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On the nonuniqueness of the factorization factors in the product singular value decomposition ☆

Delin Chu [a,*], Bart De Moor [b,1]

[a] *Department of Mathematics, National University of Singapore, Lower Kent Road, Singapore 119260, Singapore*

[b] *Department of Electrical Engineering (ESAT), Research Group SISTA, Katholieke Universiteit Leuven, Kardinaal Mercierlaan 94, B-3001 Leuven, Belgium*

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Abstract

The product singular value decomposition (PSVD) of two matrices is revisited in this paper. The nonuniqueness of the factorization factors in the PSVD is characterized in a way different from that in existing work. © 2000 Elsevier Science Inc. All rights reserved.

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* Corresponding author. Fax: +65-7795452.

E-mail addresses: matchudl@math.nus.edu.sg (D. Chu), Bart.Demoor@esat.kuleuven.ac.be (B. De Moor).

[1] Tel.: +32-16-32-1970; fax: +32-16-32-1709.

PII: S 0 0 2 4 - 3 7 9 5 (0 0) 0 0 1 3 6 - 1

1. Introduction

The product singular value decomposition (PSVD) is a generalization for two matrices of the (ordinary) singular value decomposition (SVD) for one matrix. The explicit formulation of the PSVD was given for the first time by Fernando and Hammarling [7], who called it the ΠSVD. In this paper, unless noted, we always denote

$$r_a = \operatorname{rank}(A), \quad r_b = \operatorname{rank}(B), \quad r_{ab} = \operatorname{rank}\left(AB^{\mathrm{T}}\right), \quad X^{-\mathrm{T}} = (X^{-1})^{\mathrm{T}}$$

for any given matrices A, B of appropriate dimensions and nonsingular matrix X. Let us first state the following:

Theorem 1 (The PSVD Theorem). *Given matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times n}$. Then there exist orthogonal matrices $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{p\times p}$ and a nonsingular matrix $X \in \mathbb{R}^{n\times n}$ such that*

$$\begin{array}{c} \\ UAX = \end{array} \begin{array}{c} \\ r_{ab} \\ r_a - r_{ab} \\ m - r_a \end{array} \begin{array}{c} \begin{array}{cccc} r_{ab} & r_a - r_{ab} & r_b - r_{ab} & n + r_{ab} - r_a - r_b \end{array} \\ \left[\begin{array}{cccc} S & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array},$$

$$\begin{array}{c} \\ VBX^{-\mathrm{T}} = \end{array} \begin{array}{c} \\ r_{ab} \\ r_b - r_{ab} \\ p - r_b \end{array} \begin{array}{c} \begin{array}{cccc} r_{ab} & r_a - r_{ab} & r_b - r_{ab} & n + r_{ab} - r_a - r_b \end{array} \\ \left[\begin{array}{cccc} S & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}, \tag{1}$$

where

$$S = \operatorname{diag}\{\sigma_1 I_{i_1}, \sigma_2 I_{i_2}, \ldots, \sigma_k I_{i_k}\}, \quad \sigma_1 > \sigma_2 > \cdots > \sigma_k > 0, \ \sum_{j=1}^{k} i_j = r_{ab}.$$

Observe that

$$\begin{array}{c} \\ U\left(AB^{\mathrm{T}}\right)V^{\mathrm{T}} = \end{array} \begin{array}{c} \\ r_{ab} \\ r_a - r_{ab} \\ m - r_a \end{array} \begin{array}{c} \begin{array}{ccc} r_{ab} & r_b - r_{ab} & p - r_b \end{array} \\ \left[\begin{array}{ccc} SS^{\mathrm{T}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array},$$

hence, SS^{T} contains the singular values of AB^{T}.

Algorithmic ideas to implement the PSVD in a numerically reliable way can be found in [2,7]. Applications include the orthogonal Procrustes problem [1], computing balancing transformations for state space systems [7,9], and computing the Kalman decomposition of a linear system [8]. The PSVD could also be applied in the computation of approximate intersections between subspaces in the stochastic realization problem [6], as an alternative to canonical correlation analysis.

The structure and geomety of the PSVD have been studied in [4]. In particular, the nonuniqueness of the factorization factors in the PSVD has been analyzed in detail. In this paper, we revisit the PSVD. Our purpose is to characterize the nonuniqueness of the factors in the PSVD in a way different from that in [4].

