STATE EQUATIONS OF NONLINEAR DYNAMIC SYSTEMS

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ABSTRACT

In a keynote paper [Rosenberg 1971] Rosenberg formulated a method for deriving the state equations starting from the bond graph model of a system, $\dot{\mathbf{x}}_{\mathbf{I}}(\mathbf{t}) = \mathbf{f}(\mathbf{x}_{\mathbf{I}}(\mathbf{t}), \dot{\mathbf{x}}_{\mathbf{I}}(\mathbf{t}), \mathbf{u}(\mathbf{t}), \dot{\mathbf{u}}(\mathbf{t}))$. This formulation hides certain model properties which depend on the energetical structure of the model (i.e. the a-causal bond graph) and the constraints on the dynamic variables involved (which stem from the causality analysis of the bond graph). Both aspects are useful in understanding the model behaviour. Besides this purpose, the models are derived for simulation or control design purposes. Due to practical considerations in both domains one has a strong preference for an explicit state space model description instead of the aforementioned one. It is shown that this formulation can be written in a closed form (namely a *descriptor form* which lies close to the state space formulation), $\mathbf{E}.\dot{\mathbf{x}}_{\mathbf{I}}(\mathbf{t}) = \mathbf{A}.\mathbf{x}_{\mathbf{I}}(\mathbf{t}) + \mathbf{B}.\mathbf{u}(\mathbf{t}) + \mathbf{G}.\dot{\mathbf{u}}(\mathbf{t})$, with $\mathbf{E}$, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{G}$ explicitely depending on some bond graph field matrices. The proposed paper explores also six different classes of nonlinear systems. This classification is based on the kind of causal connections of the different bond graph fields and the time dependency of the field matrices. For each class the descriptor equations are derived and, if possible, converted into a state space form. The results are also interesting from a structural and a computational point of view.

Keywords: bond graphs, system analysis, general systems, nonlinear systems, matrix description.

INTRODUCTION

When making a bond graph, the modeler focuses on the energetical structure of the physical system involved. Thereafter, a causality analysis makes it possible to achieve further engineering objectives. A design engineer for example, uses causality to investigate some modeling assumptions or deficiencies [Minten and De Moor 1993]. A control engineer wants to obtain a mathematical representation that fits into his needs (a transfer function description [Brown 1972], a state space description [Rosenberg 1971], ...). And if a digital simulation of the physical system is the only objective, causality again is a tool for obtaining an ordered set of equations ready for input in a simulation program [Cornet and Lorenz 1989].
In the past different causality procedures have been proposed to achieve these goals.
Karnopp et al. [Karnopp *et al.* 1990] used the basic causality assignment procedure, usually denoted as the Sequential Causal-Assignment Procedure (SCAP) [Rosenberg and Karnopp 1983]. The major objective of this method is to look for a minimal set of dynamical variables to formulate the dynamical characteristics by means of a set of first order differential equations (a state space description). Another method, the Relaxed Causality Assignment Procedure (RCAP) introduced by Joseph and Martens [Joseph and Martens 1974], looks for a set of sufficient variables (not necessary dynamical) resulting in a set of differential and algebraic equations (DAE's). A third method, the Lagrangian Causality Assignment Procedure (LCAP), introduced by Karnopp [Karnopp 1977], looks for other variables (generalised displacements and velocities) to formulate a set of second order differential equations (Lagrange equations). Although in certain cases a conversion between these mathematical representations can be made, it is clear that the choice of the used causal assignment procedure depends on the desired mathematical representation. In our case this representation must be useful in system design and control design tasks. Due to practical considerations in both domains one has a strong preference for an explicit *linear time invariant state space form*:

$$\dot{\mathbf{x}}(\mathbf{t}) = \mathbf{A}\mathbf{x}(\mathbf{t}) + \mathbf{B}\mathbf{u}(\mathbf{t}). \tag{1}$$

Consequently it is obvious that SCAP wil be used. However, except in unusual cases, the real system is nonlinear time varying and most often, at least a linearisation of the system behaviour around a working point is considered to arrive at Eqn.(1). Unfortunately, it is well known that not all physical systems fit into this mathematical representation. It is again SCAP that tells in advance if such situations occur. Systems which do not obey Eqn.(1) are often called *non-causal* or *degenerate*. In literature (e.g. [Luenberger 1977] and [Verghese and Levy 1981]) such linear (or linearised) systems are described with the following equation

$$\mathbf{E}(\mathbf{t})\dot{\mathbf{x}}(\mathbf{t}) = \mathbf{A}(\mathbf{t})\mathbf{x}(\mathbf{t}) + \mathbf{B}(\mathbf{t})\mathbf{u}(\mathbf{t}), \tag{2}$$

where $\mathbf{E(t)}$ can be singular (Therefore such systems are also called *singular*). The vector $\mathbf{x(t)}$ is called *the descriptor vector.*
This paper focuses on the determination of such a mathematical formulation of nonlinear physical dynamic systems. It uses the vectorised state equation generation procedure of the keynote paper [Rosenberg 1971], where the state equations are generated for linear and state dependent nonlinear multiport systems.
For most practical systems the field relations of the bond graph can be written down explicitely as matrix functions. For such systems, a closed matrix descriptor form Eqn.(3) is derived, which is different from Eqn.(2)

$$\mathbf{E(x,u,t)\dot{x}(t) = A(x,u,t)x(t) + B(x,u,t)u(t) + G(x,u,t)\dot{u}(t)}. \tag{3}$$

