Katholieke Universiteit Leuven

Departement Elektrotechniek

ESAT-SISTA/TR 1993-59

Frequency weighted H_2 model reduction[1] [2]

Geert Schelfhout, Peter Van Overschee, and Bart De Moor[3]

September 1993

Published in *"Nonlinear Numerical Methods and Rational Approximation II"*, A. Cuyt, ed., Kluwer Academic Publishers, 1994, pp. 187-199.

[1]This report is available by anonymous ftp from *ftp.esat.kuleuven.ac.be* as *pub/SISTA/naam/reports/h2uia.ps.Z*

[2]The following text presents research results obtained within the framework of the Belgian programme on interuniversity attraction poles (IUAP-17 and IUAP-50) initiated by the Belgian State - Prime Minister's Office - Science Policy Programming. The scientific responsibility is assumed by its authors.

[3]ESAT - Katholieke Universiteit Leuven,
Kardinaal Mercierlaan 94, 3001 Leuven (Heverlee), Belgium.
tel. : 32/16/32.18.08, fax : 32/16/32.19.86
geert.schelfhout@esat.kuleuven.ac.be *bart.demoor@esat.kuleuven.ac.be.*

FREQUENCY WEIGHTED H_2 MODEL REDUCTION

G. SCHELFHOUT,* P. VAN OVERSCHEE and B. DE MOOR†
ESAT–SISTA
Katholieke Universiteit Leuven
Kard. Mercierlaan 94
B-3001 Leuven
BELGIUM
Tel.: 00-32-16-220931, FAX: 00-32-16-221855
e-mail : geert.schelfhout@esat.kuleuven.ac.be

Abstract. We consider the problem of approximating a discrete time single-input single-output system by one of order less than r, minimizing the frequency weighted H_2 norm of the modeling error.
Using Lagrange multipliers, we derive a set of necessary conditions, which are solved iteratively in a way inspired by the inverse power iteration algorithm for singular value decomposition.

Keywords: H_2 norm, frequency weighting, inverse iteration

1. Introduction

H_2-optimal model reduction was one of the first model reduction techniques considered and has continued to raise interest in the last thirty years. Most approaches boil down to the calculation of a gradient and subsequent optimization using a descent algorithm, e.g. (Aplevich, 1973a; Galiana, 1973; Spanos *et al.*, 1992; Wilson *et al.*, 1979; Baratchart *et al.*, 1988a). Computing the gradient is generally costly however, therefore non-gradient algorithms (Lepschy *et al.*, 1991a) are potentially interesting. The method presented here is a novel non-gradient algorithm.

Frequency weighting is a very important asset of a model reduction method for control design purposes, and in general it is hard to name applications where the quality of the model has to be equally good for all frequencies. Ample motivation for frequency weighted model reduction in a control context can be found e.g. in (Anderson *et al.*, 1989; Doyle, 1981). Nevertheless, frequency weighted extensions for model reduction algorithms are rarely given in the literature (Noldus *et al.*, 1976; Halevi, 1990; Enns, 1984; Latham *et al.*, 1985). We have included frequency weighting with a stable and minimum phase weighting function.

* GS and PVO are research assistants with the NFWO (Belgian National Fund for Scientific Research)
† BDM is a research associate with the NFWO

2. Definitions and notation

T is the unit circle in the complex plane.

Let $x(z) \triangleq \sum_{-\infty}^{\infty} x_i z^{-i}$

$[x(z)]_i \triangleq x_i$: i-th Markov parameter of $x(z)$
$[x(z)]_+ \triangleq \sum_{i=1}^{\infty} x_i z^{-i}$: strictly proper part of $x(z)$
$[x(z)]_{0\dots r}$: first $r+1$ Markov parameters of $x(z)$
$\|x(z)\|_2 \triangleq \sqrt{\sum_{i=-\infty}^{\infty} h_i^2}$
$L_2 \triangleq \{x(z) |\ \|x(z)\|_2 < \infty\}$
$L_2^+ \triangleq \{x(z) \in L_2 |\ x(z) = [x(z)]_+\}$ (the set of strictly proper functions $\in L_2$)
$L_2^{0+} \triangleq \{x(z) \in L_2 |\ z^{-1}x(z) \in L_2^+\}$ (i.e. the Hardy space H_2)
RL_2 : $\{x(z) \in L_2 \mid x(z)$ converges to a rational function on $T\}$
RL_2^+ : $RL_2 \cap L_2^+$

It is important to note that if a series $x(z) \in L_2^+$, it converges for all z satisfying $|z| > 1$; if $x(z) \in RL_2^+$, for $|z| > \alpha$ (for some $\alpha < 1$); and if $x(z) \in RL_2$, it converges wherever $\alpha < |z| < \beta$, (where α, β are real numbers satisfying $\alpha < 1 < \beta$).

$x \star y$: convolution of series x and y ($(x \star y)_i \triangleq \sum_{j=-\infty}^{\infty} x_j y_{i-j}$)
If two series $x(z)$ and $y(z)$ converge in open regions S_x resp. S_y such that $S_x \cap S_y$ is nonempty, then the convolution $x(z)y(z)$ has well-defined coefficients and converges in $S_x \cap S_y$.

3. Necessary conditions

Let $w(z) \in RL_2^+$ be a stable and minimum phase (i.e. $w^{-1}(z) \in RL_2^+$) frequency weighting function[1] of order w. Let $h(z) \triangleq \sum_{i=0}^{\infty} h_i z^{-i} \in RL_2^+$ be the impulse response of a strictly proper stable system of order n. Suppose that $w(z)$ has no multiple poles and zeroes, that none of these coincide with a pole of $h(z)$, and that $h(z)$ has no multiple poles either[2].
We want to find the system with impulse response $\hat{h}(z) = \sum_{i=1}^{\infty} \hat{h}_i z^{-i} \in RL_2^+$ such that $\frac{1}{2}\sum_{i=1}^{\infty}([w \star h]_i - [w \star \hat{h}]_i)^2$ is minimized[3], and the $\infty \times (r+1)$ Hankel matrix $\hat{H}$ constructed from the Markov parameters $\hat{h}_i$, is not full rank, i.e. $\hat{h}(z)$ should have order less or equal to r. The rank-deficiency constraint on $\hat{H}$ is equivalent to the existence of a nonzero null vector, i.e. : $\exists y : \|y\|_2 = 1$ and $\hat{H}y = 0$.