2. Main result

Before we state our main result, we need a technical lemma.

Lemma 2. *Given $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times n}$. Then there exist orthogonal matrices $U_a \in \mathbb{R}^{m\times m}$, $V_b \in \mathbb{R}^{p\times p}$ and $Q_{ab} \in \mathbb{R}^{n\times n}$ such that*

$$U_a A Q_{ab} = \begin{array}{c} \\ r_{ab} \\ r_a - r_{ab} \\ m - r_a \end{array} \begin{array}{c} \begin{array}{cccc} r_{ab} & r_a - r_{ab} & r_b - r_{ab} & n + r_{ab} - r_a - r_b \end{array} \\ \begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ 0 & A_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array},$$

$$V_b B Q_{ab} = \begin{array}{c} \\ r_{ab} \\ r_b - r_{ab} \\ p - r_b \end{array} \begin{array}{c} \begin{array}{cccc} r_{ab} & r_a - r_{ab} & r_b - r_{ab} & n + r_{ab} - r_a - r_b \end{array} \\ \begin{bmatrix} B_{11} & 0 & B_{13} & B_{14} \\ 0 & 0 & B_{23} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}, \tag{2}$$

where A_{11}, A_{22}, B_{11}, B_{23} are nonsingular.

Proof. See Appendix A. □

Based on Lemma 2, we can prove Theorem 1. In fact, we have:

Corollary 3. *Given $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times n}$. Let orthogonal matrices U_a, V_b and Q_{ab} be defined as in Lemma 2. Assume that the SVD of $A_{11}B_{11}^{\mathrm{T}}$ is*

$$U_{11}A_{11}B_{11}^{\mathrm{T}}V_{11}^{\mathrm{T}} = S^2, \tag{3}$$

where U_{11}, V_{11} are orthogonal and S is defined as in Theorem 1. Define

$$U = \begin{bmatrix} U_{11} & \\ & I_{m-r_{ab}} \end{bmatrix} U_a, \qquad V = \begin{bmatrix} V_{11} & \\ & I_{p-r_{ab}} \end{bmatrix} V_b,$$

$$X = Q_{ab} \begin{bmatrix} A_{11}^{-1}U_{11}^{\mathrm{T}}S & -A_{11}^{-1}A_{12}A_{22}^{-1} & 0 & 0 \\ 0 & A_{22}^{-1} & 0 & 0 \\ B_{13}^{\mathrm{T}}V_{11}^{\mathrm{T}}S^{-1} & 0 & B_{23}^{\mathrm{T}} & 0 \\ B_{14}^{\mathrm{T}}V_{11}^{\mathrm{T}}S^{-1} & 0 & 0 & I \end{bmatrix}.$$

Then UAX *and* $VBX^{-\mathrm{T}}$ *are in the form* (1).

Corollary 3 provides an alternative and very simple way to characterize the PSVD, which is different from the derivation in [4,5,7]. We are now in the position to present our main result.

Theorem 4 (Main Result). *Given* $A \in \mathbb{R}^{m\times n}$ *and* $B \in \mathbb{R}^{p\times n}$. *Let orthogonal matrices* $U_a,\ V_b,\ Q_{ab},\ U_{11}$ *and* V_{11} *be defined as in Lemma* 2 *and Corollary* 3. *Assume that* $U \in \mathbb{R}^{m\times m}$ *and* $V \in \mathbb{R}^{p\times p}$ *are orthogonal and* $X \in \mathbb{R}^{n\times n}$ *is nonsingular. Then* UAX *and* $VBX^{-\mathrm{T}}$ *are in the form* (1) *if and only if*

$$U = \begin{array}{c} r_{ab} \\ r_a - r_{ab} \\ m - r_a \end{array} \overset{\begin{array}{ccc} r_{ab} & r_a - r_{ab} & m - r_a \end{array}}{\begin{bmatrix} P_{11}U_{11} & & \\ & P_{22} & \\ & & P_{33} \end{bmatrix}} U_a,$$

$$V = \begin{array}{c} r_{ab} \\ r_b - r_{ab} \\ p - r_b \end{array} \overset{\begin{array}{ccc} r_{ab} & r_b - r_{ab} & p - r_b \end{array}}{\begin{bmatrix} P_{11}V_{11} & & \\ & W_{22} & \\ & & W_{33} \end{bmatrix}} V_b, \tag{4}$$

