

On local unitary versus local Clifford equivalence of stabilizer states

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(Dated: March 22, 2005)

We study the relation between local unitary (LU) equivalence and local Clifford (LC) equivalence of stabilizer states. We introduce a large subclass of stabilizer states, such that every two LU equivalent states in this class are necessarily LC equivalent. Together with earlier results, this shows that LC, LU and SLOCC equivalence are the same notions for this class of stabilizer states. Moreover, recognizing whether two given stabilizer states in the present subclass are locally equivalent only requires a polynomial number of operations in the number of qubits.

PACS numbers: 03.67.-a

I. INTRODUCTION

Stabilizer states constitute a class of multipartite pure states that play an important role in numerous tasks in quantum information theory, such as quantum error correction [1] and measurement-based quantum computation [2]. A stabilizer state on n qubits is defined as a simultaneous eigenvector of a maximal set of commuting operators in the Pauli group on n qubits, where the latter is the group generated by all n -fold tensor products of the Pauli matrices and the identity.

In order to understand the role of stabilizer states in existing and possibly new applications, the properties of these states have recently been studied by numerous authors (see e.g. [3–10]). One major open problem is a classification of stabilizer states in local unitary (LU) equivalence classes: not only is this problem of natural importance in the study of the entanglement properties of stabilizer states, but it is also relevant both in the quantum coding aspect of stabilizer states as in their role in the one-way quantum computing model. When studying LU equivalence of stabilizer states, it is natural to consider, in a first step, only those LU operations that belong to the local Clifford (LC) group, where the latter consists of all local unitary operations that map the Pauli group to itself under conjugation. Indeed, the stabilizer formalism plus the (local) Clifford group form a closed framework which can entirely be described in terms of binary linear algebra, and this binary description simplifies the study of LC equivalence of stabilizer states to a great extent with respect to general LU equivalence. We have studied several aspects of LC equivalence of stabilizer states in earlier work [7, 8, 10]. In a second step, it is natural to raise the question whether the restriction of considering only LC operations is in fact a restriction at all. In other words, the question is asked whether every two LU equivalent stabilizer states are also LC equivalent, or, conversely, whether there exist LU equivalent stabilizer states that are not related by a local Clifford operation. This problem will be our topic of interest in the following. In particular, we will show that the answer to the above question is positive for a large subclass of stabilizer states and it is, with the present result, our aim to take the first steps towards a complete answer to the

above question.

To construct our subclass of stabilizer states, we elaborate on an approach that was used in Ref. [11] to prove that any two LU equivalent so-called GF(4)-linear stabilizer codes are also LC equivalent (this terminology is discussed below). The main concept introduced in the above reference to study the problem at hand is that of *minimal support* of a stabilizer. In the following, we study this notion in more detail and construct an extension of the class of (self-dual) GF(4)-linear stabilizer codes, such that every two LU equivalent states in this class must also be LC equivalent.

Finally, we wish to point out that the present result, and a possible general proof of the assertion that LU and LC equivalence of stabilizer states are identical notions, has several interesting implications. Firstly, we showed in earlier work [12] that any two stabilizer states related by an (invertible) stochastic local operation assisted with classical communication (SLOCC) are also LU equivalent. Together with the present result, this shows that SLOCC, LU and LC equivalence are identical notions within the present subclass of stabilizer states, and therefore there is in fact only a single notion of 'local' equivalence within this class of states. Secondly, in Ref. [8] we presented an algorithm of polynomial complexity in the number of qubits which recognizes whether two given stabilizer states are LC equivalent. This shows that all three local equivalences can be detected efficiently within the present class of stabilizer states.

II. STABILIZER FORMALISM

In this section we state the necessary preliminaries concerning the stabilizer formalism and the local Clifford group.

A. Stabilizer states, LU and LC equivalence

The Pauli group \mathcal{G}_n on n qubits consists of all 4×4^n local operators of the form $M = \alpha_M M_1 \otimes \cdots \otimes M_n$, where $\alpha_M \in \{\pm 1, \pm i\}$ is an overall phase factor and M_i is either the 2×2 identity matrix σ_0 or one of the Pauli

matrices $\sigma_x, \sigma_y, \sigma_z$. The Clifford group \mathcal{C}_1 on one qubit is the group of all 2×2 unitary operators that map σ_u to $\alpha_u \sigma_{\pi(u)}$ under conjugation, where $u = x, y, z$, for some $\alpha_u = \pm 1$ and some permutation π of $\{x, y, z\}$. The local Clifford group \mathcal{C}_n^l on n qubits is the n -fold tensor product of \mathcal{C}_1 with itself.

A stabilizer \mathcal{S} in the Pauli group is defined as an abelian subgroup of \mathcal{G}_n which does not contain $-I$ [13]. A stabilizer consists of 2^k Hermitian Pauli operators (i.e. they must have real overall phase factors ± 1), for some $k \leq n$. As the operators in a stabilizer commute, they can be diagonalized simultaneously and, what is more, if $|\mathcal{S}| = 2^n$ then there exists a unique state $|\psi\rangle$ on n qubits such that $M|\psi\rangle = |\psi\rangle$ for every $M \in \mathcal{S}$. Such a state $|\psi\rangle$ is called a stabilizer state and the group $\mathcal{S} = \mathcal{S}(\psi)$ is called the stabilizer of $|\psi\rangle$. The expansion

$$|\psi\rangle\langle\psi| = \frac{1}{2^n} \sum_{M \in \mathcal{S}(\psi)} M, \quad (1)$$

which describes a stabilizer state as a sum of all elements in its stabilizer, can readily be verified.