[1] The assumption that $w(z)$ be stable and minimum phase is not restrictive, since only $w(z^{-1})w(z)$ is significant.
[2] An extension to the case where $w(z)$ has multiple poles or zeros is possible (using the same techniques), but omitted for the sake of brevity. The same remark is in order for $h(z)$.
[3] Note that one does not obtain the same minimum if one minimizes over all $\hat{h} = \sum_{i=0}^{\infty} \hat{h}_i z^{-i}$, as one would in the unweighted case.

Using Lagrange multipliers l_k and λ, one may formulate the problem as

$$\mathrm{ext}_{\hat{h},y,l,\lambda} \frac{1}{2}\sum_{i=1}^{\infty}([w\star h]_i - [w\star\hat{h}]_i)^2 + \sum_{k=1}^{\infty} l_k \sum_{j=0}^{r} \hat{h}_{k+j} y_j + \lambda(y^T y - 1)$$

Taking the derivatives of the Lagrangian w.r.t. $\hat{h}, l, y$ and λ yields the corresponding four conditions, written in terms of formal series (see appendix A) :

$$\left[w(z^{-1})w(z)(h(z)-\hat{h}(z))\right]_+ = l(z)y(z),$$
$$\left[\hat{h}(z)y(z^{-1})\right]_+ = 0,\ \left[\hat{h}(z)l(z^{-1})\right]_{0\ldots r} = 0 \text{ and } \|y\| = 1$$

$\hat{h}(z) \in RL_2^+$, therefore also $l(z) \in RL_2^+$, so that by Lemma 1 the first condition may be rewritten as :

$$h(z) - \hat{h}(z) = w^{-1}(z)\left[w^{-1}(z^{-1})l(z)y(z)\right]_+ \tag{1}$$

By convolution with $y(z^{-1})$ and $l(z^{-1})$ respectively, and taking the causal part, $\hat{h}(z)$ (and $\hat{H}$) is eliminated from the above necessary conditions:

$$\begin{aligned}\left[h(z)y(z^{-1})\right]_+ &= \left[w^{-1}(z)\left[w^{-1}(z^{-1})l(z)y(z)\right]_+ y(z^{-1})\right]_+\\ \left[h(z)l(z^{-1})\right]_{0\ldots r} &= \left[w^{-1}(z)\left[w^{-1}(z^{-1})l(z)y(z)\right]_+ l(z^{-1})\right]_{0\ldots r}\end{aligned}$$

If these equations can be solved for $y(z)$ and $l(z)$, $\hat{h}(z)$ can be recomputed from $y(z)$ and $l(z)$ using eq. (1).

4. Iteration scheme

Note that in the unweighted case the equations above reduce to:

$$\begin{aligned}\left[h(z)y(z^{-1})\right]_+ &= \left[y(z)y(z^{-1})l(z)\right]_+\\ \left[h(z)l(z^{-1})\right]_{0\ldots r} &= \left[l(z)l(z^{-1})y(z)\right]_{0\ldots r}\end{aligned}$$

In (De Moor, 1993b), $y(z)$ and $l(z)$ are represented as vectors of Markov parameters y_i and l_i, and these two equations are written as $Hy = T_y T_y^T l \triangleq D_y l$ and $H^T l = T_l T_l^T y \triangleq D_l y$, where T_y and T_l are Toeplitz matrices depending affinely on y_i and l_i respectively, and having only zeroes below the diagonal, such that D_y and D_l are invertible. If one rescales $l = x\sigma$ such that x is a unit vector as well, the equations read : $Hy = T_y T_y^T x\sigma$ and $H^T x = T_x T_x^T y\sigma$. The error $\|h - \hat{h}\|_2$ can be expressed as $\|x * y\|_2\sigma$. The similarity with an SVD problem and the dependence of the error criterion on the 'singular value' σ then prompts the following inverse power iteration scheme :

$$\tilde{A}_{k-1}\begin{bmatrix} y_k \\ l_k \end{bmatrix} \triangleq \begin{bmatrix} H & -D_{y_{k-1}} \\ 0 & H^T \end{bmatrix}\begin{bmatrix} y_k \\ l_k \end{bmatrix} = \begin{bmatrix} 0 \\ D_{l_{k-1}} y_{k-1} \end{bmatrix}$$

Note that if l_k is represented as a vector, it must necessarily be truncated to length p (as well as H and D_y). $\tilde{A}_{k-1}$ is invertible whenever $D_{y_{k-1}}$ is, hence there is always

just one solution for y_k and l_k.
This way one solves the problem for a finite time horizon H_2 norm $\|h(z)\|^2 \triangleq \sum_i^p h_i^2$. When using this scheme to find an approximate solution for the infinite time horizon scheme, one has a tradeoff between memory and time requirements on the one hand (small p is desirable), and accuracy requirements on the other (large p desirable). Letting $p \to \infty$, these equations read (in the previous notation):

$$\begin{aligned}\left[h(z)y^{[k]}(z^{-1})\right]_+ &= \left[l^{[k]}(z)y^{[k-1]}(z)y^{[k-1]}(z^{-1})\right]_+ \\ \left[h(z)l^{[k]}(z^{-1})\right]_{0\ldots r} &= \left[l^{[k-1]}(z)y^{[k-1]}(z)l^{[k-1]}(z^{-1})\right]_{0\ldots r}\end{aligned}$$

By analogy, we propose the following iteration for the frequency weighted case :

$$\left[h(z)y^{[k]}(z^{-1})\right]_+ = \left[w^{-1}(z)\left[w^{-1}(z^{-1})l^{[k]}(z)y^{[k-1]}(z)\right]_+ y^{[k-1]}(z^{-1})\right]_+ \tag{2}$$

$$\left[h(z)l^{[k]}(z^{-1})\right]_{0\ldots r} = \left[w^{-1}(z)\left[w^{-1}(z^{-1})l^{[k-1]}(z)y^{[k-1]}(z)\right]_+ l^{[k-1]}(z^{-1})\right]_{0\ldots r} \tag{3}$$

