Coupled string guitar models.[1]

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Abstract

An important aspect of the sound of the acoustic guitar is the beating caused by string interaction. In this paper, we present a coupled string model for the acoustic guitar that includes the body in the feedback loop. Constraints on the model parameters are determined using Finite Element analysis of the saddle.

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Abstract: An important aspect of the sound of the acoustic guitar is the beating caused by string interaction. In this paper, we present a coupled string model for the acoustic guitar that includes the body in the feedback loop. Constraints on the model parameters are determined using Finite Element analysis of the saddle.

Key–Words: Music, Sound synthesis, Physical modeling, String Coupling, Waveguide

1 Introduction

One important aspect that affects the natural quality of the sound of a string instrument is the sympathetic coupling of the strings. As one string is struck, plucked or bowed, the other strings of the instrument start to vibrate and interact with the excited string, changing the global decay or adding beating to some partials of the sound. This is an important aspect as a sound without some amount of beating is almost immediately recognized as synthetic. Several mathematical structures to simulate the sympathetic coupling of strings have been proposed in the past. In this study, we propose a general guitar model that includes the guitar body. This gives the performer extra possibilities and makes the global model quite easy to calibrate. The paper starts with the common waveguide models in section 2 and builds it up to a fully coupled model, including the body resonator. In section 3, rules for the coupling parameters are derived based on Finite Element analysis and physical considerations.

2 Modeling

To build the coupled multi–string model, we start from the acoustic principles (2.1) and apply the common waveguide physical modeling technique (2.2) to build a coupled string model (2.3) and finally a fully coupled instrument model (2.4).

2.1 Acoustic principles of the guitar

The acoustic sound generation mechanism of the classical guitar is fairly straightforward: a wave on the string is reflected at the attachment points (the saddle/bridge and the nut). This results in reactive forces and a slight displacement of the attachment points. The nut is fairly rigid (almost ideal reflection), but the bridge is attached to the flexible top plate, resulting in an energy transfer to the guitar body (damping of the wave and movement of the top plate). The guitar is designed such that both the sustain (duration of the sound; lower damping results in longer sustain) and the loudness (more transfer to the guitar body) are acceptable. The guitar body acts as a resonator and couples the string movement to the surrounding air.

2.2 Waveguide model

The most widely used physical modeling technique for string instruments is based on the general solution of the wave equation describing the string movement [1]. The resulting structures (digital waveguides) are physically relevant abstractions and computationally efficient models that can be used as basic building blocks for larger models [2]. A good overview of the evolution of the digital waveguide technique applied to string instrument modeling can be found in [3].

In digital waveguide modeling, the string is represented by two fractional delay lines, connected by linear filters that model the reflection at the bridge

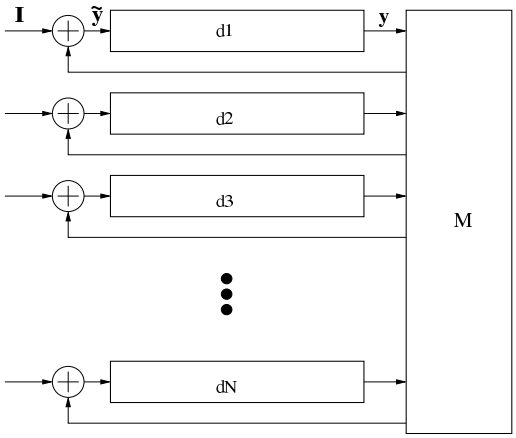

Fig. 1 A general waveguide structure with string coupling through a coupling matrix M. Elements of M are complex and frequency–dependent.

and nut. More recent models use only one delay line, with the the two reflectors lumped together in one point.

2.3 Coupled string model

At the saddle/bridge point, all the strings induce movement of the top plate. At the same time, movement of the top plate moves all the strings. This means that the strings cannot be seen as independent entities, but must be seen as parts of a larger, coupled system. Single delay–line (SDL) waveguide implementations of a guitar with coupled strings have a structure as shown in figure 1. Because of the dual polarization (vertical and horizontal movement of the strings), we need N=12 fractional delay lines to represent the complete guitar. In the most general case, the delay line input $\tilde{y}$ is calculated as the sum of the system input I and the vectorized output y of the delay lines multiplied by a coupling matrix C.