The support $\text{supp}(M)$ of an element $M = \alpha_M M_1 \otimes \dots \otimes M_n \in \mathcal{S}(\psi)$ is the set of all $i \in \{1, \dots, n\}$ such that M_i differs from the identity. Let $\omega = \{i_1, \dots, i_k\}$ be a subset of $\{1, \dots, n\}$. Tracing out all qubits of $|\psi\rangle$ outside ω yields a (generally mixed) state $\rho_\omega(\psi)$, which is equal to

$$\rho_\omega(\psi) = \frac{1}{2^{|\omega|}} \sum_{M \in \mathcal{S}, \text{supp}(M) \subseteq \omega} \alpha_M M_{i_1} \otimes \dots \otimes M_{i_k}. \quad (2)$$

This can easily be verified using the identity (1). We denote by $A_\omega(\psi)$ the number of elements $M \in \mathcal{S}(\psi)$ with $\text{supp}(M) = \omega$. It is important to note that the function $A_\omega(\cdot)$ is an LU invariant, i.e. it takes on equal values on LU equivalent stabilizer states [10].

Two stabilizer states are called LU (LC) equivalent if there exists a local unitary (local Clifford) operator which relates these two states. If $|\psi\rangle$ is a stabilizer state, the set $\text{LU}(\psi)$ consists of all stabilizer states that are LU equivalent to $|\psi\rangle$. The set $\text{LC}(\psi)$ is defined analogously.

Finally, a multipartite pure state is called *fully entangled* if it cannot be written as a tensor product of two states. It is clear that in the present context there is no restriction in considering only fully entangled stabilizer states, and we will suppose that every stabilizer state in the following is fully entangled. To avoid technical details that appear when dealing with small numbers of qubits, we will also only consider stabilizer states on $n \geq 3$ qubits [17].

B. Binary representation

It is well known that the stabilizer formalism has an equivalent formulation in terms of algebra over the field $\mathbb{F}_2 = \text{GF}(2)$, where arithmetic is performed modulo two.

The heart of this binary representation of stabilizers is the mapping

$$\begin{aligned} \sigma_0 &= \sigma_{00} \mapsto (0, 0) \\ \sigma_x &= \sigma_{01} \mapsto (0, 1) \\ \sigma_z &= \sigma_{10} \mapsto (1, 0) \\ \sigma_y &= \sigma_{11} \mapsto (1, 1), \end{aligned} \quad (3)$$

which encodes the Pauli matrices as pairs of bits. Consequently, the elements of \mathcal{G}_n can be represented as $2n$ -dimensional binary vectors as follows:

$$\sigma_{w_1 w'_1} \otimes \dots \otimes \sigma_{w_n w'_n} = \sigma_{(w, w')} \mapsto (w, w') \in \mathbb{F}_2^{2n}, \quad (4)$$

where $w := (w_1, \dots, w_n)$, $w' := (w'_1, \dots, w'_n) \in \mathbb{F}_2^n$ are n -dimensional binary vectors. Note that the information about the i th qubit is distributed over the i th components of the vectors w and w' . The parameterization (4) establishes a group homomorphism between \mathcal{G}_n, \cdot and $\mathbb{F}_2^{2n}, +$ (which disregards the overall phases of Pauli operators). In this binary representation, two Pauli operators σ_a and σ_b , where $a, b \in \mathbb{F}_2^{2n}$, commute if and only if $a^T P b = 0$, where the $2n \times 2n$ matrix

$$P = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \quad (5)$$

defines a symplectic inner product on \mathbb{F}_2^{2n} . Therefore, a stabilizer $\mathcal{S}(\psi)$ of an n -qubit stabilizer state $|\psi\rangle$ corresponds to an n -dimensional linear subspace of \mathbb{F}_2^{2n} which is self-dual with respect to this symplectic inner product, i.e., $a^T P b = 0$ for every a, b in this space. The binary stabilizer space is usually presented in terms of a $2n \times n$ binary matrix S , the columns of which form a basis of this space. The entire binary stabilizer space \mathcal{C}_S is the column space of S . As \mathcal{C}_S is its own symplectic dual, a vector $v \in \mathbb{F}_2^{2n}$ belongs to \mathcal{C}_S if and only if $S^T P v = 0$. Consequently, the generator matrix S satisfies $S^T P S = 0$.

An important subclass of stabilizer states is constituted by the graph states. Graph states are those stabilizer states that have a generator matrix of the form $[\theta \ I]^T$, where θ is the $n \times n$ adjacency matrix of a simple graph on n vertices (see e.g. [7]). Therefore, a graph state on n qubits is in a one-to-one correspondence with a graph on n vertices. It is well known that every stabilizer state is LC equivalent to some (generally non-unique) graph state [14].

When disregarding the overall phases of the elements in \mathcal{G}_1 , it is easy to see that there exists a one-to-one correspondence between the one-qubit Clifford operations and the 6 possible invertible linear transformations of \mathbb{F}_2^2 , since each one-qubit Clifford operator performs one of the 6 possible permutations of the Pauli matrices and leaves the identity fixed. Local Clifford operations $U \in \mathcal{C}_n^l$ on n qubits then correspond to nonsingular $2n \times 2n$ binary matrices Q of the block form

$$Q = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad (6)$$

where the $n \times n$ blocks A, B, C, D are diagonal [7]. We denote the diagonal entries of A, B, C, D by a_i, b_i, c_i, d_i , respectively. The n submatrices

$$Q_i := \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} \in GL(2, \mathbb{F}_2) \quad (7)$$

correspond to the tensor factors of U . We denote the group of all such Q by C_n^l . Two n -qubit stabilizer states $|\psi\rangle, |\psi'\rangle$ with generator matrices S, S' , respectively, are therefore LC equivalent if and only if there exists an operator $Q \in C_n^l$ such that $C_{QS} = C_{S'}$, i.e., Q maps the space C_S to the space $C_{S'}$. As these spaces are their own symplectic duals, this is equivalent to stating that $S'^T PQS = 0$.