It is possible to find a solution $y^{[k]}(z)$ and $l^{[k]}(z) \in RL_2^+$ for these equations, and to express the denominator of $l^{[k]}(z)$ in terms of previously computed quantities, such that all the unknowns left to be computed each iteration are 'numerator quantities' (numerator coefficients or residuals in the poles) appearing linearly in the equations above (this statement is elaborated in more detail in the sequel).
The iteration could now be implemented with the polynomial techniques used for the unweighted case in (De Moor *et al.*, 1993a) (i.e. all series $h(z)$, $y^{[k]}(z)$, $l^{[k]}(z)$, $w(z)$, $[w^{-1}(z^{-1})l^{[k]}(z)y^{[k-1]}(z)]_+$ are represented as ratios of polynomials in z) resulting in a $(3n+2w+2r)$-by-$(3n+2w+2r)$ linear system to be solved each iteration (see section 4.2). To circumvent the problem of having to solve a linear system this size, a partial fraction representation of $l(z)$ is used. This has the advantage that equating causal parts of two expressions is equivalent with equating the residuals in the stable poles. As these residuals are exactly the unknowns, far fewer unknowns are needed, and only an $(n+w+r)$-by-$(n+w+r)$ linear system[4] has to be solved each time (cf. infra).

4.1. More definitions

The following definitions are needed in the sequel: $h(z) \triangleq \frac{b(z)}{a(z)} \triangleq \sum_{i=1}^n \frac{H_i}{z-\alpha_i}$,
$\hat{h}(z) \triangleq \frac{\hat{b}(z)}{\hat{a}(z)} \triangleq \sum_{i=1}^r \frac{\hat{H}_i}{z-\hat{\alpha}_i}$, $\quad w(z) \triangleq \frac{n_w(z)}{d_w(z)} \triangleq \sum_{i=1}^w \frac{W_i}{z-\alpha_{wi}}$
$w^{-1}(z) = (w(z))^{-1}$
$a_i^k \triangleq \alpha_i \quad (i = 1, \ldots, n), \qquad a_i^k \triangleq \alpha_{wi} \quad (i = n+1, \ldots, n+r)$,
$a_i^k \triangleq \hat{\alpha}_i^k \quad (i = n+r+1, \ldots, n+r+w)$
$\beta_{wi} \triangleq$ roots of $n_w(z)$ (= zeroes of $w(z)$)

[4] With complex coefficients this time, and a solution with complex conjugate elements.

From $\left[\hat{h}(z)y(z^{-1})\right]_+ = 0$, we find that $\hat{a}(z) = \eta y(z^{-1})$ (where η is a constant), therefore define:
$\hat{a}^{[k]}(z) \triangleq y^{[k]}(z^{-1}) \Rightarrow y^{[k]}(z) = \hat{a}^{[k]}(z^{-1}) \qquad \deg \hat{a}^{[k]}(z) = r$
$\hat{a}_r^{[k]}(z) \triangleq z^r \hat{a}^{[k]}(z^{-1})$
Finally define $\xi^{[k]}(z) \triangleq [w^{-1}(z^{-1})l^{[k]}(z)y^{[k-1]}(z)]_+$

4.2. Some useful properties

Assume $l^{[k]}(z) \in RL_2$, then $\xi^{[k]}(z) \in RL_2$ as well. Since $\left[w^{-1}(z)\xi^{[k]}(z)y^{[k-1]}(z^{-1})\right]_+ = \left[h(z)y^{[k]}(z^{-1})\right]_+ \in RL_2^+$ (2), $\xi^{[k]}(z) \in RL_2^+$ as well, since if it had any unstable pole, $w^{-1}(z)\xi^{[k]}(z)$ and $\left[w^{-1}(z)\xi^{[k]}(z)y^{[k-1]}(z^{-1})\right]_+$ would have the same pole. Rewriting (2) as $\frac{d_w(z)}{n_w(z)}\xi^{[k]}(z)\hat{a}^{[k-1]}(z) = \left[\frac{b(z)}{a(z)}y^{[k]}(z^{-1})\right]_+ + p(z)$, where $p(z)$ is a polynomial of degree $r-1$, and multiplying both sides with $w(z)\frac{1}{\hat{a}^{[k-1]}(z)}$, it is found that $\xi^{[k]}(z)$ must converge to a rational function of the form[5] :

$$\xi^{[k]}(z) = \frac{n_w(z)\phi^{[k]}(z)}{a(z)\hat{a}^{[k-1]}(z)d_w(z)}, \qquad \deg \phi^{[k]}(z) \leq n + r - 1$$

at least in an open region containing T, and since $\xi^{[k]}(z) \in RL_2^+$, in an open region containing $\{|z| \geq 1\}$.
Now since $\xi^{[k]}(z) = [w^{-1}(z^{-1})y_r^{[k-1]}(z^{-1})z^{-r}l^{[k]}(z)]_+$, Lemma 1 yields:
$z^{-r}l^{[k]}(z) = \left[w(z^{-1})\frac{1}{y_r^{[k-1]}(z^{-1})}\xi^{[k]}(z)\right]_+$, so in an open region containing $\{|z| \geq 1\}$:

$$l^{[k]}(z) = \frac{z^r f^{[k]}(z)}{a(z)\hat{a}^{[k-1]}(z)d_w(z)}, \qquad \deg f^{[k]}(z) \leq n + w - 1$$

These are the representations used in the polynomial version of (De Moor *et al.*, 1993a). $[z^{-1}h(z)l^{[k]}(z^{-1})]_+$ and $[z^{-1}w^{-1}(z)[w^{-1}(z^{-1})l^{[k-1]}(z)y^{[k-1]}(z)]_+l^{[k-1]}(z^{-1})]_+$ need only be equal in the first $r+1$ Markov parameters, so the difference can be expressed as $\frac{s^{[k]}(z)}{a(z)\hat{a}^{[k-2]}(z)n_w(z)}$, where $s^{[k]}(z)$ is an unknown polynomial with $\deg s^{[k]}(z) = n + w - 2$.
So in this version, the unknowns are the coefficients of $\hat{a}^{[k]}(z), f^{[k]}(z), \phi^{[k]}(z)$ and $s^{[k]}(z)$, giving $3n + 2w + 2r$ unknowns.

4.3. Partial fraction representation

As mentioned earlier, one may express $\xi^{[k]}(z)$ and $l^{[k]}(z)$ as sums of pole-residue fractions to avoid the rather large number of unknowns and other numerical problems connected with these polynomial representations.