$$X = Q_{ab} \begin{bmatrix} A_{11}^{-1}U_{11}^{\mathrm{T}}SP_{11}^{\mathrm{T}} & -A_{11}^{-1}A_{12}A_{22}^{-1}P_{22}^{\mathrm{T}} & 0 & 0 \\ 0 & A_{22}^{-1}P_{22}^{\mathrm{T}} & 0 & 0 \\ B_{13}^{\mathrm{T}}V_{11}^{\mathrm{T}}S^{-1}P_{11}^{\mathrm{T}} & X_{32} & B_{23}^{\mathrm{T}}W_{22}^{\mathrm{T}} & X_{34} \\ B_{14}^{\mathrm{T}}V_{11}^{\mathrm{T}}S^{-1}P_{11}^{\mathrm{T}} & X_{42} & 0 & X_{44} \end{bmatrix},$$

where $P_{11},\ P_{22},\ P_{33},\ W_{22}$ *and* W_{33} *are arbitrary orthogonal matrices,* P_{11} *the block-diagonal*:

$$P_{11} = \begin{array}{c} i_1 \\ i_2 \\ \vdots \\ i_k \end{array} \overset{\begin{array}{cccc} i_1 & i_2 & \cdots & i_k \end{array}}{\begin{bmatrix} P_{11}^{(1)} & & & \\ & P_{11}^{(2)} & & \\ & & \ddots & \\ & & & P_{11}^{(k)} \end{bmatrix}}, \tag{5}$$

$X_{32},\ X_{34}$ *and* X_{42} *the real and arbitrary, and* X_{44} *is arbitrary real nonsingular matrix.*

Obviously, Theorem 4 gives a complete characterization of the nonuniqueness property of the factorization factors in the PSVD.

3. The proof of Theorem 4

In order to prove Theorem 4, we need two preliminary lemmas, in which the first one is well known and the second one is a direct consequence of QR and QL factorizations.

Lemma 5. *The SVD of AB^{T} is given by*

$$\left(\begin{bmatrix} U_{11} & \\ & I_{m-r_{ab}} \end{bmatrix} U_a\right) AB^{\mathrm{T}} \left(\begin{bmatrix} V_{11} & \\ & I_{p-r_{ab}} \end{bmatrix} V_b\right)^{\mathrm{T}} = \begin{bmatrix} S^2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Furthermore, if orthogonal matrices U and V satisfy

$$UAB^{\mathrm{T}}V^{\mathrm{T}} = \begin{bmatrix} S^2 & 0 \\ 0 & 0 \end{bmatrix},$$

then

$$U = \begin{matrix} r_{ab} \\ r_a - r_{ab} \\ m - r_a \end{matrix} \overset{\begin{matrix} r_{ab} & \quad r_a - r_{ab} & \quad m - r_a \end{matrix}}{\begin{bmatrix} P_{11}U_{11} & & \\ & P_{22} & P_{23} \\ & P_{32} & P_{33} \end{bmatrix}} U_a,$$

$$V = \begin{matrix} r_{ab} \\ r_b - r_{ab} \\ p - r_b \end{matrix} \overset{\begin{matrix} r_{ab} & \quad r_b - r_{ab} & \quad p - r_b \end{matrix}}{\begin{bmatrix} P_{11}V_{11} & & \\ & W_{22} & W_{23} \\ & W_{32} & W_{33} \end{bmatrix}} V_b, \tag{6}$$

where

$$P_{11}, \quad \begin{bmatrix} P_{22} & P_{23} \\ P_{32} & P_{33} \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} W_{22} & W_{23} \\ W_{32} & W_{33} \end{bmatrix}$$

are orthogonal, and P_{11} is of the block-diagonal form (5).