C. GF(4) representation and linearity

There is also a well known representation of stabilizers in terms of algebra over the field $\mathbb{F}_4 = \text{GF}(4)$ [15]. This is the finite field of 4 elements, which can be written as $\mathbb{F}_4 = \{0, 1, \xi, \xi^2\}$. Addition and multiplication in \mathbb{F}_4 satisfy the rules

$$\begin{aligned} 1 + 1 &= \xi + \xi = \xi^2 + \xi^2 = 0 \\ 1 + \xi &= \xi^2. \end{aligned} \quad (8)$$

Note that addition in \mathbb{F}_4 is performed modulo 2. Similar to (3), one now uses the encoding

$$\sigma_0 \mapsto 0, \sigma_z \mapsto 1, \sigma_x \mapsto \xi, \sigma_y \mapsto \xi^2, \quad (9)$$

and consequently Pauli operators on n qubits are represented as n -dimensional vectors with entries in \mathbb{F}_4 . As in the binary description, the multiplicative structure of \mathcal{G}_n becomes the additive structure of \mathbb{F}_4^n . Therefore, every stabilizer $\mathcal{S}(\psi)$ on n qubits corresponds to an *additive* subset (or *code*) of \mathbb{F}_4^n , i.e., the sum of any two vectors in this set again belongs to this set. Analogous to the binary case, this additive code is presented in terms of a generator matrix, which now has dimensions $n \times n$. Moreover, the property that a stabilizer is an abelian group can again be translated into a self-duality property of the corresponding additive code over GF(4) with respect to a certain inner product. As the details of this inner product are irrelevant in the following, we will omit them and the interested reader is referred to Ref. [15].

Below we will be interested in those specific stabilizers corresponding to codes over GF(4) that are closed under scalar multiplication with ξ . Such codes are genuine linear subspaces of \mathbb{F}_4^n , as they are closed under taking linear combinations with coefficients in \mathbb{F}_4 . They are therefore called GF(4)-linear codes.

III. MINIMAL SUPPORTS

In this section, we develop the necessary concepts for our study of LU versus LC equivalence of stabilizer states, and we prove our main result.

Let $|\psi\rangle$ be a stabilizer state on n qubits. A minimal support of $\mathcal{S}(\psi)$ is a set $\omega \subseteq \{1, \dots, n\}$ such that there exists an element in $\mathcal{S}(\psi)$ with support equal to ω , but there exist no elements with support strictly contained in ω . An element with minimal support is called a minimal element. Clearly, if $|\psi'\rangle \in \text{LU}(\psi)$ then ω is also a minimal support of $\mathcal{S}(\psi')$: this follows from the fact that the function $A_{\omega'}(\cdot)$ is an LU invariant, for every $\omega' \subseteq \{1, \dots, n\}$.

We have the following lemma:

Lemma 1 *Let $|\psi\rangle$ be a stabilizer state and let ω be a minimal support of $\mathcal{S}(\psi)$. Then $A_\omega(\psi)$ is equal to 1 or 3 and the latter case can only occur if $|\omega|$ is even.*

Proof: (i) If $A_\omega(\psi) = 1$ then we are done. If $A_\omega \geq 2$, let $M, M' \in \mathcal{S}(\psi)$ be two different elements with supports equal to ω . These elements must satisfy $\sigma_0 \neq M_i \neq M'_i \neq \sigma_0$ for every $i \in \omega$. Indeed, if this were not the case, then $\text{supp}(MM')$ would be strictly contained in ω , which contradicts the given. It follows that also $\text{supp}(MM') = \omega$ and that the set $\{M_i, M'_i, (MM')_i\}$ is equal to $\{\sigma_x, \sigma_y, \sigma_z\}$ for every $i = 1, \dots, n$. Consequently, M, M' , and MM' are the only elements in $\mathcal{S}(\psi)$ with support equal to ω : indeed, suppose there does exist a fourth element $N \in \mathcal{S}(\psi)$ with $\text{supp}(N) = \omega$; fixing any $i_0 \in \omega$, then either M_{i_0}, M'_{i_0} or $(MM')_{i_0}$ is equal to N_{i_0} , say $N_{i_0} = M_{i_0}$; but then $\text{supp}(MN)$ is strictly contained in ω , which leads to a contradiction. This proves the first part of the lemma. Secondly, $|\omega|$ must be even since M and M' commute. \square

If ω is a minimal support of $\mathcal{S}(\psi)$, it follows from the proof of lemma 1 that $\rho_\omega(\psi)$ is, up to an LC operation, one of the following two states :

$$\frac{1}{2^{|\omega|}} \left(I_{|\omega|} + \sigma_x^{\otimes |\omega|} \right) \quad (10)$$

$$\frac{1}{2^{|\omega|}} \left(I_{|\omega|} + \sigma_x^{\otimes |\omega|} + (-1)^{|\omega|/2} \sigma_y^{\otimes |\omega|} + \sigma_z^{\otimes |\omega|} \right), \quad (11)$$

where $I_{|\omega|}$ denotes the identity operator on $|\omega|$ qubits. We denote the state (11) by $\rho^{[|\omega|, |\omega|-2, 2]}$. The following property of $\rho^{[|\omega|, |\omega|-2, 2]}$ will be a central ingredient to the proof of our main result below:

Lemma 2 [11] *Let $m \in \mathbb{N}_0$, $m \geq 2$. Let ρ, ρ' be two (mixed) states on $2m$ qubits, both LC equivalent to $\rho^{[2m, 2m-2, 2]}$, and let $U \in U(2)^{\otimes 2m}$ be a local unitary operator such that $U\rho U^\dagger = \rho'$. Then $U \in C_n^l$.*

Lemma 2 is in fact a variant of a result in Ref. [11] and the reader is referred to this reference for a proof.