The explicit knowledge of these matrices could be beneficial in structural analysis of the observed physical system, robust controller design schemes,....
Mainly depending on the junction structure these dynamic equations can be written into a form without cross-terms (i.e. with $\mathbf{E(t)}$, $\mathbf{A(x,t)}$, $\mathbf{B(u,t)}$ and $\mathbf{G(u,t)}$). The proposed treatment also leads to an explicit determination of degeneracy (or singularity) of the observed system.
This paper is organized as follows. In Section 2 some necessary preliminaries are given and from a certain class of nonlinear systems the closed matrix descriptor form is proposed. Depending on the linearity and time dependency of certain parts of the bond graph, different subclasses can be defined. This is done in Section 3. Finally, in Section 4 some conclusions are given.

PRELIMINARIES

A vectorised view of a physical system described with bond graphs is shown in figure 1 [Karnopp *et al.* 1990]. This partitioning can be done after converting the bond graph into a standard and augmented form. The standard form of a bond graph classifies each power bond as *external* or *internal*, the augmenting of a bond graph involves a SCAP. This conversion can be automated [Hood *et al.* 1989], and results in the determination of the *key vectors* and the *field assignment statements.* Based on these vectors and statements Rosenberg [Rosenberg 1971] has outlined a method for deriving the state and output equations of linear and state dependent nonlinear systems.

Definition of the key vectors

The key vectors are the output variables of the different fields and involve power and powerless interaction with or within the physical system (figure 1). For convenience they are briefly explained.
The dissipation field input and output vector is $\mathbf{d_i}$ resp. $\mathbf{d_o}$.
The energy vector of the dynamic system is taken as the *descriptor vector* $\mathbf{x}$ and can be partitioned in an independent (index $\mathbf{I}$) and a dependent (index $\mathbf{D}$) part, $\mathbf{x} = [\ \mathbf{x_I} \quad \mathbf{x_D}\ ]^t$. The components of the independent part $\mathbf{x_I}$ are the generalized momenta p_{i_I} and generalized

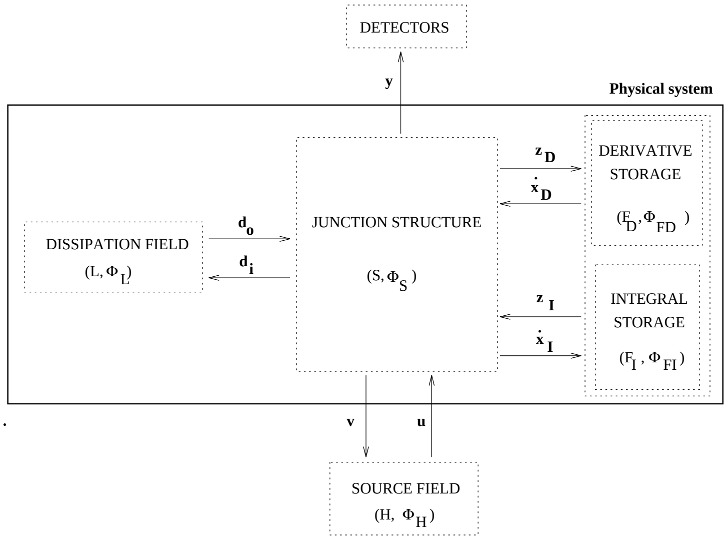

Figure 1: *Visualisation of the key vectors of a physical system.*

displacements q_{j_I} on the ports of the I-elements resp. C-elements with integral causality. The derivative with respect to time of this vector is the integral storage field input vector. The derivative with respect to time of the components of the dependent part $\mathbf{x_D}$ are the output variables e_{i_D} and f_{j_D} on the ports of the I-elements resp. C-elements with derivative causality.
Similarly *the co-energy vector* $\mathbf{z}$ can be partitioned in an independent (index $\mathbf{I}$) and a dependent (index $\mathbf{D}$) part, $\mathbf{z} = [\ \mathbf{z_I} \quad \mathbf{z_D}\ ]^t$. The components of the independent part $\mathbf{z_I}$ are the integral storage field output variables f_{i_I} and e_{j_I} on the ports of the I-elements resp. C-elements with integral causality. The components of the dependent part $\mathbf{z_D}$ are the derivative storage field input components f_{i_D} and e_{j_D} on the ports of the I-elements resp. C-elements with derivative causality.
Finally *the source field input and output vector* is denoted as $\mathbf{v}$ resp. $\mathbf{u}$, *the detector field input vector* is denoted as $\mathbf{y}$ and *the junction structure input and output vector* is $\mathbf{j_i} = [\ \mathbf{z_I} \quad \mathbf{\dot{x}_D} \quad \mathbf{d_o} \quad \mathbf{u}\ ]^t$ resp. $\mathbf{j_o} = [\ \mathbf{\dot{x}_I} \quad \mathbf{z_D} \quad \mathbf{d_i} \quad \mathbf{v} \quad \mathbf{y}\ ]^t$.