Proof. The proof is trivial. □

Lemma 6. *Let $X \in \mathbb{R}^{n \times n}$ be nonsingular. Then there exists an orthogonal matrix Q such that*

$$Q^{\mathrm{T}}X = \begin{matrix} r_{ab} \\ r_a - r_{ab} \\ r_b - r_{ab} \\ n + r_{ab} - r_a - r_b \end{matrix} \overset{\begin{matrix} r_{ab} & \quad r_a - r_{ab} & \quad r_b - r_{ab} & \quad n + r_{ab} - r_a - r_b \end{matrix}}{\begin{bmatrix} L_{11} & L_{12} & 0 & 0 \\ 0 & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & L_{34} \\ L_{41} & L_{42} & L_{43} & L_{44} \end{bmatrix}}, \tag{7}$$

where

$$L_{11},\quad L_{22}\quad and\quad \begin{bmatrix} L_{33} & L_{34} \\ L_{43} & L_{44} \end{bmatrix}$$

are nonsingular.

Proof. We partition X to be

$$X = n \begin{array}{c} \\ \left[\right. \end{array} \begin{array}{cccc} r_{ab} & r_a - r_{ab} & r_b - r_{ab} & n + r_{ab} - r_a - r_b \\ X_1 & X_2 & X_3 & X_4 \end{array} \begin{array}{c} \\ \left.\right]. \end{array}$$

Let $\tilde{Q}$ be such that

$$\tilde{Q}^{\mathrm{T}} \begin{bmatrix} X_3 & X_4 \end{bmatrix} = \begin{array}{c} r_{ab} \\ r_a - r_{ab} \\ r_b - r_{ab} \\ n + r_{ab} - r_a - r_b \end{array} \overset{\begin{array}{cc} r_b - r_{ab} & n + r_{ab} - r_a - r_b \end{array}}{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ L_{33} & L_{34} \\ L_{43} & L_{44} \end{bmatrix}},$$

and denote

$$\tilde{Q}^{\mathrm{T}} \begin{bmatrix} X_1 & X_2 \end{bmatrix} = \begin{array}{c} r_{ab} \\ r_a - r_{ab} \\ r_b - r_{ab} \\ n + r_{ab} - r_a - r_b \end{array} \overset{\begin{array}{cc} r_{ab} & r_a - r_{ab} \end{array}}{\begin{bmatrix} \tilde{L}_{11} & \tilde{L}_{12} \\ \tilde{L}_{21} & \tilde{L}_{22} \\ L_{31} & L_{32} \\ L_{41} & L_{42} \end{bmatrix}}.$$

Now, there exists an orthogonal matrix $\hat{Q}$ satisfying

$$\hat{Q}^{\mathrm{T}} \begin{bmatrix} \tilde{L}_{11} & \tilde{L}_{12} \\ \tilde{L}_{21} & \tilde{L}_{22} \end{bmatrix} = \begin{array}{c} r_{ab} \\ r_a - r_{ab} \end{array} \overset{\begin{array}{cc} r_{ab} & r_a - r_{ab} \end{array}}{\begin{bmatrix} L_{11} & L_{12} \\ 0 & L_{22} \end{bmatrix}}.$$

Set

$$Q = \tilde{Q} \begin{bmatrix} \hat{Q} & \\ & I \end{bmatrix}.$$

Then Q is such that (7) holds. □

Now we prove Theorem 4.

Proof of Theorem 4. First we prove the sufficiency, then prove the necessity.

Sufficiency: Assume that $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{p\times p}$ are orthogonal, $X \in \mathbb{R}^{n\times n}$ is nonsingular, and UAX and $VBX^{-\mathrm{T}}$ are in the form (1). Then $UAB^{\mathrm{T}}V^{\mathrm{T}}$ is the SVD of AB^{T}, and hence, by Lemma 5, U and V are of the form (6). Note that

$$\begin{bmatrix} 0 & P_{32} & P_{33} \end{bmatrix} U_a A X = 0, \qquad \begin{bmatrix} 0 & W_{32} & W_{33} \end{bmatrix} V_b B X^{-\mathrm{T}} = 0.$$

This implies that

$$\begin{bmatrix} 0 & P_{32} & P_{33} \end{bmatrix} U_a A Q_{ab} = 0, \qquad \begin{bmatrix} 0 & W_{32} & W_{33} \end{bmatrix} V_b B Q_{ab} = 0.$$

So,

$$P_{32} = 0, \qquad W_{32} = 0. \tag{8}$$

Consider that

$$\begin{bmatrix} P_{22} & P_{23} \\ P_{32} & P_{33} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} W_{22} & W_{23} \\ W_{32} & W_{33} \end{bmatrix}$$

are orthogonal. Hence, we also have

$$P_{23} = 0, \qquad W_{23} = 0. \tag{9}$$

By (8) and (9), we have that U and V are in the form (7).