Remark. If $m = 1$ then

$$\rho^{[2, 0, 2]} = \frac{1}{4} (I + \sigma_x^{\otimes 2} - \sigma_y^{\otimes 2} + \sigma_z^{\otimes 2}) \quad (12)$$

is the rank one projection operator associated with the EPR state $(|00\rangle + |11\rangle)/\sqrt{2}$ (which is a stabilizer state on 2 qubits).

We will now use the concept of minimal support to construct our class of stabilizer states $|\psi\rangle$ that satisfy $\text{LU}(\psi) = \text{LC}(\psi)$. In other words, every two LU equivalent states in this class are necessarily LC equivalent. Denoting by $\mathcal{M}(\psi)$ the subgroup of $\mathcal{S}(\psi)$ generated by all minimal elements, we are ready to state the central result of this paper:

Theorem 1 *Let $|\psi\rangle$ be a fully entangled n -qubit stabilizer such that $\sigma_x, \sigma_y, \sigma_z$ occur on every qubit in $\mathcal{M}(\psi)$. Then $\text{LU}(\psi) = \text{LC}(\psi)$.*

Proof: Let $|\psi'\rangle \in \text{LU}(\psi)$ and fix $U = U_1 \otimes \cdots \otimes U_n \in U(2)^{\otimes n}$ such that $U|\psi\rangle = |\psi'\rangle$. We will show that U_i is a Clifford operation, for every $i = 1, \dots, n$. Considering e.g. the first qubit, there exists a minimal element $M = \alpha_M M_1 \otimes \cdots \otimes M_n \in \mathcal{S}(\psi)$ such that $M_1 \neq \sigma_0$, say $M_1 = \sigma_x$. Let $\omega = \{1 = i_1, i_2, \dots, i_k\} \subseteq \{1, \dots, n\}$, where $k = |\omega|$, denote the support of M . Then $A_\omega(\psi)$ is equal to either 1 or 3 from lemma 1. We will make a distinction between these two cases.

Firstly, suppose that $A_\omega(\psi) = 3$. Then $\rho_\omega(\psi)$ is LC equivalent to $\rho^{[|\omega|, |\omega|-2, 2]}$ from the above. Moreover, as $|\psi'\rangle \in \text{LU}(\psi)$, the set ω is also a minimal support of $\mathcal{S}(\psi')$ with $A_\omega(\psi') = A_\omega(\psi) = 3$. Therefore, $\rho_\omega(\psi')$ is LC equivalent to $\rho^{[|\omega|, |\omega|-2, 2]}$ as well. Using the notation $U_\omega = U_{i_1} \otimes \cdots \otimes U_{i_k}$, it follows from $U|\psi\rangle = |\psi'\rangle$ that U_ω maps $\rho_\omega(\psi)$ to $\rho_\omega(\psi')$ under conjugation. Now, note that we must have $|\omega| > 2$; indeed, if $|\omega|$ were equal to 2 then it follows from the remark below lemma 2 that $\rho_\omega(\psi)$ is a pure state; this is however impossible as $|\psi\rangle$ is fully entangled. Moreover, $|\omega|$ is even from lemma 1, and we can therefore conclude that $|\omega| \geq 4$. We can now use lemma 2, finding that $U_\omega \in \mathcal{C}_k^t$, and, in particular, U_1 is a Clifford operation.

Secondly, let $A_\omega(\psi) = 1$. Then there exists another minimal element $N \in \mathcal{M}(\psi)$ such that $1 \in \text{supp}(N)$ and $M_1 \neq N_1$ from the assumption in the theorem, say $N_1 = \sigma_z$. Let $\mu = \text{supp}(N)$. Note that if $A_\mu(\psi) = 3$, we can apply the same argument as above and conclude that U_1 is a Clifford operation. On the other hand, suppose that $A_\mu(\psi) = 1$. Denoting $M_\omega = \alpha_M M_{i_1} \otimes \cdots \otimes M_{i_k}$, and N_μ analogously, we can write

$$\begin{aligned} \rho_\omega(\psi) &= \frac{1}{2^{|\omega|}} (I_{|\omega|} + M_\omega) \\ \rho_\mu(\psi) &= \frac{1}{2^{|\mu|}} (I_{|\mu|} + N_\mu). \end{aligned} \quad (13)$$

