity of the structure corresponding to mode i and $h_i(j\omega)$ the transfer function relating $\psi_i(j\omega)$ to the sound pressure on the surface. The angles θ and ϕ describe the integration points on the surrounding surface and ω denotes the frequency in radians per second. Active reduction of the radiated noise can now be achieved by controlling the modes $\psi_i(j\omega)$ of the structure in such a way that the total radiated energy E(40) is minimized. To solve this problem, an analytic expression of $M(j\omega)$ as a decomposition $G(j\omega)G^T(-j\omega)$ is needed. This is achieved by numerically evaluating equation (41) in a number of frequency points $M_0 = M(j\omega_0)$,

* The total acoustic energy is the time-integral of the acoustic power radiated through a surface enclosing the vibrating structure. It depends on the squared sound pressure on the considered enclosing surface, which in turn depends, via the Rayleigh-integral expression, on the velocity distribution of the structure.

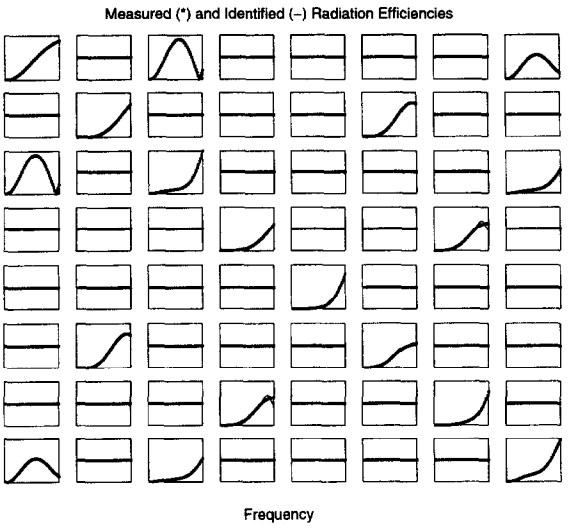

Fig. 5. Original (stars) and tenth order identified (full line) radiation efficiencies. There is hardly any difference between the numerical integral solution (equation (41)) and the model (only the elements (7, 4) and (4, 7) differ a bit at higher frequencies). The subspace identification algorithm identifies these data very well.

$M_1 = M(j\omega_1), \ldots, M_N = M(j\omega_N)$. With the Nyquist frequency equal to $\omega_N/2\pi$, we can now model the points M_i as a discrete* power spectrum with the algorithm of Fig. 2.

As an example, consider a rectangular plate modeled by its first 8 modes ($P = 8$). We evaluate equation (41) in 50 frequency points which results in 50 8×8 matrices M_i. These matrices are used as the input to the algorithm of Fig. 2. Once again, we used the ARPACK software (Lehoucq *et al.*, 1995) to compute the 40 most significant singular values and vectors. The speed up factor attained with the Lanczos algorithm in the SVD step was equal to 6 for this example. The singular value spectrum is shown in Fig. 4. We choose the order to be equal to 10. For this example, the identified power spectrum was not positive, so we had to apply the corrections and non-linear least squares optimizations of the algorithm in Fig. 2. The identified spectrum is shown in Fig. 5.

6. CONCLUSIONS

In this paper we presented a new subspace algorithm for the identification of multi-input multi-output linear discrete time systems from measured power spectrum data. We showed how the inverse discrete Fourier transform of the given data can be used in a new realization algorithm which determines the system matrices. Special attention was paid to the positivity of the identified power spectrum. The algorithm was illustrated with two practical examples.

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* After modeling, the discrete power spectrum can be transformed back to the continuous domain using the inverse ZOH transformation. When the bilinear transform is preferred, the original frequency axis has to be pre-warped. See also McKelvey et al. (1996).

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APPENDIX A. PROOF OF THEOREM 1

The proof makes extensive use of the results of McKelvey *et al.* (1996). From (9) we find that $S(z)$ can be split into the sum of two transfer matrices $H(z)$ and $H^T(z^{-1})$. The sampled values of these spectra are denoted ($k = 0, \ldots, 2N-1$) as follows:

$$H_k = H(z)|_{z=e^{j(2\pi k/2N)}} \tag{A.1}$$

$$H_k = H^T(z^{-1})|_{z=e^{j(2\pi k/2N)}} \tag{A.2}$$

$$= H_k^*. \tag{A.3}$$

We denote the inverse discrete Fourier transform of the signals H_k and $\tilde{H}_k$ with h_k and $\tilde{h}_k$, respectively. Through linearity of the inverse discrete Fourier transform, we thus find:

$$s_k = h_k + \tilde{h}_k. \tag{A.4}$$

In McKelvey *et al.* (1996) it is proven that h_k can be written as

$$h_k = CA^{k-1}(I_n - A^{2N})^{-1}G, \quad k > 0, \tag{A.5}$$

$$h_0 = \tfrac{1}{2}\Lambda_0 + CA^{2N-1}(I_n - A^{2N})^{-1}G. \tag{A.6}$$

From standard discrete Fourier transform properties it also follows that:

$$\tilde{h}_k = h_{2N-k}^* = G^T(A^T)^{2N-k-1}(I_n - (A^T)^{2N})^{-1}C^T, \tag{A.7}$$

$$\tilde{h}_0 = h_0^* = \tfrac{1}{2}\Lambda_0 + G^T(A^T)^{2N-1}(I_n - (A^T)^{2N})^{-1}C^T. \tag{A.8}$$