The field assignment statements

Different bond graph elements are used in the fields shown in figure 1. The implicit constituent relation of such a field can be characterized by assembling the underlying implicit element relations. For most nonlinear fields an explicit matrix function can be derived. In general these functions are not restricted to linear functions, as shown in the sequel. As an example, we take the storage field. Similar propositions can be made for the other field statements. *The (energy) storage field* is constructed of **C**- and **I**-elements. Their constituent relation is different:

$$\begin{aligned} \mathbf{C}\text{-element} &: \quad \mathbf{\Phi_C(e,q)} = 0; \\ \mathbf{I}\text{-element} &: \quad \mathbf{\Phi_I(f,p)} = 0; \\ \text{mixed } \mathbf{IC}\text{-element} &: \quad \mathbf{\Phi_{IC}(f_1,p_1,e_2,q_2)} = 0, \end{aligned}$$

where $\mathbf{q}$ is the generalized displacement vector and $\mathbf{p}$ is the generalized momentum vector of the corresponding element. They can be calculated as the time integral of the vectors $\mathbf{f}$ resp. $\mathbf{e}$.

Provided explicit relations of the storage elements exist, we may assemble them into the following explicit field statement:

$$\begin{bmatrix} \mathbf{z_I} \\ \mathbf{x_D} \end{bmatrix} = \mathbf{\Phi_F}(\begin{bmatrix} \mathbf{x_I} \\ \mathbf{z_D} \end{bmatrix}). \tag{4}$$

If one assumes only one port storage elements or if there are no mixed causality storage fields in the system involved, then the independent and dependent state vectors $\mathbf{x_I}$ and $\mathbf{x_D}$ are not coupled, and Eqn. (4) can be written as:

$$\begin{cases} \mathbf{z_I} = \mathbf{\Phi_{FI}}(\mathbf{x_I}) \\ \mathbf{x_D} = \mathbf{\Phi_{FD}^{-1}}(\mathbf{z_D}) \end{cases}. \tag{5}$$

In a large class of physical systems these explicit nonlinear field functions can be written as matrix functions:

$$\begin{cases} \mathbf{z_I} = \mathbf{F_I x_I} \\ \mathbf{x_D} = \mathbf{F_D^{-1} z_D} \end{cases}, \tag{6}$$

with $\mathbf{F_I}$ the independent storage field matrix and $\mathbf{F_D}$ the dependent storage field matrix. Notice that these matrices could depend on $\mathbf{x_I}$ and $\mathbf{z_D}$, but for reason of conciseness this dependency is not shown explicitely.

These results can be generalized for the *dissipation* and *source field*, resulting in

$$\mathbf{d_o} = \mathbf{L d_i}, \tag{7}$$
$$\mathbf{u} = \mathbf{H v}, \tag{8}$$

with $\mathbf{L}$ the dissipation field matrix and $\mathbf{H}$ the source field matrix.

Finally, if one assumes (a) the use of SCAP, (b) only *solvable junction structures* [Rosenberg and Andry 1979], and (c) the direction of the power arrows towards or outwards the junction structure, it follows that the junction structure matrix $\mathbf{S}$ can be written as

$$\begin{bmatrix} \dot{\mathbf{x}}_\mathbf{I} \\ \mathbf{z_D} \\ \mathbf{d_i} \\ \mathbf{v} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{S_{11}} & \mathbf{S_{12}} & \mathbf{S_{13}} & \mathbf{S_{14}} \\ -\mathbf{S_{12}^T} & \mathbf{0} & \mathbf{0} & \mathbf{S_{24}} \\ -\mathbf{S_{13}^T} & \mathbf{0} & \mathbf{S_{33}} & \mathbf{S_{34}} \\ -\mathbf{S_{14}^T} & -\mathbf{S_{24}^T} & -\mathbf{S_{34}^T} & \mathbf{S_{44}} \\ \mathbf{S_{51}} & \mathbf{S_{52}} & \mathbf{S_{53}} & \mathbf{S_{54}} \end{bmatrix} \begin{bmatrix} \mathbf{z_I} \\ \dot{\mathbf{x}}_\mathbf{D} \\ \mathbf{d_o} \\ \mathbf{u} \end{bmatrix}, \tag{9}$$

where the submatrices $\mathbf{S_{11}}$, $\mathbf{S_{33}}$ and $\mathbf{S_{44}}$ on the diagonal are square and skew symmetric.