Moreover, since the sets ω and μ are also minimal supports of $\mathcal{S}(\psi')$ with $A_\mu(\psi') = A_\omega(\psi') = 1$, there exist unique $M', N' \in \mathcal{S}(\psi')$ such that

$$\begin{aligned} \rho_\omega(\psi') &= \frac{1}{2^{|\omega|}} (I_{|\omega|} + M'_\omega) \\ \rho_\mu(\psi') &= \frac{1}{2^{|\mu|}} (I_{|\mu|} + N'_\mu), \end{aligned} \quad (14)$$

where M'_ω and N'_μ are defined analogously to M_ω and

N_μ . Now, as

$$\begin{aligned} U_\omega \rho_\omega(\psi) U_\omega^\dagger &= \rho_\omega(\psi') \\ U_\mu \rho_\mu(\psi) U_\mu^\dagger &= \rho_\mu(\psi'), \end{aligned} \quad (15)$$

where again we have used the notation $U_\omega = U_{i_1} \otimes \cdots \otimes U_{i_k}$ and analogously for U_μ , we have

$$\begin{aligned} U_1 M_1 U_1^\dagger &= \pm M'_1 \\ U_1 N_1 U_1^\dagger &= \pm N'_1 \end{aligned} \quad (16)$$

from (13)-(14). Using $M_1 = \sigma_x$ and $N_1 = \sigma_z$, The identities (16) show that U_1 is a Clifford operation.

Repeating the above arguments for all n qubits yields the result. \square

Theorem 1 shows that it is sufficient that the group $\mathcal{M}(\psi)$ has a sufficiently rich structure in order for $\text{LU}(\psi)$ to be equal to $\text{LC}(\psi)$. To gain insight in which states meet the requirement of theorem 1, it is instructive to consider some sufficient conditions for this criterion to hold. Several sufficient conditions are summarized in corollary 1. We note that case (ii) of corollary 1 has already been proved in Ref. [11].

Corollary 1 *Let $|\psi\rangle$ be a stabilizer state on n -qubits such that one of the following assertions (i)-(iv) is true. Then $\text{LU}(\psi) = \text{LC}(\psi)$.*

- (i) $\mathcal{S}(\psi) = \mathcal{M}(\psi)$, i.e. $\mathcal{S}(\psi)$ is generated by its minimal elements.
- (ii) The stabilizer $\mathcal{S}(\psi)$ corresponds to a $GF(4)$ -linear code.
- (iii) For every $M \in \mathcal{S}(\psi)$ with nonminimal support, there exists a minimal support $\omega' \subset \text{supp}(M)$ such that $A_{\omega'}(\psi) = 3$.
- (iv) There exists minimal supports $\omega_1, \omega_2, \dots$ satisfying $3 = A_{\omega_1}(\psi) = A_{\omega_2}(\psi) = \dots$, such that every $i \in \{1, \dots, n\}$ belongs to at least one ω_j .

Proof: We will show that (i)-(iv) imply that σ_x, σ_y and σ_z occur on every qubit in $\mathcal{M}(\psi)$.

Firstly, using lemma 3 (stated below), one immediately finds that assertion (i) implies the desired result.

Secondly, suppose (ii) holds. To prove the result, note that every linear subspace of $GF(q)^n$, with q any prime power, is generated by its minimal elements [16] (here, the notions of (minimal) support and minimal element are defined in the natural way). Using (i) then yields the result.

Thirdly, we show that (iii) also implies (i): let $M \in \mathcal{S}(\psi)$ be an arbitrary nonzero stabilizer element. We have to show that M is a product of minimal elements. If M is minimal then we are done. For nonminimal M , we will prove the assertion by induction on $|\text{supp}(M)|$. Let ω' be a minimal support such that $\omega' \subset \text{supp}(M)$ and $A_{\omega'}(\psi) = 3$. As $A_{\omega'}(\psi) = 3$, there exists a minimal codeword $M' \in \mathcal{M}(\psi)$ with support equal to ω' such that M and M' are equal on the first qubit (see proof of lemma 1). Consequently, $|\text{supp}(MM')|$ is strictly smaller than $|\text{supp}(M)|$ and therefore $MM' \in \mathcal{M}(\psi)$ by induction.

But then also $M \in \mathcal{M}(\psi)$, since M' is minimal. This shows that (iii) implies (i).

Fourthly, suppose that assertion (iv) holds. It then immediately follows from the proof of lemma 1 that σ_x , σ_y and σ_z occur on every qubit in $\mathcal{M}(\psi)$. \square

Lemma 3 *Let $|\psi\rangle$ be a (fully entangled) stabilizer state on $n \geq 2$ qubits. Then all three Pauli matrices σ_x , σ_y , σ_z occur on every qubit in $\mathcal{S}(\psi)$.*

The proof of lemma 3 is given in appendix A.

Clearly, conditions (iii) and (iv) are much more operational than theorem 1, as it is sufficient to know (only part of) the list of invariants $A_\omega(\psi)$ of a given state $|\psi\rangle$ in order to conclude that $\text{LU}(\psi)$ is equal to $\text{LC}(\psi)$. Also condition (ii) is easy to check: indeed, we will show in the next section that one can characterize stabilizers that correspond to GF(4)-linear codes through a very simple constraint on their binary generator matrix.

IV. EXAMPLE: GF(4)-LINEAR CODES

In this section we give a simple characterization of those stabilizers $\mathcal{S}(\psi)$ that correspond to GF(4)-linear stabilizer spaces.