[5] During the first few iterations of the algorithm, $\hat{a}^{[k-1]}(z)$ may have unstable roots, in this case only the 'stable root' factors of $\hat{a}^{[k-1]}(z)$ should be included in the denominator of $\xi^{[k]}(z)$. In such a case, $l^{[k]}(z)$ has the 'stable root' factors of $\hat{a}^{[k-1]}(z)$ and $\hat{a}_r^{[k-1]}(z)$ in its denominator.

- First define the polynomials $p_i(z) \triangleq \frac{a(z)}{z-\alpha_i}$, then write

$$\hat{a}^{[k]}(z) = \sum_i^n A_i^k p_i(z), \qquad \Rightarrow \qquad \begin{bmatrix} 0_{n-r-1} \\ \hat{a}_k \end{bmatrix} = \Gamma A_k$$

Here the columns of Γ store the coefficients of $p_i(z)$
Note that $\deg p_i(z) = n-1$, so $\deg \hat{a}^{[k]}(z) = r$ gives $n-r-1$ conditions on A_k :

$$\begin{bmatrix} I_{n-r-1} & 0 \end{bmatrix} \Gamma A_k \triangleq \Gamma_r A_k = 0 \tag{4}$$

$$\begin{bmatrix} 0 & I_{r+1} \end{bmatrix} \Gamma A_k \triangleq \Gamma_a A_k = \hat{a}_k \tag{5}$$

- We write $l^{[k]}(z)$ as:

$$l^{[k]}(z) \triangleq \sum_{i=1}^n \frac{C_{\alpha i}^k}{z-\alpha_i} + \sum_{i=1}^r \frac{C_{\hat{\alpha} i}^k}{z-\hat{\alpha}_i^{[k-1]}} + \sum_{i=1}^w \frac{C_{wi}^k}{z-\alpha_{wi}} \triangleq \sum_i \frac{C_i^k}{z-a_i^{k-1}}$$

- By splitting $w^{-1}(z^{-1})l^{[k]}(z)y^{[k-1]}(z) \in RL_2$ into partial fractions and retaining only the stable terms, one finds that $\xi^{[k]}(z)$ converges to (see Lemma 2) :

$$\begin{aligned} \xi^{[k]}(z) &= \Big[w^{-1}(z^{-1})l^{[k]}(z)y^{[k-1]}(z)\Big]_+ = \sum_{i=1}^n \frac{w^{-1}(\alpha_i^{-1})\hat{a}^{[k-1]}(\alpha_i^{-1})C_{\alpha i}^k}{z-\alpha_i} \\ &+ \sum_{i=1}^r \frac{w^{-1}(\hat{\alpha}_i^{-1})\hat{a}^{[k-1]}(\hat{\alpha}_i^{-1})C_{\hat{\alpha} i}^k}{z-\hat{\alpha}_i} + \sum_{i=1}^w \frac{w^{-1}(\alpha_{wi}^{-1})\hat{a}^{[k-1]}(\alpha_{wi}^{-1})C_{wi}^k}{z-\alpha_{wi}} \end{aligned} \tag{6}$$

- The dependence of A_i^k on $C_{\alpha i}^k$ is found using Lemma 2 :

$$\begin{aligned} \Big[h(z)y^{[k]}(z^{-1})\Big]_+ &= \left[\frac{b(z)}{a(z)}\hat{a}^{[k]}(z)\right]_+ = \left[b(z)\sum_i \frac{A_i^k}{z-\alpha_i}\right]_+ \\ &= \left[w^{-1}(z)\hat{a}^{[k-1]}(z)\left[w^{-1}(z^{-1})l^{[k]}(z)\hat{a}^{[k-1]}(z^{-1})\right]_+\right]_+ \end{aligned}$$

i.e. defining $D^{k-1} \triangleq \operatorname{diag}\left(\frac{w^{-1}(\alpha_i)w^{-1}(\alpha_i^{-1})\hat{a}^{[k-1]}(\alpha_i)\hat{a}^{[k-1]}(\alpha_i^{-1})}{b(\alpha_i)}\right)$, one finds :
$A_k = D^{k-1}C_\alpha^k$, which yields (combined with (4)(5)) :

$$\Gamma_a D^{k-1} C_\alpha^k = \hat{a}_k \tag{7}$$

$$\Gamma_r D^{k-1} C_\alpha^k = 0 \tag{8}$$

Equation (8) gives $n-r-1$ conditions on C^k.

- The fact that $l^{[k]}(z)$ has an r-multiple zero in $z=0$ yields a further r conditions on C^k (with an obvious definition for P_α^{k-1}, $P_{\hat{\alpha}}^{k-1}$, P_w^{k-1}) :

$$\frac{d^{s-1}}{dz^{s-1}} l^{[k]}(z)\Big|_{z=0} = 0, \qquad s = 1\ldots r \qquad \Rightarrow \sum_i \frac{C_{\alpha i}^k}{\alpha_i^s} + \sum_i \frac{C_{\hat{\alpha} i}^k}{\hat{\alpha}_i^s} + \sum_i \frac{C_{wi}^k}{\alpha_{wi}^s} = 0$$

$$\text{i.e.} \qquad P_\alpha^{k-1}C_\alpha^k + P_{\hat{\alpha}}^{k-1}C_{\hat{\alpha}}^k + P_w^{k-1}C_w^k = 0 \tag{9}$$

– Likewise, the knowledge of w zeroes of $\xi^{[k]}(z)$ gives w conditions on C^k, i.e. plugging eq. (6) into $\xi^{[k]}(\beta_{wi}) = 0$ yields : $\sum_{j=1}^{n} \frac{w^{-1}(\alpha_j^{-1})\hat{a}^{[k-1]}(\alpha_j^{-1})C_{\alpha j}^k}{\beta_{wi}-\alpha_j} + \sum_{j=1}^{r} \frac{w^{-1}(\hat{\alpha}_j^{-1})\hat{a}^{[k-1]}(\hat{\alpha}_j^{-1})C_{\hat{\alpha} j}^k}{\beta_{wi}-\hat{\alpha}_j} + \sum_{j=1}^{w} \frac{w^{-1}(\alpha_{wj}^{-1})\hat{a}^{[k-1]}(\alpha_{wj}^{-1})C_{wj}^k}{\beta_{wi}-\alpha_{wj}} = 0$

$$\text{i.e.} \qquad G_\alpha^{k-1}C_\alpha^k + G_{\hat{\alpha}}^{k-1}C_{\hat{\alpha}}^k + G_w^{k-1}C_w^k = 0 \tag{10}$$