According to Lemma 6, there exists an orthogonal matrix Q such that $Q^{\mathrm{T}}X$ is of the form (7). Set

$$Q = \begin{bmatrix} Q_1 & Q_2 & Q_3 & Q_4 \end{bmatrix}.$$

Then

$$A \begin{bmatrix} Q_3 & Q_4 \end{bmatrix} = 0. \tag{10}$$

But, let us partition Q_{ab} in Lemma 2 into

$$Q_{ab} = \begin{array}{c} \\ \left[\right. \end{array}\!\!\begin{array}{cccc} r_{ab} & r_a - r_{ab} & r_b - r_{ab} & n + r_{ab} - r_a - r_b \\ Q_{ab1} & Q_{ab2} & Q_{ab3} & Q_{ab4} \end{array}\!\!\begin{array}{c} \\ \left.\right]. \end{array}$$

Then we know

$$A \begin{bmatrix} Q_{ab3} & Q_{ab4} \end{bmatrix} = 0.$$

Hence,

$$\begin{bmatrix} Q_3 & Q_4 \end{bmatrix} = \begin{bmatrix} Q_{ab3} & Q_{ab4} \end{bmatrix} \begin{bmatrix} \tilde{Q}_{33} & \tilde{Q}_{34} \\ \tilde{Q}_{43} & \tilde{Q}_{44} \end{bmatrix},$$

$$\begin{bmatrix} Q_1 & Q_2 \end{bmatrix} = \begin{bmatrix} Q_{ab1} & Q_{ab2} \end{bmatrix} \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} \\ \tilde{Q}_{21} & \tilde{Q}_{22} \end{bmatrix}$$

with

$$\begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} \\ \tilde{Q}_{21} & \tilde{Q}_{22} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \tilde{Q}_{33} & \tilde{Q}_{34} \\ \tilde{Q}_{43} & \tilde{Q}_{44} \end{bmatrix}$$

orthogonal. Since we also have

$$BQ_2 = 0, \qquad BQ_{ab2} = 0,$$

so,

$$\tilde{Q}_{21} = 0, \qquad \tilde{Q}_{12} = 0.$$

Hence, we have

$$Q = Q_{ab} \times \begin{matrix} r_{ab} \\ r_a - r_{ab} \\ r_b - r_{ab} \\ n + r_{ab} - r_a - r_b \end{matrix} \overset{\begin{matrix} r_{ab} & r_a - r_{ab} & r_b - r_{ab} & n + r_{ab} - r_a - r_b \end{matrix}}{\begin{bmatrix} \tilde{Q}_{11} & & & \\ & \tilde{Q}_{22} & & \\ & & \tilde{Q}_{33} & \tilde{Q}_{34} \\ & & \tilde{Q}_{43} & \tilde{Q}_{44} \end{bmatrix}}$$

with

$$\tilde{Q}_{11}, \quad \tilde{Q}_{22} \quad \text{and} \quad \begin{bmatrix} \tilde{Q}_{33} & \tilde{Q}_{34} \\ \tilde{Q}_{43} & \tilde{Q}_{44} \end{bmatrix}$$

orthogonal. We can write X to be

$$X = Q_{ab} \begin{bmatrix} \tilde{Q}_{11} & & & \\ & \tilde{Q}_{22} & & \\ & & \tilde{Q}_{33} & \tilde{Q}_{34} \\ & & \tilde{Q}_{43} & \tilde{Q}_{44} \end{bmatrix} \begin{bmatrix} L_{11} & L_{12} & 0 & 0 \\ 0 & L_{22} & 0 & 0 \\ L_{31} & L_{32} & L_{33} & L_{34} \\ L_{41} & L_{42} & L_{43} & L_{44} \end{bmatrix}$$

$$= Q_{ab} \times \begin{matrix} r_{ab} \\ r_a - r_{ab} \\ r_b - r_{ab} \\ n + r_{ab} - r_a - r_b \end{matrix} \overset{\begin{matrix} r_{ab} & r_a - r_{ab} & r_b - r_{ab} & n + r_{ab} - r_a - r_b \end{matrix}}{\begin{bmatrix} X_{11} & X_{12} & 0 & 0 \\ 0 & X_{22} & 0 & 0 \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{bmatrix}}.$$