It follows from the discussion in section II that every element in \mathbb{F}_4 has the form $a1 + b\xi \equiv a + b\xi$, where $(a, b) \in \mathbb{F}_2^2$. Using the identity $\xi^2 = \xi + 1$, multiplication of $a + b\xi$ with a second element $a' + b'\xi$, where also $(a', b') \in \mathbb{F}_2^2$, yields

$$(a + b\xi)(a' + b'\xi) = aa' + bb' + (ab' + ba' + bb')\xi. \quad (17)$$

Consequently, the set \mathbb{F}_2^2 inherits a multiplication law $*$ from \mathbb{F}_4 , defined by

$$(a, b) * (a', b') = (aa' + bb', ab' + ba' + bb'), \quad (18)$$

and \mathbb{F}_2^2 , with the multiplication $*$ and the standard addition modulo 2, is a field isomorphic to \mathbb{F}_4 . It follows that the set \mathbb{F}_2^{2n} with the standard addition of vectors modulo 2 and the scalar multiplication $*$, is an n -dimensional vector space over \mathbb{F}_4 ; here, the scalar multiplication is defined as follows: letting $v \in \mathbb{F}_2^{2n}$ and $(a, b) \in \mathbb{F}_2^2$, the vector $w := (a, b) * v$ is defined by

$$(w_i, w_{n+i}) = (a, b) * (v_i, v_{n+i}), \quad (19)$$

for every $i = 1, \dots, n$. With these definitions, it follows from the discussion in section II that a binary stabilizer space \mathcal{C}_S is GF(4)-linear if and only if $(0, 1) * v \in \mathcal{C}_S$ for every $v \in \mathcal{C}_S$. Note that the action $(a, b) \rightarrow (0, 1) * (a, b)$ is linear transformation on the vector $(a, b) \in \mathbb{F}_2^2$. Indeed, one has

$$(0, 1) * (a, b) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}. \quad (20)$$

Consequently, if $v \in \mathbb{F}_2^{2n}$ then $(0, 1) * v$ corresponds to

$$\begin{bmatrix} 0 & I \\ I & I \end{bmatrix} v, \quad (21)$$

where I is the $n \times n$ identity matrix and 0 is here the $n \times n$ zero matrix. This leads to a simple characterization of GF(4)-linear stabilizer spaces: let \mathcal{C}_S be a binary stabilizer space with generator matrix S . Denoting the columns of S by s^j ($j = 1, \dots, n$), the space \mathcal{C}_S is GF(4)-linear if and only if

$$(0, 1) * s^j \in \mathcal{C}_S \quad (22)$$

for every $j = 1, \dots, n$, as the columns of S form a basis of \mathcal{C}_S (regarded as a vector space over \mathbb{F}_2). Using (21) and the fact that any vector $v \in \mathbb{F}_2^{2n}$ belongs to \mathcal{C}_S if and only if $S^T P v = 0$, we find that \mathcal{C}_S is GF(4)-linear if and only if

$$S^T P \begin{bmatrix} 0 & I \\ I & I \end{bmatrix} S = 0. \quad (23)$$

Denoting $S = [S_z^T \ S_x^T]^T$, where S_z, S_x are $n \times n$ blocks, (23) is equivalent to

$$0 = S_z^T S_z + S_x^T S_x + S_z^T S_x. \quad (24)$$

We have proven the following theorem:

Theorem 2 *Let $|\psi\rangle$ be a stabilizer state on n qubits with generator matrix $S = [S_z^T \ S_x^T]^T$. Then \mathcal{C}_S is GF(4)-linear if and only if*

$$S_z^T S_z + S_x^T S_x + S_z^T S_x = 0. \quad (25)$$

In particular, if $S = [\theta \ I]^T$ is the generator matrix of a graph state, this is equivalent to $\theta^2 + \theta + I = 0$.

Theorem 2 shows that it is indeed easy to check whether a stabilizer corresponds to a GF(4)-linear code. It is interesting to note that the $2n \times 2n$ matrix

$$\begin{bmatrix} 0 & I \\ I & I \end{bmatrix} \quad (26)$$

in (23) belongs to the group \mathcal{C}_n^l and is therefore the binary representation of a local Clifford operation $V \in \mathcal{C}_n^l$ [18]. What is more, every tensor factor of V corresponds to the same binary operator, namely the 2×2 matrix which appears in (20), which in turn corresponds to a cyclic permutation $\sigma_x \rightarrow \sigma_y \rightarrow \sigma_z \rightarrow \sigma_x$ of the three Pauli matrices. Thus, (23) expresses that the operator (26) maps the space \mathcal{C}_S to itself or, equivalently, that $V|\psi\rangle = |\psi\rangle$.

V. DISCUSSION AND CONCLUSION

We will now discuss the results in this paper. Our main result was the presentation of a subclass of stabilizer states, such that any two LU equivalent stabilizer states in this class must also be LC equivalent. To construct our class, a central concept was that of minimal support. In particular, we showed that if the subgroup

$\mathcal{M}(\psi)$ generated by all minimal elements in a stabilizer $\mathcal{S}(\psi)$ has a sufficiently rich structure, then the LU equivalence class and the LC equivalence class of the state $|\psi\rangle$ coincide.

The main objectives of this work were to give support to the conjecture that $\text{LU}(\psi) = \text{LC}(\psi)$ for every stabilizer state and to investigate the relevance of the notion of minimal support in this problem. The next insight that needs to be gained in the present research, is what constraints are imposed on those stabilizer states that do not meet the requirement of theorem 1 and what the structure and the size of this remaining set of states is. It is in fact not easy to find many examples of states outside of our class, and it is not unlikely that the constraints imposed on such states are sufficiently strict, such that the question whether $\text{LU}(\psi) = \text{LC}(\psi)$ for these remaining states can be settled by considering these states case by case. We note that it is not our hope that our class covers all stabilizer states. Indeed, there do exist states that do not belong to the class we have considered. An important example is the generalized GHZ state $|GHZ_n\rangle$ on n qubits, which has a stabilizer $\mathcal{S}(GHZ_n)$ generated by the elements

$$\begin{aligned} & \sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \sigma_x \otimes \cdots \otimes \sigma_x, \\ & \sigma_z \otimes \sigma_z \otimes \sigma_0 \otimes \sigma_0 \otimes \cdots \otimes \sigma_0, \\ & \sigma_0 \otimes \sigma_z \otimes \sigma_z \otimes \sigma_0 \otimes \cdots \otimes \sigma_0, \\ & \sigma_0 \otimes \sigma_0 \otimes \sigma_z \otimes \sigma_z \otimes \cdots \otimes \sigma_0, \\ & \dots \end{aligned} \tag{27}$$

