

An efficient algorithm to recognize local Clifford equivalence of graph states

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In [Phys. Rev. A 69, 022316 (2004)] we presented a description of the action of local Clifford operations on graph states in terms of a graph transformation rule, known in graph theory as *local complementation*. It was shown that two graph states are equivalent under the local Clifford group if and only if there exists a sequence of local complementations which relates their associated graphs. In this short note we report the existence of a polynomial time algorithm, published in [Combinatorica 11 (4), 315 (1991)], which decides whether two given graphs are related by a sequence of local complementations. Hence an efficient algorithm to detect local Clifford equivalence of graph states is obtained.

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Graph states have been studied extensively and have been employed in a number of applications in quantum information theory and quantum computing (see e.g. [1–4]). This is mainly due to the fact that these states can be described in a relatively transparent way, while they maintain a sufficiently rich structure. In this note we consider the problem of recognizing local Clifford (LC) equivalence between graph states. This issue is of natural importance in multipartite entanglement theory [4, 5] and in the development of the one-way quantum computer, which is a universal measurement-based model of quantum computation [3]. In the following, we present an efficient algorithm which recognizes whether two given graph states are LC-equivalent. At the heart of this algorithm lies an earlier result [5] of ours, which is a translation of the action of local Clifford operations on graph states in terms of a graph transformation rule known in graph theory as *local complementation*. It was shown that two graph states are equivalent under the local Clifford group if and only if there exists a sequence of local complementations which relates their associated graphs. As it turns out, local graph complementation is well known in graph theory (see e.g. [6] and references within). What is more, in ref. [7] a polynomial time algorithm is derived which detects whether two given graphs are related by a sequence of local complementations. This yields for our purposes an efficient algorithm which recognizes LC-equivalence of graph states. We repeat this algorithm below. Note that the present result immediately yields an efficient algorithm to recognize LC-equivalence of *all* stabilizer states (and not just the subclass of graph states). Indeed, it is well known that any stabilizer state is LC-equivalent to a graph state [8, 9]. Moreover, if a particular stabilizer state is given then an LC-equivalent graph state can be found in polynomial time, as the typical existing algorithms used to produce this graph state essentially use pivoting methods, which can be implemented efficiently.

Before presenting the algorithm, we state some definitions and introduce some notations. *Graph states* are special cases of stabilizer states. They are defined as follows: let G be a simple graph on n vertices with adjacency matrix θ [11] and define n commuting correlation operators K_j ($j = 1, \dots, n$), which act on $\mathbb{C}_2^{\otimes n}$, by

$$K_j := \sigma_x^{(j)} \prod_{k=1}^n \left(\sigma_z^{(k)} \right)^{\theta_{kj}}.$$

Here $\sigma_x^{(i)}, \sigma_y^{(i)}, \sigma_z^{(i)}$ are the Pauli matrices which act on the i th copy of \mathbb{C}_2 . The graph state $|G\rangle$ is the unique eigenvector (up to an overall phase) with eigenvalue one of the n operators K_j . The *Clifford group* \mathcal{C}_1 on one qubit is the group of all 2×2 unitary operators which map σ_u to $\alpha_u \sigma_{\pi(u)}$ under conjugation, where $u = x, y, z$, for some $\alpha_u = \pm 1$ and some permutation π of $\{x, y, z\}$. The n -qubit local Clifford group \mathcal{C}_n^l is the n -fold tensor product of \mathcal{C}_1 with itself. The action of local Clifford operations on graph states can be translated elegantly in terms of graph transformations. In ref. [5] we give such a translation and show that two states $|G\rangle, |G'\rangle$ are LC-equivalent if and only if G' can be obtained from G by a finite sequence of *local complementations*. The local complement $g_i(G)$ of a graph G at one of its vertices $i \in V$ is defined by its adjacency matrix $g_i(\theta)$ as follows:

$$g_i(\theta) = \theta + \theta_i \theta_i^T + \Lambda, \quad (1)$$

where θ is the adjacency matrix of G , θ_i is its i th column and Λ is a diagonal matrix such as to yield zeros on the diagonal of $g_i(\theta)$. Addition in (1) is to be performed modulo two. In graph theoretical terminology, $g_i(G)$ is obtained by replacing the subgraph of G induced by the neighborhood of i by its complement.