The nonlinear matrix descriptor and output equation

In the sequel the class of nonlinear systems for which the field relations can be written in explicit matrix form are considered. No constraint is made on the time dependency or linearity of the matrix relations involved.

In order to derive the matrix descriptor equation the vector $\dot{\mathbf{x}}_\mathbf{D}$ and $\mathbf{d_o}$ has to be rewritten as a function of $\mathbf{z_I}$ and $\mathbf{u}$. Putting Eqn.(9) (b) into Eqn.(6) (b) yields after differentiation:

$$\begin{aligned} \dot{\mathbf{x}}_\mathbf{D} = & -\left\{\dot{\mathbf{F}}_\mathbf{D}^{-1}\mathbf{S_{12}^T} + \mathbf{F_D^{-1}}\dot{\mathbf{S}}_\mathbf{12}^\mathbf{T}\right\}\mathbf{z_I} - \mathbf{F_D^{-1}S_{12}^T}\dot{\mathbf{z}}_\mathbf{I} \\ & + \left\{\dot{\mathbf{F}}_\mathbf{D}^{-1}\mathbf{S_{24}} + \mathbf{F_D^{-1}}\dot{\mathbf{S}}_\mathbf{24}\right\}\mathbf{u} + \mathbf{F_D^{-1}S_{24}}\dot{\mathbf{u}}. \end{aligned} \tag{10}$$

Remark that $\dot{\mathbf{F}}_\mathbf{D}^{-1}$ denotes the time derivative of the inertia matrix $\mathbf{F_D^{-1}}$. Putting Eqn.(9) (c) in Eqn.(7) yields:

$$\mathbf{d_o} = (\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{L}\left[-\mathbf{S_{13}^T z_I} + \mathbf{S_{34} u}\right]. \tag{11}$$

Of course, this equation only holds for an invertible $(\mathbf{I} - \mathbf{LS_{33}})$. For non-dissipative systems this is always true, since $\mathbf{L} = \mathbf{0}$. Finally the matrix descriptor formulation can be obtained by putting Eqns.(10), (11) and (6) into Eqn.(9) (a):

$$\mathbf{E}\dot{\mathbf{x}}_\mathbf{I} = \mathbf{A x_I} + \mathbf{B u} + \mathbf{G}\dot{\mathbf{u}}, \tag{12}$$

with

$$\mathbf{E} = \mathbf{I} + \mathbf{S_{12}F_D^{-1}S_{12}^T F_I}, \tag{13}$$

$$\begin{aligned} \mathbf{A} = & \left(\mathbf{S_{11}} - \mathbf{S_{12}}\left[\dot{\mathbf{F}}_\mathbf{D}^{-1}\mathbf{S_{12}^T} + \mathbf{F_D^{-1}}\dot{\mathbf{S}}_\mathbf{12}^\mathbf{T}\right]\right. \\ & \left. -\mathbf{S_{13}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{13}^T}\right)\mathbf{F_I} \\ & -\mathbf{S_{12}F_D^{-1}S_{12}^T}\dot{\mathbf{F}}_\mathbf{I}, \end{aligned} \tag{14}$$

$$\begin{aligned} \mathbf{B} = & \mathbf{S_{14}} + \mathbf{S_{12}}\left(\dot{\mathbf{F}}_\mathbf{D}^{-1}\mathbf{S_{24}} + \mathbf{F_D^{-1}}\dot{\mathbf{S}}_\mathbf{24}\right) \\ & +\mathbf{S_{13}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{34}}, \end{aligned} \tag{15}$$

$$\mathbf{G} = \mathbf{S_{12}F_D^{-1}S_{24}}. \tag{16}$$

If $\mathbf{E}$ is regular then the *standard output equation* can be derived. By putting Eqns.(10), (11) and (6) into Eqn.(9) (e), wherafter $\dot{\mathbf{x}}_\mathbf{I}$ is substituted by Eqn.(12), one obtains

$$\mathbf{y} = \mathbf{C x_I} + \mathbf{D u} + \mathbf{Q}\dot{\mathbf{u}}, \tag{17}$$

with

$$\mathbf{C} = \mathbf{C}^* + \mathbf{PE^{-1}A}, \tag{18}$$
$$\mathbf{D} = \mathbf{D}^* + \mathbf{PE^{-1}B}, \tag{19}$$
$$\mathbf{Q} = \mathbf{Q}^* + \mathbf{PE^{-1}G}, \tag{20}$$

in which $\mathbf{E}$, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{G}$ are defined by Eqns.(13), (14), (15) and (16), and