The combination of (A.4), (A.6) and (A.8) leads to the proof of formula (14), while the combination of (A.4), (A.5) and (A.7) leads to the proof of formula (15).

APPENDIX B. PROOF OF THEOREM 2

Using Theorem 1, and the fact that for any $r, s \geq 0$ and for any $t \geq \max(r, s)$

$$A^r M A^{(t-r)} = A^s M A^{(t-s)},$$

we can rewrite and decompose the block-Hankel matrix (16) as follows:

$$\mathbb{S} = \begin{pmatrix} CMG & CAMG & \cdots & CA^{r-1}G \\ CAMG & CA^2MG & \cdots & CA^rG \\ \cdots & \cdots & \cdots & \cdots \\ CA^{q-1}MG & CA^qMG & \cdots & CA^{r+q-2}G \end{pmatrix} + \begin{pmatrix} G^T(A^T)^{r+q-2}M^TC^T & \cdots & G^T(A^T)^{r-2}M^TC^T & G^T(A^T)^{r-1}M^TC^T \\ \cdots & \cdots & \cdots & \cdots \\ G^T(A^T)^qM^TC^T & \cdots & G^T(A^T)^2M^TC^T & G^TA^TM^TC^T \\ G^T(A^T)^{q-1}M^TC^T & \cdots & G^TA^TM^TC^T & G^TM^TC^T \end{pmatrix}$$

$$= \begin{pmatrix} C \\ CA \\ \cdots \\ CA^{q-1} \end{pmatrix} M(G \;\; AG \ldots A^{r-1}G) = \begin{pmatrix} G^T(A^T)^{q-1} \\ \cdots \\ G^TA^T \\ G^T \end{pmatrix} M^T((A^T)^{r-1}C^T \ldots A^TC^TC^T)$$

$$= \begin{pmatrix} C & G^T(A^T)^{q-1} \\ CA & \cdots \\ \cdots & G^TA^T \\ CA^{q-1} & G^T \end{pmatrix} \begin{pmatrix} M & 0 \\ 0 & M^T \end{pmatrix} \begin{pmatrix} G & AG & \cdots & A^{r-1}G \\ (A^T)^{r-1}C^T & \cdots & A^TC^T & C^T \end{pmatrix} = \Gamma_q . M . \Delta_r + \tilde{\Delta}_q^T . M^T . \tilde{\Gamma}_r^T = (\Gamma_q \;\; \tilde{\Delta}_q^T) \cdot \begin{pmatrix} M & 0 \\ 0 & M^T \end{pmatrix} \cdot \begin{pmatrix} \Delta_r \\ \tilde{\Gamma}_r^T \end{pmatrix}. \tag{B.1}$$

This coincides exactly with equation (17). The other claims of Theorem 2 follow trivially from equation (B.1).

APPENDIX C. PROOF OF EQUATION (26)

From equation (19) we find:

$$\mathscr{U}(T_2 \;\; T_1)\begin{pmatrix} M^T & 0 \\ 0 & M^T \end{pmatrix} = (\tilde{\Delta}_q^T \;\; \Gamma_q)\begin{pmatrix} M^T & 0 \\ 0 & M^T \end{pmatrix} \tag{C.1}$$

From equation (20) on the other hand, we find:

$$\begin{pmatrix} M & 0 \\ 0 & M^T \end{pmatrix} \cdot \begin{pmatrix} \tilde{\Delta}_r \\ \Gamma_r^T \end{pmatrix} = T^{-1}\mathscr{V}\Pi. \tag{C.2}$$

Combining (C.1) and (C.2) leads to

$$\mathscr{U}(T_2 \;\; T_1)\begin{pmatrix} M^T & 0 \\ 0 & M \end{pmatrix} = \Pi\mathscr{V}^T(T^T)^{-1}.$$

This can be rewritten as

$$\mathscr{U}^+\Pi\mathscr{V}^T = (T_2 \;\; T_1)\begin{pmatrix} MT & 0 \\ 0 & M \end{pmatrix}\begin{pmatrix} T_1^T \\ T_2^T \end{pmatrix}$$

$$= T_1MT_2^T + T_2M^TT_1^T,$$

which is equation (26).