The subgroup $\mathcal{M}(GHZ_n)$ consists of all possible elements $M = M_1 \otimes \cdots \otimes M_n$ with $M_i \in \{\sigma_0, \sigma_z\}$, for every $i = 1, \dots, n$, and therefore $|GHZ_n\rangle$ does not meet the requirement of theorem 1. However, also for the GHZ state (and therefore for every state in its LC equivalence class) one has $\text{LU}(GHZ_n) = \text{LC}(GHZ_n)$. This is stated in the following proposition, which is proven in appendix B.

Proposition 1 *Let $|\psi\rangle$ be a fully entangled stabilizer state. Then $|\psi\rangle \in \text{LC}(GHZ_n)$ if and only if $A_\omega(\psi) = 1$ for every $\omega \in \{\{1, 2\}, \{2, 3\}, \dots\}$. Consequently, $\text{LU}(GHZ_n) = \text{LC}(GHZ_n)$.*

Therefore, while the GHZ state does not belong to our subclass of stabilizer states, it does still satisfy $\text{LU}(GHZ_n) = \text{LC}(GHZ_n)$, which can be proven with simple arguments. It is to date not clear if there exist other stabilizer states that do not satisfy the conditions of theorem 1 and hence do not belong to our subclass.

In conclusion, we have studied the relation between local unitary equivalence and local Clifford equivalence of stabilizer states. In particular, we have investigated the question whether every two LU equivalent stabilizer states are also LC equivalent. We have shown that the answer to this question is positive for a large class of stabilizer states, which can be regarded as an extension of the set of those stabilizer states that correspond to

GF(4)-linear codes. We have given sufficient conditions for a state to belong to our specific class and, in particular, we have given a simple characterization of stabilizer states corresponding to GF(4)-linear codes in terms of the binary stabilizer formalism.

Acknowledgments

MVDN thanks H. Briegel for interesting discussions concerning stabilizer states and for inviting him to the Techn. Un. Innsbrück; MVDN thanks M. Hein, J. Eisert and D. Schlingemann for interesting discussions concerning LU and LC equivalence of stabilizer states. The authors thank the referee for a thorough reading of the manuscript. This research is supported by several funding agencies: Research Council KUL: GOA-Mefisto 666, GOA-Ambiorics, several PhD/postdoc and fellow grants; Flemish Government: - FWO: PhD/postdoc grants, projects, G.0240.99 (multilinear algebra), G.0407.02 (support vector machines), G.0197.02 (power islands), G.0141.03 (Identification and cryptography), G.0491.03 (control for intensive care glycemia), G.0120.03 (QIT), G.0452.04 (QC), G.0499.04 (robust SVM), research communities (ICCoS, ANMMM, MLDM); - AWI: Bil. Int. Collaboration Hungary/ Poland; - IWT: PhD Grants, GBOU (McKnow) Belgian Federal Government: Belgian Federal Science Policy Office: IUAP V-22 (Dynamical Systems and Control: Computation, Identification and Modelling, 2002-2006), PODO-II (CP/01/40: TMS and Sustainability); EU: FP5-Quprodis; ERNSI; Eureka 2063-IMPACT; Eureka 2419-FliTE; Contract Research/agreements: ISMC/IPCOS, Data4s, TML, Elia, LMS, IPCOS, Mastercard; QUIPROCONE; QUPRODIS.

APPENDIX A: PROOF OF LEMMA 3

Let $|\psi\rangle$ be a (fully entangled) stabilizer state on $n \geq 2$ qubits. Consider e.g. the first qubit. Denote $\omega = \{2, \dots, n\}$ and let \mathcal{S}_ω be the set consisting of all $M \in \mathcal{S}(\psi)$ such that $M_1 = \sigma_0$, which is a subgroup of $\mathcal{S}(\psi)$. Firstly, note that it is impossible that $\mathcal{S}_\omega = \mathcal{S}(\psi)$; indeed, if this were the case, then the set

$$\mathcal{S}_\omega^\times = \{M_\omega \mid M \in \mathcal{S}(\psi)\} \tag{A1}$$

would be a stabilizer on $n - 1$ qubits with cardinality 2^n , which is a contradiction (in (A1), we have used the notation $M_\omega = \alpha_M M_2 \otimes \cdots \otimes M_n$ as before). Secondly, suppose that there exists an $a \in \{x, y, z\}$ such that $M_1 \in \{\sigma_0, \sigma_a\}$ for every $M \in \mathcal{S}(\psi)$ (and both σ_0 and σ_a occur). It is then easy to verify that

$$\mathcal{S}(\psi) = \mathcal{S}_\omega \cup M^a \mathcal{S}_\omega \tag{A2}$$

where M^a is an arbitrary element in $\mathcal{S}(\psi)$ satisfying $M_1^a = \sigma_a$, and $M^a \mathcal{S}_\omega$ is the coset of \mathcal{S}_ω determined by the