$$\begin{aligned} \mathbf{C}^* = & \left(\mathbf{S_{51}} - \mathbf{S_{52}}\left[\dot{\mathbf{F}}_\mathbf{D}^{-1}\mathbf{S_{12}^T} + \mathbf{F_D^{-1}}\dot{\mathbf{S}}_\mathbf{12}^\mathbf{T}\right]\right. \\ & \left. -\mathbf{S_{53}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{13}^T}\right)\mathbf{F_I} \\ & -\mathbf{S_{52}F_D^{-1}S_{12}^T}\dot{\mathbf{F}}_\mathbf{I}, \end{aligned} \tag{21}$$

$$\mathbf{P} = -\mathbf{S_{52}F_D^{-1}S_{12}^T F_I}, \tag{22}$$

$$\begin{aligned} \mathbf{D}^* = & \mathbf{S_{54}} + \mathbf{S_{52}}\left(\dot{\mathbf{F}}_\mathbf{D}^{-1}\mathbf{S_{24}} + \mathbf{F_D^{-1}}\dot{\mathbf{S}}_\mathbf{24}\right) \\ & +\mathbf{S_{53}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{34}}, \end{aligned} \tag{23}$$

$$\mathbf{Q}^* = \mathbf{S_{52}F_D^{-1}S_{24}}. \tag{24}$$

Again, Eqn.(17) holds for an invertible $(\mathbf{I} - \mathbf{LS_{33}})$ only. From these equations a classification of nonlinear systems can be made. In the next section some general considerations are given and several subclasses are proposed.

SUBCLASSES OF NONLINEAR SYSTEMS WITH FIELD MATRICES

In the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ derivatives with respect to time of some submatrices of the matrix $\mathbf{S}$ and $\mathbf{F_D^{-1}}$ appear. In certain cases, like in two or three dimensional mechanical systems major parts of the matrix $\mathbf{S}$ are coordinate transformation matrices. Then these submatrices of $\mathbf{S}$ can be written as functions of generalized coordinates, which can be seen as the time integral of some descriptor variable. For such systems, Eqns.(12) and (17) contain cross-terms because products of descriptor variables with themselves and input variables appear.
It is also possible that supplementary algebraic constraints arise in the calculation of some time dependent components of the matrices involved, mainly the junction structure matrix $\mathbf{S}$. An example of this is the external modulation of a modulated transducer (MTF or MGY).
It is easy to see that $\mathbf{F_D^{-1}}$ plays a crucial role in the time behaviour of the descriptor vector. If no derivative causality appears in the storage field, then $\mathbf{E} \equiv \mathbf{I}$ and $\mathbf{G} \equiv \mathbf{0}$. As a consequence Eqn.(12) is in fact a state space description and the descriptor vector equals the state vector. Conversely, if a derivative causality appears in the storage field a state space description is only possible iff $\mathbf{E}$ is not singular, or

$$\det(\mathbf{I} + \mathbf{S_{12}F_D^{-1}S_{12}^T}) \neq 0. \tag{25}$$

However, at the same time the state vector is no longer equal to the independent energetic physical state vector. An artificial state vector $\tilde{\mathbf{x}}$ can be defined as $\tilde{\mathbf{x}} = \mathbf{x_I} - \mathbf{Gu}$. Bringing this vector into Eqn.(12) yields the matrix state equation:

$$\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{A}}\tilde{\mathbf{x}} + \tilde{\mathbf{B}}\mathbf{u}, \tag{26}$$

with

$$\begin{aligned} \tilde{\mathbf{A}} &= \mathbf{E^{-1}A}, & (27)\\ \tilde{\mathbf{B}} &= \mathbf{E^{-1}}\left(\mathbf{B} + \mathbf{AE^{-1}G} + \dot{\mathbf{E}}\mathbf{E^{-1}} - \dot{\mathbf{G}}\right). & (28) \end{aligned}$$

Note that the nature of a system obeying state equations is totally different to those obeying descriptor equations and thus influences further control schemes [Verghese and Levy 1981].
Note also that the matrix descriptor form Eqn.(12) differs form its standard form defined in [Luenberger 1977]. However when defining the block matrix $\tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{B} & \mathbf{G} \end{bmatrix}$ and the artificial input vector $\tilde{\mathbf{u}} = \begin{bmatrix} \mathbf{u} & \dot{\mathbf{u}} \end{bmatrix}^{\mathbf{t}}$ Eqn.(12) can be rewritten in *standard form:*

$$\mathbf{E}\dot{\mathbf{x}}_\mathbf{I} = \mathbf{A}\mathbf{x_I} + \tilde{\mathbf{B}}\tilde{\mathbf{u}} \tag{29}$$

Consequently the set of independent energetical states can always serve as a descriptor vector. But this implies constraints on the artificial input vector $\tilde{\mathbf{u}}$, since its components cannot vary independently and hence $\tilde{\mathbf{u}}$ cannot really be called an *input*.

