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Convergence of an algorithm for the Riemannian SVD

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1 Problem statement

The Riemannian SVD of a given matrix $A \in \mathbf{R}^{p\times q}$ is a nonlinear generalization of the SVD:

Riemannian SVD

$$\begin{aligned} A\,v &= D_v\,u\,\tau\,, & u^T D_v u = 1\,,\ \ v^T v = 1\,, \\ A^T\,u &= D_u\,v\,\tau\,, & v^T D_u v = 1\,. \end{aligned} \qquad (7.1)$$

Here $u \in \mathbf{R}^p$ and $v \in \mathbf{R}^q$ are a left, resp. right singular vector and $\tau \in \mathbf{R}$ is a singular value. The matrices $D_u \in \mathbf{R}^{q\times q}$ and $D_v \in \mathbf{R}^{p\times p}$ are symmetric positive definite matrix functions, the elements of which are quadratic in the components of u, resp. v.

The singular triplet (u, τ, v) corresponding to the smallest singular value τ, provides the solution to a so-called stuctured and/or weighted total least squares problems (STLS and/or WTLS), which, for a given data matrix $A \in \mathbf{R}^{p\times q}$ and given nonnegative weights w_{ij} (including the limiting cases 0 and $+\infty$), is the following

$$\min_{B\in\mathbf{R}^{p\times q}\,,\ \mathrm{rank}(B)=q-1\,,\ B\ \mathrm{structured}} \sum_{i=1}^{p}\sum_{j=1}^{q} (a_{ij} - b_{ij})^2 w_{ij}\ . \qquad (7.2)$$

By 'B structured', we mean so-called affine structures, i.e. $B = B_0 + B_1\beta_1 + \ldots + B_N\beta_N$, where the matrices $B_i \in \mathbf{R}^{p\times q}$ are fixed and the scalars $\beta_i \in \mathbf{R}$ are the parameters. Examples of such structured matrices are (centro- and per-) symmetric matrices, (block-)Hankel, (block-)Toeplitz, circulant, Brownian, Hankel + Toeplitz matrices, matrices with a certain zero structure (sparsity pattern), etc ...[1] Structured/weighted total least squares problems have a wide variety of applications in signal processing and system identification (see e.g. [3] [4] [6] [5] or consult the search engine at http://www.esat.kuleuven.ac.be/sista/publications). The precise structure of the matrices D_u and D_v in the Riemannian SVD depends on the structure of A and B and the given weights. As an example, when B is required to be a $p \times q$ rank deficient Hankel matrix, and when all weights in (7.2) are 1, we have

$$D_u = T_u T_u^T \ , \ D_v = T_v T_v^T \ ,$$

where $T_u \in \mathrm{R}^{p\times(p+q-1)}$ is a banded Toeplitz matrix with the components of u as

$$T_u = \begin{pmatrix} u_1 & u_2 & \dots & \dots & u_{p-1} & u_p & 0 & \dots & 0 & 0 \\ 0 & u_1 & u_2 & \dots & u_{p-2} & u_{p-1} & u_p & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & u_1 & \dots & \dots & u_p \end{pmatrix} \qquad (7.3)$$

and $T_v \in \mathrm{R}^{p\times(p+q-1)}$ is constructed similarly from the components of v. We refer to [4] for a derivation and details.

The quest for the minimal singular value τ in the Riemannian SVD, originates in the fact that τ^2 is equal to the objective function in (7.2) (see e.g. [3] [4] for a derivation). For invertible D_v, we have $D_v^{-1}Av = u\tau$, hence the generalized (nonlinear) eigenvalue problem

$$A^T D_v^{-1} A v = D_u v \tau^2 \ . \qquad (7.4)$$

Premultiplication with v^T, shows that finding the minimal singular value τ, is equivalent with finding the minimizing solution v of the constrained optimization problem[2]:

$$\min_v \ \tau^2 = v^T A^T D_v^{-1} A v \ \text{ subject to } \ v^T v = 1 \ . \qquad (7.5)$$