Obviously,

$$X_{11}, \quad X_{22} \quad \text{and} \quad \begin{bmatrix} X_{33} & X_{34} \\ X_{43} & X_{44} \end{bmatrix}$$

are nonsingular. Now we have

$$\begin{bmatrix} P_{11}U_{11} & & \\ & P_{22} & \\ & & P_{33} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ 0 & A_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} & 0 & 0 \\ 0 & X_{22} & 0 & 0 \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{bmatrix}$$
$$= \begin{bmatrix} S & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{11}$$

and

$$\begin{bmatrix} P_{11}V_{11} & & \\ & W_{22} & \\ & & W_{33} \end{bmatrix} \begin{bmatrix} B_{11} & 0 & B_{13} & B_{14} \\ 0 & 0 & B_{23} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} S & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_{11} & X_{12} & 0 & 0 \\ 0 & X_{22} & 0 & 0 \\ X_{31} & X_{32} & X_{33} & X_{34} \\ X_{41} & X_{42} & X_{43} & X_{44} \end{bmatrix}^{\mathrm{T}}. \tag{12}$$

A simple calculation yields that (11) and (12) hold if and only if

$$\begin{aligned} &P_{11}U_{11}A_{11}X_{11} = S, \\ &P_{22}A_{22}X_{22} = I, \\ &P_{11}U_{11}(A_{11}X_{12} + A_{12}X_{22}) = 0 \end{aligned} \tag{13}$$

and

$$\begin{aligned} &P_{11}V_{11}B_{11} = SX_{11}^{\mathrm{T}}, \\ &P_{11}V_{11}B_{13} = SX_{31}^{\mathrm{T}}, \\ &W_{22}B_{23} = X_{33}^{\mathrm{T}}, \\ &P_{11}V_{11}B_{14} = SX_{41}^{\mathrm{T}}, \\ &X_{43}^{\mathrm{T}} = 0. \end{aligned} \tag{14}$$

Equivalently, (11) and (12) hold if and only if

$$\begin{aligned} &X_{11} = A_{11}^{-1}U_{11}^{\mathrm{T}}SP_{11}^{\mathrm{T}}, \\ &X_{22} = A_{22}^{-1}P_{22}^{\mathrm{T}}, \\ &X_{12} = -A_{11}^{-1}A_{12}A_{22}^{-1}P_{22}^{\mathrm{T}}, \\ &X_{33} = B_{23}^{\mathrm{T}}W_{22}^{\mathrm{T}}, \\ &X_{31} = B_{13}^{\mathrm{T}}V_{11}^{\mathrm{T}}P_{11}^{\mathrm{T}}S^{-1} = B_{13}^{\mathrm{T}}V_{11}^{\mathrm{T}}S^{-1}P_{11}^{\mathrm{T}}, \\ &X_{41} = B_{14}^{\mathrm{T}}V_{11}^{\mathrm{T}}P_{11}^{\mathrm{T}}S^{-1} = B_{14}^{\mathrm{T}}V_{11}^{\mathrm{T}}S^{-1}P_{11}^{\mathrm{T}}, \\ &X_{43} = 0. \end{aligned} \tag{15}$$

Moreover, (15) also implies that $X_{32},\ X_{42}$, X_{34} and X_{44} are arbitrarily, and X_{44} is nonsingular because

$$\begin{bmatrix} X_{33} & X_{34} \\ X_{43} & X_{44} \end{bmatrix}$$

is nonsingular and $X_{43} = 0$. Therefore, X is also in the form (4). Up to now, we have completed the proof of sufficiency in Theorem 4.

Necessity: Let orthogonal matrices $U \in \mathbb{R}^{m\times m},\ V \in \mathbb{R}^{p\times p}$ and nonsingular matrix $X \in \mathbb{R}^{n\times n}$ are in the form (4). Then a simple calculation gives that

$$UAX = \begin{bmatrix} P_{11}SP_{11}^{\mathrm{T}} & 0 & 0 & 0 \\ 0 & P_{22}P_{22}^{\mathrm{T}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} S & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{16}$$