It is well known (see e.g. [10]) that the stabilizer formalism has an equivalent formulation in terms of binary linear algebra. In this binary formulation, a stabilizer state on n -qubits corresponds to an n -dimensional self-dual linear subspace of \mathbb{F}_2^{2n} . Here \mathbb{F}_2 is the finite field of two elements (0 and 1), where arithmetics are performed modulo 2. The self-duality of the subspace is with respect to a symplectic inner product $\langle \cdot, \cdot \rangle$ defined by

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$\langle u, v \rangle := u^T P v$, where $u, v \in \mathbb{F}_2^{2n}$ and

$$P = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$

The binary stabilizer subspace is usually presented in terms of a full rank $2n \times n$ generator matrix S , the columns of which form a basis of the subspace. This generator matrix satisfies $S^T P S = 0$ from the self-duality of the space. The entire binary stabilizer space, which we denote by \mathcal{C}_S , is the column space of S . It can easily be shown that a graph state with adjacency matrix θ has a generator matrix

$$S = \begin{bmatrix} \theta \\ I \end{bmatrix}.$$

In the binary framework, local Clifford operations $U \in \mathcal{C}_n^l$ correspond to nonsingular $2n \times 2n$ binary matrices Q of the block form

$$Q = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where the $n \times n$ blocks A, B, C, D are diagonal [5]. We denote the diagonal entries of A, B, C, D , respectively, by a_i, b_i, c_i, d_i , respectively. The n submatrices

$$Q^{(i)} := \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}$$

correspond to the tensor factors of U . It follows that each of the matrices $Q^{(i)}$ is invertible. Equivalently, the determinants $a_i d_i + b_i c_i$ are equal to one. We denote the group of all such Q by \mathcal{C}_n^l .

We are now in a position to state the algorithm. Let $|G\rangle, |G'\rangle$ be two states with adjacency matrices θ, θ' , respectively, and generator matrices

$$S := \begin{bmatrix} \theta \\ I \end{bmatrix}, \quad S' := \begin{bmatrix} \theta' \\ I \end{bmatrix},$$

respectively. Then $|G\rangle$ and $|G'\rangle$ are LC-equivalent if and only if there exists $Q \in \mathcal{C}_n^l$ such that $\mathcal{C}_{QS} = \mathcal{C}_{S'}$. Equivalently, this occurs iff there exists an invertible $n \times n$ matrix R over \mathbb{F}_2 such that

$$QSR = S'. \quad (2)$$

If θ and θ' are given, (2) is a matrix equation in the unknowns Q and R . Note that we can get rid of the unknown R , as (2) is equivalent to

$$S^T Q^T P S' = 0. \quad (3)$$

Indeed, (3) expresses that $u^T P v = 0$ for every $u \in \mathcal{C}_{QS}$ and $v \in \mathcal{C}_{S'}$, which implies that \mathcal{C}_{QS} and $\mathcal{C}_{S'}$ are each other's symplectic orthogonal complement. These spaces

must therefore be equal, as any n -dimensional binary stabilizer space is its own symplectic dual, and (2) is obtained. More explicitly, (3) is the system of n^2 linear equations

$$\left(\sum_{i=1}^n \theta_{ij} \theta'_{ik} c_i \right) + \theta_{jk} a_k + \theta'_{jk} d_j + \delta_{jk} b_j = 0, \quad (4)$$

for all $j, k = 1, \dots, n$, where the $4n$ unknowns a_i, b_i, c_i, d_i must satisfy the quadratic constraints

$$a_i d_i + b_i c_i = 1. \quad (5)$$

The set \mathcal{V} of solutions to the linear equations (4), with disregard of the constraints, is a linear subspace of \mathbb{F}_2^{4n} . A basis $B = \{b_1, \dots, b_d\}$ of \mathcal{V} can be calculated efficiently in $\mathcal{O}(n^4)$ time by standard Gauss elimination over \mathbb{F}_2 . Then we can search the space \mathcal{V} for a vector which satisfies the constraints (5). As (4) is for large n a highly overdetermined system of equations, the space \mathcal{V} is typically low-dimensional. Therefore, in the majority of cases this method gives a quick response. Nevertheless, in general one cannot exclude that the dimension of \mathcal{V} is of order $\mathcal{O}(n)$ and therefore the overall complexity of this approach is nonpolynomial. However, it was shown in [7] that it is sufficient to enumerate a specified subset $\mathcal{V}' \subseteq \mathcal{V}$ with $|\mathcal{V}'| = \mathcal{O}(n^2)$ in order to find a solution which satisfies the constraints, if such a solution exists. Indeed, the following lemma holds:

Lemma 1 [7] *If $\dim(\mathcal{V}) > 4$, then the system (4)-(5) of linear equations plus constraints has a solution if and only if the set*

$$\mathcal{V}' := \{b + b' \mid b, b' \in B\} \subseteq \mathcal{V}$$

contains a vector which satisfies the constraints.

The proof of lemma 1 is involved and makes extensive use of local graph complementation. The reader is referred to ref. [7] for more details. Lemma 1 shows that, if a solution to (4)-(5) exists, this solution can be found by enumerating either all $|\mathcal{V}| \leq 16$ elements of \mathcal{V} if $\dim(\mathcal{V}) \leq 4$ or the $\mathcal{O}(n^2)$ elements of \mathcal{V}' if $\dim(\mathcal{V}) > 4$ and checking these vectors against the constraints (5). Hence, a polynomial time algorithm to check the solvability of (4)-(5) is obtained. The overall complexity of the algorithm is $\mathcal{O}(n^4)$. Note that, whenever LC-equivalence occurs, this algorithm provides an explicit local unitary operator in the Clifford group which maps the one state to the other, as a solution $(a_1, b_1, c_1, d_1, \dots, a_n, b_n, c_n, d_n)$ to (4)-(5) immediately yields an operator $Q \in \mathcal{C}^l$.

In conclusion, we have presented an algorithm of polynomial complexity which detects whether two given graph states are equivalent under the local Clifford group. This algorithm leans heavily on a former result of ours, which is a description of the action of local Clifford operations on graph states in terms of local graph complementation. Whenever equivalence of two graph states is

recognized, the algorithm provides an explicit local Clifford operator which maps the one state to the other. Moreover, together with existing algorithms, this result yields an efficient algorithm which recognizes local Clifford equivalence of all stabilizer states.

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- [1] D. Gottesman. *Stabilizer codes and quantum error correction*. PhD thesis, Caltech, 1997. quant-ph/9705052.
- [2] W. Dür, H. Aschauer, and H.J. Briegel. Multiparticle entanglement purification for graph states. *Phys. Rev. Lett.*, 91:107903, 2003. quant-ph/0303087.
- [3] R. Raussendorf, D.E. Browne, and H.J. Briegel. Measurement-based quantum computation with cluster states. *Phys. Rev. A*, 68:022312, 2003. quant-ph/0301052.
- [4] M. Hein, J. Eisert, and H.J. Briegel. Multi-party entanglement in graph states. quant-ph/0307130.
- [5] M. Van den Nest, J. Dehaene, and B. De moor. Graphical description of the action of local clifford operations on graph states. *Phys. Rev. A*, 69:022316, 2004. quant-ph/0308151.
- [6] A. Bouchet. Recognizing locally equivalent graphs. *Discrete Math.*, 114:75–86, 1993.
- [7] A. Bouchet. An efficient algorithm to recognize locally equivalent graphs. *Combinatorica*, 11(4):315–329, 1991.
- [8] D. Schlingemann. Stabilizer codes can be realized as graph codes. quant-ph/0111080.
- [9] M. Grassl, A. Klappenecker, and M. Roetteler. Graphs, quadratic forms and quantum codes. In *IEEE international symposium on information theory*, Lausanne, 2001.
- [10] I. Chuang and M. Nielsen. *Quantum computation and quantum information*. Cambridge University press, 2000.
- [11] A simple graph G has no loops or multiple edges. Therefore, it can be described by a $n \times n$ symmetric matrix θ where θ_{ij} is equal to 1 whenever there is an edge between vertices i and j and zero otherwise. As G has no loops, $\theta_{ii} = 0$ for every $i = 1, \dots, n$