element M^a . Therefore, $|\mathcal{S}_\omega| = 2^{n-1}$ and the stabilizer (A1) now has cardinality 2^{n-1} , consequently defining a stabilizer state $|\psi_\omega\rangle$ on $n-1$ qubits. Using the identities

$$\begin{aligned} |\psi\rangle\langle\psi| &= \frac{1}{2^n} \sum_{M \in \mathcal{S}} M \\ |\psi_\omega\rangle\langle\psi_\omega| &= \frac{1}{2^{n-1}} \sum_{N \in \mathcal{S}_\omega^\times} N \end{aligned} \quad (\text{A3})$$

and (A2), it easily follows that

$$|\psi\rangle\langle\psi| = \frac{1}{2}(I + M^a)(\sigma_0 \otimes |\psi_\omega\rangle\langle\psi_\omega|) \quad (\text{A4})$$

and therefore $\text{Tr}_1\{|\psi\rangle\langle\psi|\} = |\psi_\omega\rangle\langle\psi_\omega|$. This shows that $\text{Tr}_1|\psi\rangle\langle\psi|$ is a pure state, which leads to a contradiction, as $|\psi\rangle$ is fully entangled. It follows that $\sigma_x, \sigma_y, \sigma_z$ must occur on the first qubit of $\mathcal{S}(\psi)$. Repeating the above argument for all qubits yields the result. \square

APPENDIX B: PROOF OF PROPOSITION 1

If $|\psi\rangle \in \text{LU}(GHZ_n)$ then clearly $A_\omega(\psi) = 1$ for every $\omega \in \{\{1, 2\}, \{2, 3\}, \dots\}$. Conversely, suppose that $|\psi\rangle$ is a fully entangled stabilizer state on n qubits such that $A_\omega(\psi) = 1$ for every $\omega \in \{\{1, 2\}, \{2, 3\}, \dots\}$. Then $\mathcal{S}(\psi)$ has $n-1$ elements

$$\begin{aligned} &\alpha_1 \sigma_{k_1} \otimes \sigma_{k_2} \otimes \sigma_0 \otimes \sigma_0 \otimes \dots \otimes \sigma_0, \\ &\alpha_2 \sigma_0 \otimes \sigma_{l_2} \otimes \sigma_{l_3} \otimes \sigma_0 \otimes \dots \otimes \sigma_0, \\ &\alpha_3 \sigma_0 \otimes \sigma_0 \otimes \sigma_{m_3} \otimes \sigma_{m_4} \otimes \dots \otimes \sigma_0, \\ &\dots \end{aligned} \quad (\text{B1})$$

where $\alpha_1, \alpha_2, \alpha_3, \dots \in \{\pm 1\}$ and $k_1, k_2, l_2, l_3, \dots \in \{x, y, z\}$. Moreover, $k_2 = l_2, l_3 = m_3, \dots$ since $\mathcal{S}(\psi)$ is an abelian group. This yields $n-1$ independent [19] elements of $\mathcal{S}(\psi)$. Therefore, to obtain a complete set of n generators, we need one additional element. We claim that any other generator $M = \alpha_n M_1 \otimes \dots \otimes M_n$ of $\mathcal{S}(\psi)$ must have full support and additionally satisfy $M_1 \neq \sigma_{k_1}, M_2 \neq \sigma_{l_2}, \dots$: indeed, this readily follows from lemma 3. We therefore obtain a set of generators of the form

$$\begin{aligned} &\gamma_1 \sigma_{a_1} \otimes \sigma_{a_2} \otimes \sigma_{a_3} \otimes \sigma_{a_4} \otimes \dots \otimes \sigma_{a_n}, \\ &\gamma_2 \sigma_{b_1} \otimes \sigma_{b_2} \otimes \sigma_0 \otimes \sigma_0 \otimes \dots \otimes \sigma_0, \\ &\gamma_3 \sigma_0 \otimes \sigma_{b_2} \otimes \sigma_{b_3} \otimes \sigma_0 \otimes \dots \otimes \sigma_0, \\ &\gamma_4 \sigma_0 \otimes \sigma_0 \otimes \sigma_{b_3} \otimes \sigma_{b_4} \otimes \dots \otimes \sigma_0, \\ &\dots, \end{aligned} \quad (\text{B2})$$

with $\gamma_i \in \{\pm 1\}$, $a_i, b_i \in \{x, y, z\}$ and $a_i \neq b_i$ for every $i = 1, \dots, n$. One can now always find a local Clifford operator which maps these operators to the set (27), which shows that $|\psi\rangle \in \text{LC}(GHZ_n)$. Finally, suppose that $|\psi\rangle$ is a stabilizer state LU equivalent to $|GHZ_n\rangle$. Then $|\psi\rangle$ is fully entangled and $A_\omega(\psi) = A_\omega(GHZ) = 1$ for every $\omega \in \{\{1, 2\}, \{2, 3\}, \dots\}$. But then $|\psi\rangle \in \text{LC}(GHZ_n)$ from the above. This ends the proof. \square

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- [17] We note that $\text{LU}(\psi) = \text{LC}(\psi)$ for every stabilizer state $|\psi\rangle$ on $n \leq 2$ qubits; this can trivially be verified for the 1-qubit case; for the 2-qubit case, this result essentially follows from the fact that, up to LC equivalence, the only fully entangled stabilizer state is the EPR state $(|00\rangle + |11\rangle)/\sqrt{2}$.
- [18] To be precise, it is the binary representation of a set of local Clifford operations, since a local Clifford operation is defined by an element in C_n^l and a set of $2n$ phases ± 1 .
- [19] By "independent" is meant that no element in this set can be written as a product of the others.