$$\tilde{y} = C \cdot y + I \qquad (1)$$

The elements c_{km} of the coupling matrix represent the (lumped) transmission from delay line m to delay line k, and the elements c_{kk} represent the reflection for delay line k. Note that, in general, the c_{km} are complex and frequency–dependent and should, therefore, be implemented as linear filters. A physically sound value for the elements of the coupling matrix can be derived from the bridge admittance (bridge coupled model) [4]. In this model, the admittance measured at the bridge is seen by all the strings and results in a

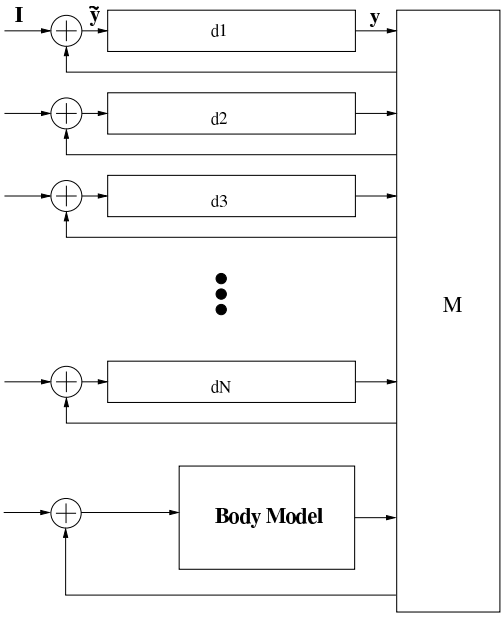

Fig. 2 Adding the body itself as a resonator with full coupling to strings. The coupling M now effectively represents the saddle/bridge.

common movement. As a result, the elements of the coupling matrix are quite complicated expressions.

2.4 Fully coupled instrument model

The general structure basically consists of a coupling node and a series of resonators. We can add an extra resonator to the structure: the body itself, as can be seen on figure 2. This is fairly similar to the N–dimensional loaded waveguide junction [5], but with feedback to the same junction. Adding the body as a "pure" resonator has a great advantage. It is now possible to give an independent input to the body (e.g. a slap, as in *flamenco* playing) that will cause all the strings to vibrate. The coupling matrix now has a slightly different interpretation.

It is clear from the bridge–coupled model that the coupling matrix does not represent the bridge as such, but the combination of body and bridge (the admittance of the whole system measured at the bridge). Existing models include the guitar body only as a pre– or postfiltering operation [3]. If we include the body as an independent resonator, we effectively simplify the coupling matrix as we represent *only* the bridge. In the acoustic guitar case, the saddle/bridge

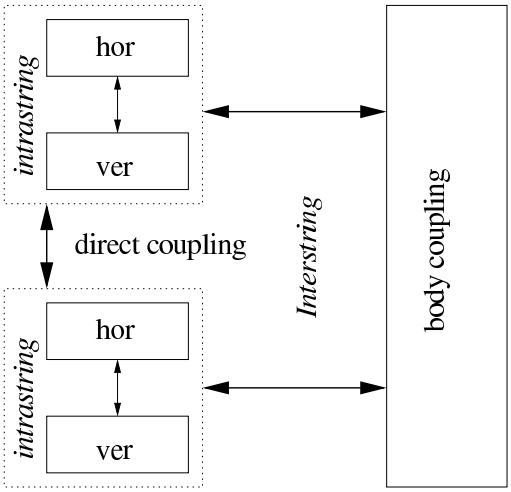

Fig. 3 The different classes of coupling possible with the fully coupled instrument model.

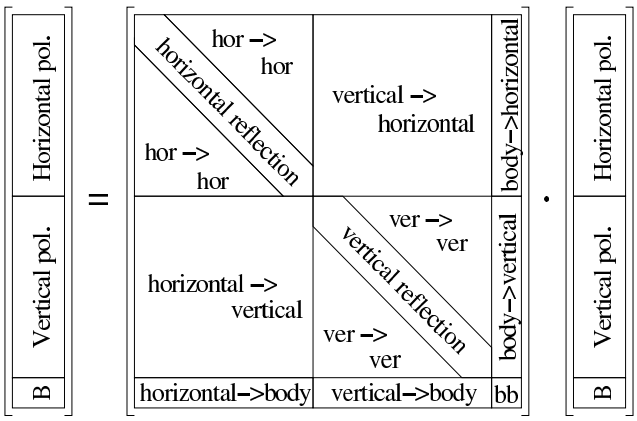

Fig. 4 The coupling matrix C after inclusion of the body. The polarizations are separated. The distinction between direct bridge coupling and body coupling is now clear.

is a fairly rigid construction, with vibration modes of rather high frequency compared to the body: this will lead to nearly frequency independent coefficients. We have an added degree of freedom to describe the coupling: it is possible to couple strings through direct bridge coupling, without influence of the body (see figure 3). Grouping the horizontal and vertical polarizations gives the global coupling matrix structure shown in figure 4.