The reduced formulations of Eqns.(12) and (17) can be classified in the following 6 classes:

CLASS 1: there is no causal connection between the dependent storage field and the detectors.

Then in Eqn.(9) the submatrix $\mathbf{S_{52}} = \mathbf{0}$ and the output equation (17) becomes

$$\mathbf{y} = \mathbf{Cx_I} + \mathbf{Du}, \tag{30}$$

with

$$\begin{aligned} \mathbf{C} &= \left(\mathbf{S_{51}} - \mathbf{S_{53}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{13}^T}\right)\mathbf{F_I}, & (31)\\ \mathbf{D} &= \mathbf{S_{54}} + \mathbf{S_{53}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{34}}. & (32) \end{aligned}$$

If in addition there is no causal connection between the dissipation field and the detectors, then $\mathbf{S_{53}} = \mathbf{0}$ and the output matrices of Eqn.(30) become simply

$$\begin{aligned} \mathbf{C} &= \mathbf{S_{51}F_I}, & (33)\\ \mathbf{D} &= \mathbf{S_{54}}. & (34) \end{aligned}$$

CLASS 2: the storage field matrices are time invariant.

If both the matrices $\mathbf{F_D^{-1}}$ and $\mathbf{F_I}$ are constant, then the descriptor equation matrices $\mathbf{A}$ and $\mathbf{B}$, and the output equation matrices $\mathbf{C}^*$ and $\mathbf{D}^*$, Eqns.(14), (15), (21) and (23) simplify into

$$\mathbf{A} = \left(\mathbf{S_{11}} - \mathbf{S_{12}F_D^{-1}\dot{S}_{12}^T} - \mathbf{S_{13}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{13}^T}\right)\mathbf{F_I}, \tag{35}$$

$$\mathbf{B} = \mathbf{S_{14}} + \mathbf{S_{12}F_D^{-1}\dot{S}_{24}} + \mathbf{S_{13}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{34}}, \tag{36}$$

$$\mathbf{C}^* = \left(\mathbf{S_{51}} - \mathbf{S_{52}F_D^{-1}\dot{S}_{12}^T} - \mathbf{S_{53}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{13}^T}\right)\mathbf{F_I}, \tag{37}$$

$$\mathbf{D}^* = \mathbf{S_{54}} + \mathbf{S_{52}F_D^{-1}\dot{S}_{24}} + \mathbf{S_{53}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{34}}. \tag{38}$$

CLASS 3: the junction structure field matrix is time invariant.

This is the case when no MTF of MGY elements appear in the junction structure. Then also the descriptor equation matrices $\mathbf{A}$ and $\mathbf{B}$, and the output equation matrices $\mathbf{C}^*$ and $\mathbf{D}^*$, Eqns.(14), (15), (21) and (23) simplify into

$$\begin{aligned} \mathbf{A} = {} & \left(\mathbf{S_{11}} - \mathbf{S_{12}\dot{F}_D^{-1}S_{12}^T} - \mathbf{S_{13}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{13}^T}\right)\mathbf{F_I} \\ & - \mathbf{S_{12}F_D^{-1}S_{12}^T\dot{F}_I}, \end{aligned} \tag{39}$$

$$\mathbf{B} = \mathbf{S_{14}} + \mathbf{S_{12}\dot{F}_D^{-1}S_{24}} + \mathbf{S_{13}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{34}}, \tag{40}$$

$$\begin{aligned} \mathbf{C}^* = {} & \left(\mathbf{S_{51}} - \mathbf{S_{52}\dot{F}_D^{-1}S_{12}^T} - \mathbf{S_{53}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{13}^T}\right)\mathbf{F_I} \\ & - \mathbf{S_{52}F_D^{-1}S_{12}^T\dot{F}_I}, \end{aligned} \tag{41}$$

$$\mathbf{D}^* = \mathbf{S_{54}} + \mathbf{S_{52}}\dot{\mathbf{F}}_{\mathbf{D}}^{-1}\mathbf{S_{24}} + \mathbf{S_{53}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{34}}. \tag{42}$$

CLASS 4: the storage and junction structure field matrices are time invariant.

Again the descriptor equation matrices $\mathbf{A}$ and $\mathbf{B}$ and the output equation matrices $\mathbf{C}^*$ and $\mathbf{D}^*$, Eqns.(14), (15), (21) and (23) simplify into

$$\mathbf{A} = \left\{\mathbf{S_{11}} - \mathbf{S_{13}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{13}^T}\right\}\mathbf{F_I}, \tag{43}$$
$$\mathbf{B} = \mathbf{S_{14}} + \mathbf{S_{13}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{34}}, \tag{44}$$
$$\mathbf{C}^* = \left\{\mathbf{S_{51}} - \mathbf{S_{53}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{13}^T}\right\}\mathbf{F_I}, \tag{45}$$
$$\mathbf{D}^* = \mathbf{S_{54}} + \mathbf{S_{53}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{34}}. \tag{46}$$

Remark that formally this result is also valid for linear systems. However, here $\mathbf{L}$ can be time varying or state dependent.