and

$$\begin{bmatrix} S & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_{11}^{-1}U_{11}^{\mathrm{T}}SP_{11}^{\mathrm{T}} & -A_{11}^{-1}A_{12}A_{22}^{-1}P_{22}^{\mathrm{T}} & 0 & 0 \\ 0 & A_{22}^{-1}P_{22}^{\mathrm{T}} & 0 & 0 \\ B_{13}^{\mathrm{T}}V_{11}^{\mathrm{T}}S^{-1}P_{11}^{\mathrm{T}} & X_{32} & B_{23}^{\mathrm{T}}W_{22}^{\mathrm{T}} & X_{34} \\ B_{14}^{\mathrm{T}}V_{11}^{\mathrm{T}}S^{-1}P_{11}^{\mathrm{T}} & X_{42} & 0 & X_{44} \end{bmatrix}^{\mathrm{T}}$$

$$= \begin{bmatrix} SP_{11}SU_{11}A_{11}^{-\mathrm{T}} & 0 & SP_{11}S^{-1}V_{11}B_{13} & SP_{11}S^{-1}V_{11}B_{14} \\ 0 & 0 & W_{22}B_{23} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} P_{11}V_{11}B_{11} & 0 & P_{11}V_{11}B_{13} & P_{11}V_{11}B_{14} \\ 0 & 0 & W_{22}B_{23} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} P_{11}V_{11} & & \\ & W_{22} & \\ & & W_{33} \end{bmatrix} \begin{bmatrix} B_{11} & 0 & B_{13} & B_{14} \\ 0 & 0 & B_{23} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{17}$$

equivalently, we have

$$VBX^{-\mathrm{T}} = \begin{bmatrix} S & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{18}$$

In (17), we have used the following equalities:

$$SP_{11} = P_{11}S, \qquad S^{\mathrm{T}} = S, \qquad V_{11}B_{11} = S^2U_{11}A_{11}^{-\mathrm{T}}.$$

Hence, the “necessity” follows directly from (16) and (18). □

4. Conclusion

In this paper, the nonuniqueness of the factorization factors in the PSVD has been characterized in a way different from that in [4].

Appendix A

Before we prove Lemma 2, we need to recall the QR factorization with column pivoting and URV decomposition [1], which will be the building blocks of our constructive proof of Lemma 2.

It is well known that any matrix $A \in \mathbb{R}^{m\times n}$ can be factorized as

$$UA = \begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix} \Pi, \tag{A.1}$$

where U and Π are orthogonal matrix and permutation matrix, respectively, R_1 is nonsingular and upper triangular. The factorization (A.1) is called the QR factorization of A with column pivoting.

If we continue to squeeze $[R_1 \;\; R_2]$ into upper triangular form by applying a sequence of Householder transformations, then we have the following URV decomposition of A, i.e., we get an orthogonal matrix V such that

$$UAV = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix} \tag{A.2}$$

with R nonsingular and upper triangular.

Now we are ready to present a constructive proof for Lemma 2.

Proof of Lemma 2. We prove Lemma 2 constructively by following six steps:

Step 1: Compute the URV decomposition of A:

$$\hat{U}_1 A \hat{Q}_1 = \begin{matrix} r_a \\ m - r_a \end{matrix} \overset{\begin{matrix} r_a & \quad n - r_a \end{matrix}}{\begin{bmatrix} A_1^{(1)} & 0 \\ 0 & 0 \end{bmatrix}},$$

where $A_1^{(1)}$ is nonsingular.

Step 2: Compute the QR factorization of $B\hat{Q}_1$ with column pivoting:

$$\hat{V}_1(B\hat{Q}_1) = \begin{matrix} r_b \\ p - r_b \end{matrix} \overset{\begin{matrix} r_a & \quad n - r_a \end{matrix}}{\begin{bmatrix} B_1^{(1)} & B_3^{(1)} \\ 0 & 0 \end{bmatrix}},$$

where $\begin{bmatrix} B_1^{(1)} & B_3^{(1)} \end{bmatrix}$ is of full row rank. Note that

$$r_{ab} = \operatorname{rank}\left(AB^{\mathrm{T}}\right) = \operatorname{rank}\left(A_1^{(1)} B_1^{(1)}\right) = \operatorname{rank}\left(B_1^{(1)}\right).$$