We will now build several waveguide models, by using the complete coupling matrix and by approximating it without sacrificing too much quality.

3 Calibration

A valid approximation of the coupling matrix can be found if we have a better understanding of the coupling mechanism itself. To analyze the coupling between the strings, we construct a finite element model of the saddle of the guitar using the *Aladdin* software package [6], and calculate the displacement and internal stresses for the static load and for unit forces in X and Y directions. Based on these results, we simplify the coupling matrix and estimate the coefficients.

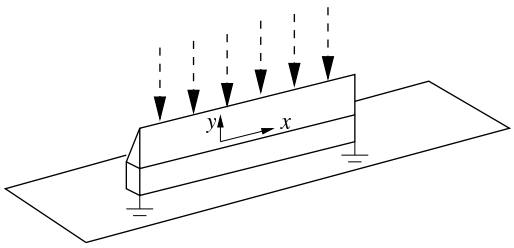

Fig. 5 Schematic view of the saddle. The two lower corners of the finite element model of the saddle are fixed nodes (allow no movement in X or Y direction).

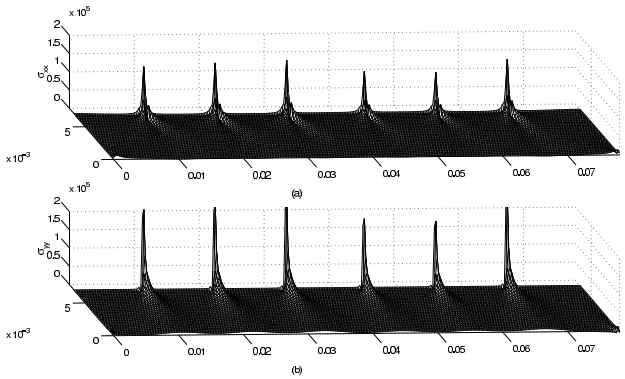

Fig. 6 The static stresses inside the saddle obtained by finite element analysis. The model is based on measurements of the saddle /bridge of a Yamaha C70 acoustic guitar using d'Addario Pro Arte 45 strings under normal tension. The saddle material was assumed bone ($E=1.7GN/m^2$). Its dimensions are 78mm× 7mm × 2.5mm. (a) σ_{xx} stress in X direction, (b) σ_{yy} stress in Y direction.

3.1 Finite element analysis

Figure 5 shows a schematic view of the analyzed structure. The lower two corners of the saddle are fixed in X and Y direction (this is needed for a valid finite element simulation). We assume that a displacement in Y direction at the bottom of the saddle results in movement of the top plate in the Y direction only. This is a valid assumption because the top plate is flexible compared to the saddle/bridge (in Y direction) and as rigid as the saddle/bridge (in the X direction).

We first determined the stresses at rest with six strings attached. The forces caused by the strings on the saddle were calculated for d'Addario Pro Arte 45 normal tension strings. Figure 6 shows the results. The experiment was repeated for a unit force in the Y direction (figure 7) and in the X direction (figure 8).

The results for a load in Y direction (string moving in vertical polarization) show that this load results in stress in both the X and Y direction and that the effect of the load is fairly localized around the load position (load point x=33mm, substantial stress between 26mm<x<40mm). This means that almost all of the force applied by the string is transmitted to the top plate, without *directly* influencing the other strings. Of course, the displacement of the top plate due to the load will influence *all* the strings. We conclude that interstring vertical–vertical coupling proceeds almost entirely through body coupling.

A load in the X direction (string moving in horizontal polarization) gives a different picture. Again, there is stress in both X and Y direction, but the effect is not localized around the load point. The whole bridge sees a stress (and the related displacement). The X component of the stress results in a *direct* coupling in the X direction for all the strings. As the Y component in this case is rather small compared to the Y component when loading in the Y direction, it results in less transmission to the top plate and thus a lower damping. We conclude that interstring horizontal–horizontal coupling happens through direct coupling and interstring horizontal–vertical coupling through a combination of direct and body coupling.