CLASS 5: there is no causal connection between the sources and the dependent storage field.

Then in Eqn.(9) the submatrix $\mathbf{S_{24}} = \mathbf{0}$, and as result the influence of the time derivative of the input vector $\mathbf{u}$ disappears

$$\mathbf{E}\dot{\mathbf{x}}_{\mathbf{I}} = \mathbf{A}\mathbf{x_I} + \mathbf{B}\mathbf{u}, \tag{47}$$

with

$$\mathbf{E} = \mathbf{I} + \mathbf{S_{12}}\mathbf{F}_{\mathbf{D}}^{-1}\mathbf{S_{12}^T}\mathbf{F_I}, \tag{48}$$
$$\mathbf{A} = \Big(\mathbf{S_{11}} - \mathbf{S_{12}}\left[\dot{\mathbf{F}}_{\mathbf{D}}^{-1}\mathbf{S_{12}^T} + \mathbf{F}_{\mathbf{D}}^{-1}\dot{\mathbf{S}}_{\mathbf{12}}^{\mathbf{T}}\right] - \mathbf{S_{13}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{13}^T}\Big)\mathbf{F_I} - \mathbf{S_{12}}\mathbf{F}_{\mathbf{D}}^{-1}\mathbf{S_{12}^T}\dot{\mathbf{F}}_{\mathbf{I}}, \tag{49}$$
$$\mathbf{B} = \mathbf{S_{14}} + \mathbf{S_{13}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{34}}, \tag{50}$$
$$\tag{51}$$

and

$$\mathbf{y} = \mathbf{C}\mathbf{x_I} + \mathbf{D}\mathbf{u}, \tag{52}$$

with

$$\mathbf{C} = \mathbf{C}^* + \mathbf{P}\mathbf{E}^{-1}\mathbf{A}, \tag{53}$$
$$\mathbf{D} = \mathbf{D}^* + \mathbf{P}\mathbf{E}^{-1}\mathbf{B}, \tag{54}$$

in which $\mathbf{E}$ is defined in Eqn.(48) and

$$\mathbf{C}^* = \Big(\mathbf{S_{51}} - \mathbf{S_{52}}\left[\dot{\mathbf{F}}_{\mathbf{D}}^{-1}\mathbf{S_{12}^T} + \mathbf{F}_{\mathbf{D}}^{-1}\dot{\mathbf{S}}_{\mathbf{12}}^{\mathbf{T}}\right] - \mathbf{S_{53}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{13}^T}\Big)\mathbf{F_I} - \mathbf{S_{52}}\mathbf{F}_{\mathbf{D}}^{-1}\mathbf{S_{12}^T}\dot{\mathbf{F}}_{\mathbf{I}}, \tag{55}$$
$$\mathbf{P} = -\mathbf{S_{52}}\mathbf{F}_{\mathbf{D}}^{-1}\mathbf{S_{12}^T}\mathbf{F_I}, \tag{56}$$
$$\mathbf{D}^* = \mathbf{S_{54}} + \mathbf{S_{53}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{34}}. \tag{57}$$

The sources of such a system can be called *natural* because they don't force a derivative causality to a part of the storage field and hence do not have the property of producing instantaneously any amount of power when necessary.

CLASS 6: there is no dependent storage field.

This is the case when the bond graph can be made fully causal. As already shown it turns out that the descriptor and output equations, Eqns.(12) and (17), change fundamentally; they reduce into a state space description

$$\dot{\mathbf{x}}_{\mathbf{I}} = \mathbf{A}\mathbf{x_I} + \mathbf{B}\mathbf{u}, \tag{58}$$

with

$$\mathbf{A} = \left\{\mathbf{S_{11}} - \mathbf{S_{13}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{13}^T}\right\}\mathbf{F_I}, \tag{59}$$
$$\mathbf{B} = \left\{\mathbf{S_{14}} + \mathbf{S_{13}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{34}}\right\}, \tag{60}$$

and

$$\mathbf{y} = \mathbf{C}\mathbf{x_I} + \mathbf{D}\mathbf{u}, \tag{61}$$

with

$$\mathbf{C} = \left\{\mathbf{S_{51}} - \mathbf{S_{53}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{13}^T}\right\}\mathbf{F_I}, \tag{62}$$
$$\mathbf{D} = \mathbf{S_{54}} + \mathbf{S_{53}}(\mathbf{I} - \mathbf{LS_{33}})^{-1}\mathbf{LS_{34}}. \tag{63}$$

Only in this case the velocity of the physical energetical state vector is a function of itself and the power input variable.
The same formulation is also obtained in [Rosenberg 1971], where the linearity of the physical system was demanded.