Step 3: Compute the QR factorization of $(B_1^{(1)})^{\mathrm{T}}$:

$$B_1^{(1)} \hat{Q}_2 = r_b \overset{\begin{matrix} r_{ab} & \quad r_a - r_{ab} \end{matrix}}{\begin{bmatrix} B_1^{(2)} & 0 \end{bmatrix}}$$

with $B_1^{(2)}$ of full column rank. Set

$$A_1^{(1)} \hat{Q}_2 = r_a \overset{\begin{matrix} r_{ab} & \quad r_a - r_{ab} \end{matrix}}{\begin{bmatrix} A_1^{(2)} & A_2^{(2)} \end{bmatrix}}.$$

Step 4: Compute the QR factorizations of $A_1^{(2)}$ and $B_1^{(2)}$ with column pivoting:

$$\hat{U}_3 \begin{bmatrix} A_1^{(2)} & A_2^{(2)} \end{bmatrix} = \begin{matrix} r_{ab} \\ r_a - r_{ab} \end{matrix} \overset{\begin{matrix} r_{ab} & \quad r_a - r_{ab} \end{matrix}}{\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}},$$

$$\hat{V}_3 \begin{bmatrix} B_1^{(2)} & B_3^{(1)} \end{bmatrix} = \begin{matrix} r_{ab} \\ r_b - r_{ab} \end{matrix} \overset{\begin{matrix} r_{ab} & n - r_a \end{matrix}}{\begin{bmatrix} B_{11} & B_{13}^{(3)} \\ 0 & B_{23}^{(3)} \end{bmatrix}},$$

where $A_{11},\ A_{22}$ and B_{11} are nonsingular. We also have that $B_{23}^{(3)}$ is of full row rank.

Step 5: Compute the QR factorization of $(B_{23}^{(3)})^{\mathrm{T}}$ with pivoting:

$$\begin{bmatrix} B_{13}^{(3)} \\ B_{23}^{(3)} \end{bmatrix} \hat{Q}_4 = \begin{matrix} r_{ab} \\ r_b - r_{ab} \end{matrix} \overset{\begin{matrix} r_b - r_{ab} & n + r_{ab} - r_a - r_b \end{matrix}}{\begin{bmatrix} B_{13} & B_{14} \\ B_{23} & 0 \end{bmatrix}},$$

where B_{23} is nonsingular.

Step 6: Set

$$U_a = \begin{bmatrix} \hat{U}_3 & \\ & I_{m-r_a} \end{bmatrix} \hat{U}_1,$$

$$V_b = \begin{bmatrix} \hat{V}_3 & \\ & I_{p-r_b} \end{bmatrix} \hat{V}_1,$$

$$Q_{ab} = \hat{Q}_1 \begin{bmatrix} \hat{Q}_2 & \\ & I_{n-r_a} \end{bmatrix} \begin{bmatrix} I_{r_a} & \\ & \hat{Q}_4 \end{bmatrix}.$$

Now we have

$$U_a A Q_{ab} = \begin{matrix} r_{ab} \\ r_a - r_{ab} \\ m - r_a \end{matrix} \overset{\begin{matrix} r_{ab} & r_a - r_{ab} & r_b - r_{ab} & n + r_{ab} - r_a - r_b \end{matrix}}{\begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ 0 & A_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}},$$

$$V_b B Q_{ab} = \begin{matrix} r_{ab} \\ r_b - r_{ab} \\ p - r_b \end{matrix} \overset{\begin{matrix} r_{ab} & r_a - r_{ab} & r_b - r_{ab} & n + r_{ab} - r_a - r_b \end{matrix}}{\begin{bmatrix} B_{11} & 0 & B_{13} & B_{14} \\ 0 & 0 & B_{23} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

with $A_{11},\ A_{22},\ B_{11}$ and B_{23} nonsingular. Therefore, Lemma 2 follows. □

In general, the size of A_{11} (i.e., the size of B_{11}) is much smaller than those of A and B. Moreover, the condensed form (2) can be computed via numerically stable ways. Hence, similar to [3], the condensed form (2) can be considered to be an efficient preprocessing algorithm for computing the PSVD of matrix pair (A, B). This preprocessing algorithm will reduce the complexity of the Kogbetliantz-type algorithm in [2]. Therefore, the PSVD of (A, B) can be computed in the following two phases:

- Reduce matrix pair (A, B) to the condensed form (2).
- Compute the PSVD of matrix pair (A_{11}, B_{11}) using the Kogbetliantz-type algorithm in [2].

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