– Finally, the first $r+1$ Markov parameters of $\left[h(z)l^{[k]}(z^{-1})\right]_+$ and $\left[w^{-1}(z)\left[w^{-1}(z^{-1})l^{[k-1]}(z)y^{[k-1]}(z)\right]_+ l^{[k-1]}(z^{-1})\right]_+$ are equal :

$$\left[h(z)l^{[k]}(z^{-1})\right]_{0\ldots r} = \left[\sum_i \frac{H_i}{z-\alpha_i} l^{[k]}(z^{-1})\right]_{0\ldots r} = \sum_i l^{[k]}(\alpha_i^{-1})H_i \left[\frac{1}{z-\alpha_i}\right]_{0\ldots r}$$

The second expression $\left[w^{-1}(z)\left[w^{-1}(z^{-1})l^{[k-1]}(z)y^{[k-1]}(z)\right]_+ l^{[k-1]}(z^{-1})\right]_+$ does not contain any unknowns. First its partial fraction decomposition $\sum_i \frac{R_{\beta i}}{z-\beta_{wi}} + \sum_i \frac{R_{\alpha i}}{z-\alpha_i} + \sum_i \frac{R_{\hat{\alpha} i}}{z-\hat{\alpha}_i^{k-2}}$ is computed by application of Lemma 2 in the poles β_{wi}, α_i and $\hat{\alpha}_i$, which yields :

$$\begin{aligned}
R_{\beta i} &\triangleq l^{[k-1]}(\beta_{wi}^{-1}) \sum_j \frac{w^{-1}((a_j^{k-1})^{-1})\hat{a}^{[k-1]}((a_j^{k-1})^{-1})C_j^{k-1}}{\beta_{wi}^{-1} - a_j^{k-1}} W_i \\
R_{\alpha i} &\triangleq w^{-1}(\alpha_i)w^{-1}(\alpha_i^{-1})l^{[k-1]}(\alpha_i^{-1})\hat{a}^{[k-1]}(\alpha_i^{-1})C_{\alpha i}^{k-1} \\
R_{\hat{\alpha} i} &\triangleq w^{-1}(\hat{\alpha}_i^{k-2})w^{-1}((\hat{\alpha}_i^{k-2})^{-1})l^{[k-1]}((\hat{\alpha}_i^{k-2})^{-1})\hat{a}^{[k-1]}((\hat{\alpha}_i^{k-2})^{-1})C_{\hat{\alpha} i}^{k-1}
\end{aligned}$$

So we find for $s = 0, \ldots, r$ (with an obvious definition for J_*) :

$$\sum_i \frac{H_i}{\alpha_i}\alpha_i^s \sum_j \frac{C_j^k}{\alpha_i^{-1} - a_j^k} = \sum_i R_{\beta i}\beta_{wi}^{s-1} + \sum_i R_{\alpha i}\alpha_i^{s-1} + \sum_i R_{\hat{\alpha} i}(\hat{\alpha}_i^{k-2})^{s-1} \triangleq V_s^{k-1}$$

$$\text{i.e.} \qquad J_\alpha^{k-1}C_\alpha^k + J_{\hat{\alpha}}^{k-1}C_{\hat{\alpha}}^k + J_w^{k-1}C_w^k = V^{k-1} \tag{11}$$

These are $r+1$ conditions still required to be able to solve for C^k .

– Summing it all up, we find equations (8)(9)(10)(11) yield $n+r+w$ linear conditions on $n+r+w$ unknowns C_i^k, i.e. the following $(n+r+w)$-by-$(n+r+w)$ linear system with complex coefficients has to be solved each time :

$$\begin{bmatrix} \Gamma_r D^{k-1} & 0 & 0 \\ P_\alpha^{k-1} & P_{\hat{\alpha}}^{k-1} & P_w^{k-1} \\ G_\alpha^{k-1} & G_{\hat{\alpha}}^{k-1} & G_w^{k-1} \\ J_\alpha^{k-1} & J_{\hat{\alpha}}^{k-1} & J_w^{k-1} \end{bmatrix} \begin{bmatrix} C_\alpha^k \\ C_{\hat{\alpha}}^k \\ C_w^k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ V^{k-1} \end{bmatrix} \tag{12}$$

5. Computation of $\hat{h}(z)$

The computation of $\hat{h}(z)$ is extremely simple:

$$\hat{h}(z) = \sum_{i=1}^{n} \frac{\hat{H}_i}{z - \hat{\alpha}_i} = h(z) - w^{-1}(z)\left[w^{-1}(z^{-1})l(z)y(z)\right]_+$$

$$\Rightarrow \hat{H}_i = -w^{-1}(\hat{\alpha}_i)w^{-1}(\hat{\alpha}_i^{-1})\hat{a}(\hat{\alpha}_i)C_{\hat{\alpha}i} \qquad (13)$$

6. Initialization

For the initial $\hat{h}(z)$ one can take the model produced by any model reduction method, e.g. frequency weighted balanced truncation (FWBT) à la Enns (Enns, 1984). This immediately yields the initial $y(z) = \hat{a}(z^{-1})$.

As in the optimum one has : $\left[w(z^{-1})w(z)(h(z) - \hat{h}(z))\right]_+ = l(z)y(z)$, one could hope that this relation will hold approximately for a good approximation of the optimal model. So the initial $l(z)$ is computed as $l_0(z) \triangleq \left[y^{-1}(z)w(z^{-1})w(z)\left(h(z) - \hat{h}(z)\right)\right]_+$.