CONCLUSIONS

In this paper nonlinear systems are modeled and analysed with bond graphs. It has been shown that a large class of nonlinear systems can be described with matrix field functions. The main results of this paper are closed formulae for the descriptor and output equation for this class of nonlinear systems. The method also provides a necessary condition for degeneracy of the descriptor equation. From these results different subclasses of nonlinear systems are treated, and an example of such a subclass is given.
It should be stressed that the descriptor variables all have a physical meaning since they are in fact the generalized momenta and displacements of the independent storage field. As a consequence this approach can be seen as a generalization of the classical derivation of the state space formulation of linear systems [Rosenberg 1971] for which the junction structure matrix $\mathbf{S}$ is constant or only dependent on the state vector $\mathbf{x_I}$.
A second contribution of this paper is the conclusion that a descriptor form defined by [Luenberger 1977] can be found. However, the derived equations in this paper differ in two ways: (a) when using the independent energetical states as descriptor vector, the input vector is constrained, and (b) it should be clear that a descriptor formulation without cross-terms is not always guaranteed. These differences depend not on the proposed method,

but on the physical system involved; especially on the existence of a dependent storage field, **MTF**- and **MGY**-transducer moduli in the junction structure matrix and the dissipation field characteristics. Due to space limitations an example of the proposed formal method is not given, but can be found in [Minten *et al.* 1993].
All this can be advantageous from a structural and a computational point of view, and in modern control applications for which a state space description (linear - not linear, constant - time varying) is needed. As a consequence it is for example possible to derive exactly the differences between the linear and the proposed exact nonlinear state space description. This structured knowledge can be beneficial in robust control schemes.

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References

[Brown 1972] Brown F.T. "Direct application of the loop rule to bondgraphs." *ASME, Journal of Dynamic Systems, Measurement and Control,* appeared in Specal Issue on Bondgraph Modelling for Engineering Systems, Vol.94, No.3, pp.253-261, 1972.

[Cornet and Lorenz 1989] Cornet A. and Lorenz F. "Equation Ordering Using Bond Graph Causality Analysis." In *IMACS Trans. on Scientific Computation-88, Vol.3: Modeling and Simulation of Systems,* pp.55-58, 1989.

[Hood *et al.* 1989] Hood S.J., Palmer E.R., and Dantzig P.M. "A Fast Complete Method for Automatically Assigning Causality to Bond Graphs." *J. Franklin Institute,* Vol.326, No.1, 1989.

[Joseph and Martens 1974] Joseph B.J. and Martens H.R "The Method of Relaxed Causality in the Bond Graph Analysis of Nonlinear Systems." *ASME, Journal of Dynamic Systems, Measurement and Control,* Vol.96, No.1, 1974.

[Karnopp 1977] Karnopp D.C. "Lagrange's Equations for Complex Bond Graph Systems." *ASME, Journal of Dynamic Systems, Measurement and Control,* Vol.99, No.4, 1977.

[Karnopp *et al.* 1990] Karnopp D.C., Margolis D.L., and Rosenberg R.C. *System dynamics : a unified approach. Second edition.* John Wiley & Sons, New York, USA, 1990.

[Luenberger 1977] Luenberger D.G. "Dynamic Equations in Descriptor Form." *IEEE Transactions on Automatic Control,* Vol.AC-20, No. 3, pp. 239-246, June 1977.

[Minten and De Moor 1993] Minten W. and De Moor B. "Structured Physical Modeling by Means of Bond Graphs: a Case Study." ESAT-SISTA, TR 93-61J, 1993, K.U.Leuven, Belgium.

[Minten *et al.* 1993] Minten W., De Moor B., and Vandewalle J. "Matrix Descriptor Formulation of a Large Class of Nonlinear Physical Dynamic Systems." ESAT-SISTA, TR 93-78I, 1993, K.U.Leuven, Belgium.

[Rosenberg 1971] Rosenberg R.C. "State Space Formulation for Bond Graph Models of Multiport Systems." *ASME, Journal of Dynamic Systems, Measurement and Control,* Vol.93, No.1, 1971.

[Rosenberg and Karnopp 1983] Rosenberg R.C. and Karnopp D.C. *Introduction to Physical System Dynamics.* McGraw-Hill Book Company, New York, USA, 1983.

[Rosenberg and Andry 1979] Rosenberg R.C. and Andry A.N. jr. "Solvability of Bond Graph Junction Structures with Loops." *IEEE Transactions on Circiuts and Systems,* Vol.cas-26, No.2, 1979.

[Singh and Liu 1979] Singh S.P. and, Liu R.W. "Existence of State Equation Representation of Linear Large-Scale Dynamical Systems." *IEEE Transactions on Circiut Theory,* Vol.CT-20, pp. 239-246, May 1979.

[Verghese and Levy 1981] Verghese G.C. and Levy B.C. "A Generalized State Space for Singular Systems." *IEEE Transactions on Automatic Control,* Vol.AC-26, pp. 811-831, 1981.