Intrastring coupling is somewhat more difficult to

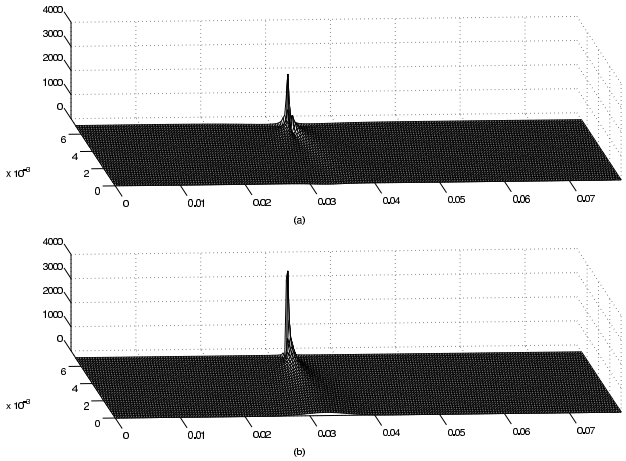

Fig. 7 The effect of unit force in the Y direction at the third string. The internal stresses are symmetric in the (a) X and (b) Y direction. The influence at y=0 (on the top board) is very small in the X direction but quite large in the Y direction.

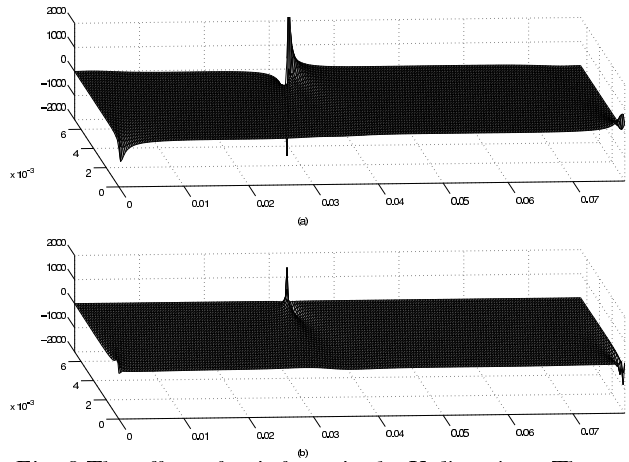

Fig. 8 The effect of unit force in the X direction. The internal stresses are asymmetric in the (a) X and (b) Y direction. Note that the stress in X or Y direction is not localized around the load point.

determine. A load in the Y direction results in a symmetric stress pattern in the X direction. A load in the X direction creates an asymmetric stress in the Y direction. We believe that this leads to stronger horizontal–vertical intrastring coupling than vertical–horizontal intrastring coupling.

These calculations are also valid for the dynamic case with changing load. The time constant to achieve stress equilibrium in the saddle is very small

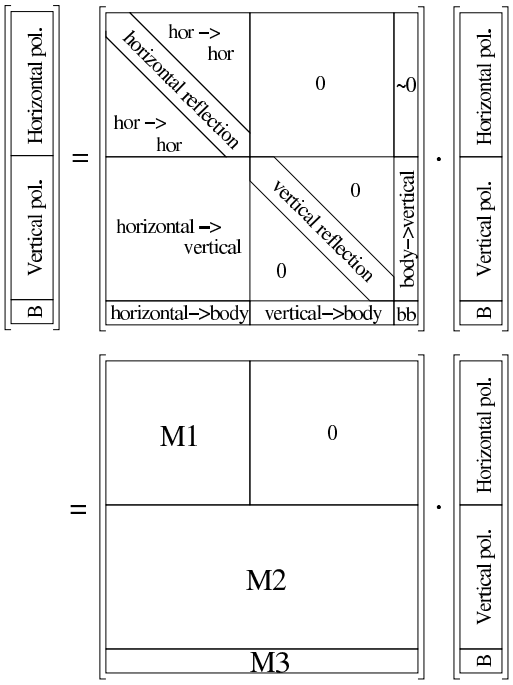

Fig. 9 Including the results of the finite element analysis simplifies the coupling matrix.

compared to the frequency of the load change (this is valid up to 10kHz). This is due to the high speed of sound in bone, combined with the small dimensions of the saddle.