7. Algorithm

Initialization:

Compute a good approximation $\hat{h}_0(z) = \frac{\hat{b}_0(z)}{\hat{a}_0(z)}$

Take $y(z) = \hat{a}_0(z^{-1})$ and $l(z) = \left[y^{-1}(z)w(z^{-1})w(z)(h(z) - \hat{h}(z))\right]_+$

Do {

1. Solve the linear system (12) $\Rightarrow l^{[k]}(z)$
2. Find $\hat{a}_k$ from (7) $\Rightarrow y^{[k]}(z)$
3. $l^{[k]} \leftarrow \frac{l^{[k]}}{\|\hat{a}_k\|}$
4. $\hat{a}_k \leftarrow \frac{\hat{a}_k}{\|\hat{a}_k\|}$
5. $k \leftarrow k + 1$

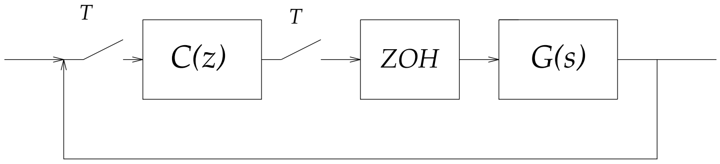
} **while** $\|\hat{a}_k - \hat{a}_{k-1}\| \geq \epsilon$ [6]

Compute $\hat{h}(z)$ using (13)

8. Example

Consider the following hydraulic actuator control system :

[6] other convergence tests may be used, e.g. one may stop when the decrease of the object function, or when the quantity $\hat{H}^T l$ becomes smaller than a certain tolerance.



$$\text{Actuator transfer function } G(s) = \frac{1000}{s^3 + 40s^2 + 800s + 1000}$$

With a sampling period of $T = 0.01$s, one finds that the discrete-time system describing the zero order hold and the plant $G(s)$ is

$$G(z) = 10^{-4}\frac{1.51z^2 + 5.44z + 1.23}{z^3 - 2.6z^2 + 2.28z - 0.67}$$

An H_∞ design with suitable weighting functions on the sensitivity function $(I + GC)^{-1}$ and the complementary sensitivity function $GC(I + GC)^{-1}$ yields the following controller :

$$C(z) = \frac{160z^6 - 373.6z^5 + 108.7z^4 + 334.7z^3 - 279.1z^2 + 29.8z + 19.5}{z^6 - 1.522z^5 + 0.166z^4 + 0.29z^3 + 0.045z^2 + 0.018z + 0.0037}$$

Choosing $w(z) = 1$, the reduced-order controller of smallest order that is still stabilizing has $r = 3$:

$$\text{The } H_2 \text{ optimally reduced controller : } \hat{C}_{H_2} = \frac{160z^3 - 95.6z^2 - 141.6z + 107.9}{z^3 + 0.21z^2 + 0.014z + 0.066}$$

$$\text{The balanced truncation (BT) reduced controller } \hat{C}_{BT} = \frac{160z^3 - 186.1z^2 - 139.3z + 171.8}{z^3 - 0.34z^2 - 0.38z - 0.23}$$

is not stabilizing.
Convergence is reached in 11 iterations and after 317660 flops, compared to 73100 flops for BT.

$$\text{Let } H_{H_2} \triangleq (I + GC_{H_2})^{-1}GC_{H_2}, \text{ then } \frac{\|H_{H_2} - H\|_\infty}{\|H\|_\infty} = 0.25$$

The Bode plots, error magnitude versus frequency and convergence of $\|\hat{a}_k - \hat{a}\|$ are shown in figure 1.
If we choose $w(z)$ to be[7] $(I + GC)^{-2}G$ (reduced to order 5 by BT, to avoid common poles with G), the smallest order yielding a stabilizing controller is $r = 2$, both for the optimal controller $\hat{C}_{wH_2}$ and the controller $\hat{C}_{wBT}$ reduced by frequency weighted balanced truncation (FWBT) (Enns, 1984).

$$\hat{C}_{wH_2} = \frac{160z^2 - 272.6z + 123.3}{z^2 - 0.544z + 0.01} \Rightarrow \frac{\|H_{wH_2} - H\|_\infty}{\|H\|_\infty} = 0.12$$

[7] This weighting function aims at minimizing the variation in the closed loop transfer function $(I + GC)^{-1}GC$

$$\hat{C}_{wBT} = \frac{160z^2 - 294z + 140.1}{z^2 - 0.853z + 0.0074} \Rightarrow \frac{\|H_{wBT} - H\|_\infty}{\|H\|_\infty} = 0.36$$

This time convergence is reached in 26 iterations and after 1.5 10^6 flops, compared to 325000 flops needed for FWBT.
The closed loop magnitude Bode plots and errors are shown in figure 2.
In general, (frequency weighted) H_2-reduced systems yield better closed loop behavior than FWBT, the advantage however decreases as r approaches n.

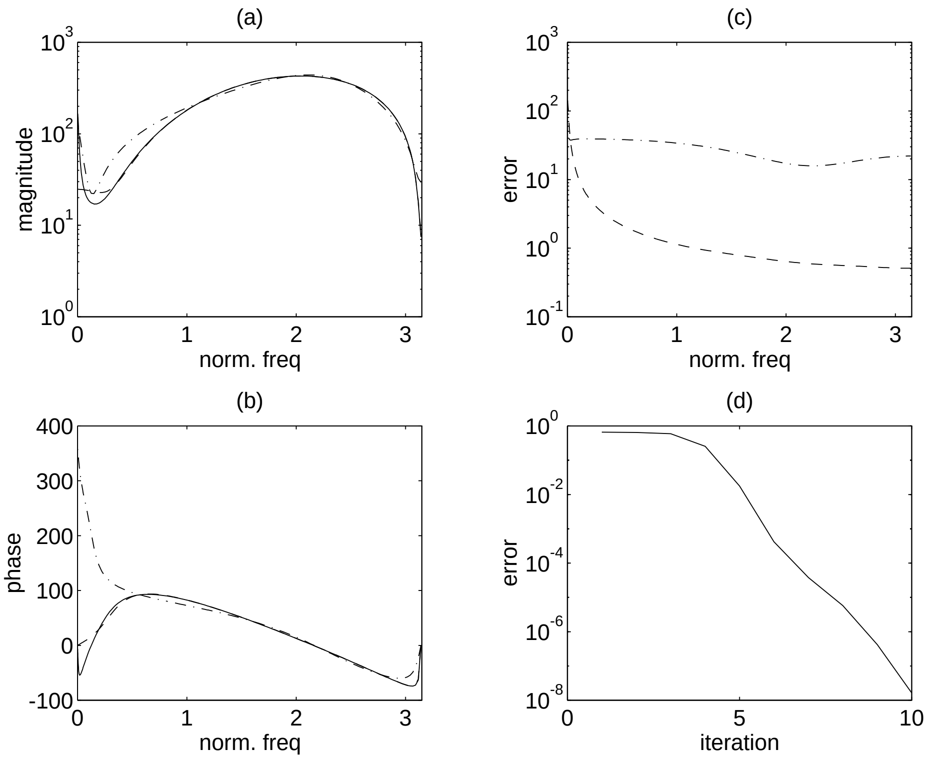


Fig. 1. Dashed line : H_2 reduced model, dash-dotted line : BT model . (a) magnitude Bode plots, (b) phase Bode plots, (c) error magnitude, (d) convergence of $\|\hat{a}_k - \hat{a}\|$

9. Conclusions

We have presented an algorithm for frequency weighted H_2 model reduction with a computational cost which is comparable with that of frequency weighted balanced truncation (the most widely used frequency weighted model reduction method), but which often yields better reduced order controllers in the sense that stabilization is more likely to be maintained and the loss of performance is smaller.