2 A convergent algorithm ?

Of course, since finding the minimal singular value τ of the Riemannian SVD is equivalent with finding the minimum of the constrained nonlinear optimization problem (7.5), one could apply 'classical' optimization techniques such as Gauss-Newton and its variants (see e.g. [1] where STLS/WTLS problems are treated this way). But when applying this approach, we ignore the 'SVD-like' character of the solution. If for instance we calculate the minimal singular value of a matrix, we use a 'dedicated' algorithm for doing so, even if we can formulate that problem as a nonlinear constrained optimization problem. There are indications that suggest that a 'dedicated' algorithm for the Riemannian SVD might be conceivable. Indeed, observe that, for fixed D_u and D_v (i.e. D_u and D_v are not a function of u, resp. v), the Riemannian SVD reduces to the so-called *restricted SVD* described in [2]. For this case, the minimal singular value may be found by the classical century-old (inverse) power method (see e.g. [7]). This simple observation leads to the following proposal for a heuristic power method approach to find the minimal singular value of the Riemannian SVD, which is based on the generalized eigenvalue problem (7.4), in which we fix the matrices D_u and D_v in each iteration step:

(Inverse) power method for Riemannian SVD **Initialize:**
$u^{[0]}$, $v^{[0]}$, $D_{u^{[0]}}, D_{v^{[0]}}, k = 0$.
Iterate until convergence:
$v^{[k+1]} = (A^T D_{v^{[k]}} A)^{-1} \ D_{u^{[k]}} v_k$
$v^{[k+1]} = v^{[k+1]}/||v^{[k+1]}||$
Update $D_{v^{[k+1]}}$
$u^{[k+1]} = D^{-1}_{v^{[k+1]}} A v^{[k+1]}$
$u^{[k+1]} = u^{[k+1]}/||(u^{[k+1]})^T D_{v^{[k+1]}} u^{[k+1]}||$
Update $D_{u^{[k+1]}}$

On numerical try outs, this algorithm seems to work well (we omit implementation details here). It convergence rate seems to be linear. The following questions now apply:

- Can something rigoreus be proven about the convergence behavior and rate of this 'heuristic' method ?

- In particular, the method we propose is like a power method, in which in each iteration the 'metric', as represented by D_u and D_v, is updated. The adaptation of the positive definite matrices D_u and D_v in each iteration step is reminiscent of so-called 'variable metric' methods. Can these observations be exploited in a convergence analysis?

- What about local and global minima ? (Observe that for fixed positive definite D_u and D_v, there is only one minimum, which is global. In this case, the other stationary points of (7.4) and (7.5) are saddle-points).

[1] When B is unstructured, and all weights are 1, we have the well-known *total linear least squares problem* (TLLS), the solution of which is given by the singular value decomposition (SVD) of A (put $D_u = I_q$ and $D_v = I_p$ in (7.1)); For references on TLLS see e.g. [6] [7] [8].

[2] When v is a minimizer, we can find u as follows: Call $z = D_v^{-1}Av$. Then it follows from the objective function (7.5) that $z^T D_v z = \tau^2$. Call $u = z/\tau$. It follows that $Av = D_v u\tau$. Furthermore, we always have the equality $v^T D_u v = u^T D_v u$. Hence the equivalence.

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Motion planning for some parameter distributed systems

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1 Motivation

In [1, 2] a method has been proposed for obtaining the motion planning of some infinite-dimensional systems governed by linear partial differential equations with constant coefficients and a single space variable. It has been successfully applied to a flexible robot in [3]. We would like to discuss its possible extension to variable coefficients and several space variables.

2 Description of the problem

Consider the Euler-Bernoulli partial differential equation with constant coefficients and one space variable:

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} = 0 \tag{8.1}$$

Classic operational calculus replaces (8.1) by the following linear ordinary differential equation

$$\hat{w}^{(4)} + s^2 \hat{w} = 0 \tag{8.2}$$

if the initial conditions are zero. The general solution of (8.2) reads

$$\hat{w}(s,x) = ae^{x\frac{1+i}{4}\sqrt{s}} \tag{8.3}$$

where the coefficients are determined from the boundary conditions which involve the control variables.