Including these results in the general structure of the coupling matrix yields the simplified matrix shown in figure 9. Hence:

$$\begin{aligned} H_{input} &= M1_{6\times6}\cdot H_{output} + I_{hor} \\ V_{input} &= M2_{13\times6}\cdot\begin{pmatrix} H_{output} \\ V_{output} \\ B_{output} \end{pmatrix} + I_{ver} \\ B_{input} &= M3_{13\times1}\cdot\begin{pmatrix} H_{output} \\ V_{output} \\ B_{output} \end{pmatrix} + I_{body} \end{aligned} \qquad (3)$$

where *H*, *V* and *B* are the inputs and outputs of respectively the horizontal polarization, the vertical polarization and the body. The bridge displacements are in phase with the string movement, so $0<c_{jk}<1$.

3.2 Parameter estimation

We now have to determine the numerical values for the coupling coefficients. The first step is to determine the reflection coefficients for the different polarizations. A good approximation is to calculate the damping for the fundamental frequency of each open string, using either Short–Time Fourier Transform or subband Hankel Singular Value Decomposition based methods [3,7,8]. The amplitude of each polarization of the fundamental can be approximated by

$$A = \exp(-a\cdot t) \quad , \qquad (3)$$

and the corresponding reflection coefficient is

$$c_{kk} = \exp(-a\cdot L_k / F_s), \qquad (4)$$

with c_{kk} the reflection coefficient for waveguide *k*, *a* the damping, L_k the length of the waveguide and F_s the sampling frequency.

If we consider the bridge as lossless, the sum of the "outgoing" displacements equals the "incoming" displacement. In terms of coupling coefficients, this means that

$$\sum_k c_{km} = 1 \quad , \qquad (5)$$

or the column sums of the coupling matrix are one. The coupling matrix elements c_{kk} are the result of lumping the reflection at both string ends together. An approximation for the reflection at the saddle/bridge only is the square root of c_{kk}. As we already know the value of c_{kk}, we can easily determine the values for the vertical–to–body coupling.

$$c_{V\to B} = 1 - \sqrt{c_{V\,refl.}} \quad . \qquad (6)$$

The $c_{H\to H}$ coefficients depend on the distance of the "source" string to the "target" string as can be seen on figure 8. We model this by taking the coefficient (linearly) proportional to the distance. The same holds for $c_{H\to V}$. According to the finite element analysis, the displacement in the horizontal dimension is about three times larger than in the vertical dimension, which leads to the ratio

$$c_{H\to H} = 3\cdot c_{H\to V} \quad . \qquad (7)$$

The transfer from the horizontal polarization to the instrument body is more difficult to determine. Note that coupling to the body is the only way of generating output (the body couples the string movement to the surrounding air). It has been found that

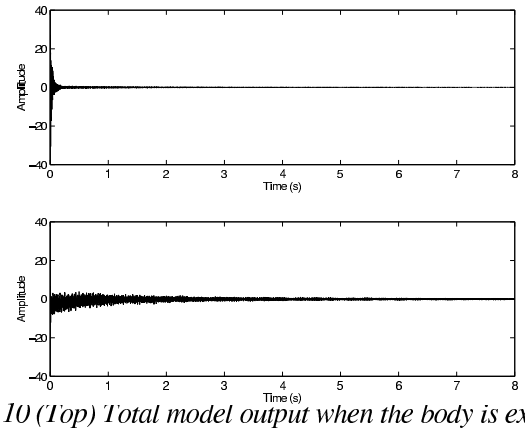

Fig. 10 (Top) Total model output when the body is excited with an impulse. (bottom) The sympathetic vibration of the strings.

$$c_{H\rightarrow B}=0.1\cdot(1-\sqrt{c_{H\,refl.}}) \qquad (8)$$

gives acceptable results. Given equation (8) and equation (5), $c_{H\rightarrow V}$ and $c_{H\rightarrow H}$ must now comply to

$$\sum_k c_{H\rightarrow V}+\sum_k c_{H\rightarrow H}=0.9\cdot(1-\sqrt{c_{H\,refl.}}) \qquad (10)$$

The last coefficients to be determined are $c_{B\rightarrow V}$. These can be estimated by exciting the guitar body with an impulse and recording the output. The ratio of string sound after the attack vs. body impulse amplitude gives an estimate of the coupling. Figure 10 shows the model output when the body is excited with an impulse.

4 Conclusion

The generation of natural acoustic guitar sounds requires a coupled string model. In this work, we included the guitar body as a resonator in a coupled string digital waveguide model of the acoustic guitar. This simplifies the coupling matrix to scalar values. Finite element analysis of the saddle/bridge structure provided a better insight of the coupling and enabled the determination of the coupling coefficients. The resulting model has a very natural sound and is easier to calibrate than bridge–coupled models.

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