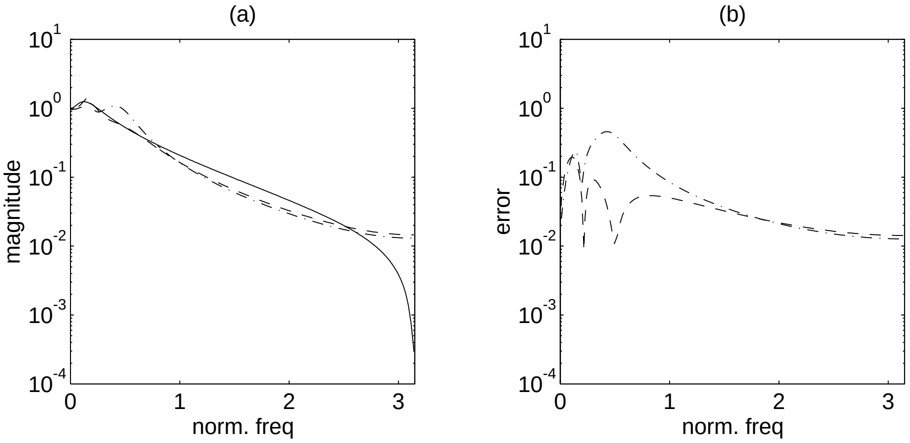


Fig. 2. Closed loop magnitude Bode plots and errors (dashed line : FW H_2 reduced model, dash-dotted line : FWBT model).

References

ANDERSON, B.D.O., and J.B. MOORE, 1989, Optimal Control - Linear Quadratic Methods. *Prentice-Hall*, Chapter 10

APLEVICH, J.D., 1973a, Approximation of discrete linear systems. *Int. J. Control*, **17**, 565-575

APLEVICH, J.D., 1973b, Gradient methods for optimal linear systems reduction. *Int. J. Control*, **18**, 767-772

BARATCHART, L., and M. OLIVI, 1987, Recent and new results in rational L_2 approximation. *Modeling, Robustness and Sensitivity Reduction in Control Systems, R.F. Curtain Ed.*, 119-126

BARATCHART, L., and M. OLIVI, 1988a, New tools in rational L_2 approximation. *Proc. IFAC Symp. Identification and System Parameter Estimation, Beijing*, **2**, 1014-1019

BARATCHART, L., M. OLIVI, and F. WIELONSKI, 1990, Asymptotic properties in rational L_2 approximation. *Analysis and Optimization of Systems, A. Bensoussan and J.L. Lions, ed., Springer Verlag*, **144**, 477-486

BRYSON, A.E., and A. CARRIER, 1990, Second-order algorithm for optimal model order reduction. *J. Guidance Control Dynam.*, **13**, 887-892

DE MOOR, B., P. VAN OVERSCHEE, and G. SCHELFHOUT, 1993a, Rational H_2 model reduction for SISO systems. *Proc. IFAC 1993, Sydney, Vol. II-227*

DE MOOR, B., 1993b, Total least squares for affinely structured matrices and the noisy realization problem. *To appear in IEEE Trans. Sign. Proc.*

DOYLE, J.C., and G. STEIN, 1981, Multivariable feedback design: Concepts for a classical/modern synthesis. *IEEE Tr AC*, **26**, 4-16

ENNS, D.F., 1984, Model reduction with balanced realization: An error bound and frequency weighted generalization. *Proc. IEEE CDC, Las Vegas*, 127-132

GALIANA, F.D., 1973, On the Approximation of Multiple-Input Multiple-Output Constant Linear Systems. *Int. J. Control*, **17**, 1313-1324

HALEVI, Y., 1990, Frequency weighted model reduction via optimal projection. *Proc. 29th CDC, Honolulu, Hawaii*, 2906-2911

HYLAND, D.C., and D.S. BERNSTEIN, 1985, The Optimal Projection Equations for Model Reduction and the Relationships Among the Methods of Wilson, Skelton, and Moore *IEEE Tr AC*, **30**, 1201-1211

KABAMBA, P.T., 1985, Balanced Gains and Their Significance for L_2 Model Reduction. *IEEE Tr AC*, **30**, 690-693

LATHAM, G.A., and B.D.O. ANDERSON, 1985, Frequency-weighted optimal Hankel-norm ap-

proximation of stable transfer functions. *Syst. Control Lett.*, **5**, 229-236
LEPSCHY, A., S. LORO, G.A. MIAN, and U. VIARO, 1991a, A numerical procedure for rational L_2 approximation. *MTNS 91, Kobe, Japan*, 271-272
LEPSCHY A., G.A. MIAN, G. PINATO, and U. VIARO, 1991b, Rational L_2 approximation: A non-gradient algorithm. *Proc. 30th CDC, Brighton, UK*, 2321-2323
MEIER, L., and D.G. LUENBERGER, 1967, Approximation of linear constant systems. *IEEE Tr AC*, **12**, 585-587
MISHRA, R.N., and D.A. WILSON, 1980, A new algorithm for optimal reduction of multivariable systems. *Int. J. Control*, **31**, 443-466
NOLDUS, E., and M. DECOSTER, 1976, Reduction of linear systems by weighted frequency response matching. *Regelungstechnik*, 122-128
SPANOS, J.T., M.H. MILMAN, and D.L. MINGORI, 1992, A New Algorithm for L_2 Optimal Model Reduction. *Automatica*, **28**, 897-909
WILSON, D.A., 1970, Optimum solution of model-reduction problem. *Proc. IEE-D*, **117**, 1161-1165
WILSON, D.A., 1974, Model Reduction of Multivariable Systems. *Int. J. Control*, **20**, 57-64
WILSON, D.A., and R.N. MISHRA, 1979, Optimal Reduction of Multivariable Systems. *Intl. J. Control*, **29**, 267-278

Appendix

A. Taking the derivative of the Lagrangian

Taking the derivative w.r.t. l, y and λ is straightforward.
Taking the derivative w.r.t. $\hat{h}_i$:
$\frac{\partial}{\partial \hat{h}_i}\left(l^T \hat{H} y\right) = [l(z)y(z)]_i$

$$\begin{aligned}
&\frac{\partial}{\partial \hat{h}_i}\left(\frac{1}{2}\sum_{i=1}^{\infty}\left([w\star h]_i - \left[w\star\hat{h}\right]_i\right)^2\right)\\
&= \frac{\partial}{\partial \hat{h}_i}\left(\frac{1}{4j\pi}\oint_T z^{-1}w(z^{-1})w(z)\left(h(z^{-1})-\hat{h}(z^{-1})\right)\left(h(z)-\hat{h}(z)\right)dz\right)\\
&= -\frac{1}{4j\pi}\oint_T w(z^{-1})w(z)\left(z^{i-1}\left(h(z)-\hat{h}(z)\right)+z^{-i-1}\left(h(z^{-1})-\hat{h}(z^{-1})\right)\right)dz\\
&= -\frac{1}{2j\pi}\oint_T z^{i-1}w(z^{-1})w(z)\left(h(z)-\hat{h}(z)\right)dz\\
&= -\left[w(z^{-1})w(z)\left(h(z)-\hat{h}(z)\right)\right]_i \quad \text{(using Laurent's theorem)}
\end{aligned}$$

Therefore we get $-\left[w(z^{-1})w(z)\left(h(z)-\hat{h}(z)\right)\right]_+ + l(z)y(z) = 0$

From this we can derive a generalized orthogonality condition between the approximant and the error, as in (De Moor *et al.*, 1993a) :

$$\begin{aligned}
&\frac{1}{2j\pi}\oint_T z^{-1}\hat{h}(z^{-1})w(z^{-1})w(z)\left(h(z)-\hat{h}(z)\right)dz\\
&= \frac{1}{2j\pi}\sum_{i=1}^{\infty}\hat{h}_i\oint_T z^{i-1}w(z^{-1})w(z)\left(h(z)-\hat{h}(z)\right)dz\\
&= \sum_{i=1}^{\infty}\hat{h}_i\,[l(z)y(z)]_i = \left[\hat{h}(z)l(z^{-1})y(z^{-1})\right]_0 = \left[\left[\hat{h}(z)y(z^{-1})\right]_+ l(z^{-1})\right]_0 = 0
\end{aligned}$$

$$\Rightarrow \sum_{i=1}^{\infty} \left(w \star \hat{h}\right)_i \left(w \star (h - \hat{h})\right)_i = 0$$

B. Lemma's

Lemma 1

$$x(z) \in L_2^+,\ w(z),\ w^{-1}(z) \in RL_2^{0+} \ \Rightarrow\ \left[w^{-1}(z^{-1})\left[w(z^{-1})x(z)\right]_+\right]_+ = x(z),\ |z| > 1$$

Proof :
Since $x(z) \in L_2^+$, it converges in $\{z|\ 1 < \|z\|\}$; since $w(z), w^{-1}(z) \in RL_2^{0+}$, $\exists \beta > 1$ such that $w(z^{-1})$, $w^{-1}(z^{-1})$ converge in $\{z|\ \|z\| < \beta\}$.
$w(z^{-1})$ is bounded on T, hence $w(z^{-1})x(z) \in L_2$ (conv. in $\{z|\ 1 < \|z\| < \beta\}$) and $w(z^{-1})x(z) = q(z) + l(z^{-1}) \quad (1 < \|z\| < \beta)$, where $q(z) \in L_2^+$, and $l(z) \in L_2^{0+}$.
Therefore $w^{-1}(z^{-1})q(z) = x(z) - w^{-1}(z^{-1})l(z^{-1}) \qquad (1 < \|z\| < \beta)$,
and $\left[w^{-1}(z^{-1})q(z)\right]_+ = \left[w^{-1}(z^{-1})\left[w(z^{-1})x(z)\right]_+\right]_+ = x(z)$, whenever $\|z\| > 1$. $\square$

Lemma 2 *Let $|\alpha| < 1$, $x(z), y(z) \in L_2^+$, where $y(z) = \sum_{i=1}^{l} \frac{Y_i}{(z-\alpha)^i}$ in $\{|z| > |\alpha|\}$*

Then $\left[x(z^{-1})y(z)\right]_+ = \sum_{i=1}^{l} \frac{C_i}{(z-\alpha)^i}$ *for* $|z| > |\alpha|$, $C_i \triangleq \sum_{r=0}^{l-i} \frac{Y_{i+r}}{r!}\left[\frac{d^r}{dz^r}(x(z^{-1}))\right]_{z=\alpha}$

Proof :
Expanding $x(z^{-1})$ (conv. $|z| < 1$) as a Taylor series of $(z - \alpha)$, we find:

$$\begin{aligned}
x(z^{-1}) &= \sum_{i=0}^{\infty} \frac{1}{i!}\frac{d^i}{dz^i}\left(x(z^{-1})\right)_{z=\alpha}(z-\alpha)^i \qquad (|z-\alpha| < 1 - |\alpha|) \\
\frac{x(z^{-1})}{(z-\alpha)^s} &= \sum_{i=0}^{\infty} \frac{1}{i!}\frac{d^i}{dz^i}\left(x(z^{-1})\right)_{z=\alpha}(z-\alpha)^{i-s} \qquad (0 < |z-\alpha| < 1 - |\alpha|) \\
\left[\frac{x(z^{-1})}{(z-\alpha)^s}\right]_+ &= \sum_{i=1}^{s} \frac{1}{(s-i)!}\frac{d^{s-i}}{dz^{s-i}}\left(x(z^{-1})\right)_{z=\alpha}\frac{1}{(z-\alpha)^i} \qquad (|z| > |\alpha|) \\
\Rightarrow \left[x(z^{-1})y(z)\right]_+ &= \sum_{s=1}^{l} Y_s \sum_{i=1}^{s} \frac{1}{(s-i)!}\frac{d^{s-i}}{dz^{s-i}}\left(x(z^{-1})\right)_{z=\alpha}\frac{1}{(z-\alpha)^i} \\
&= \sum_{i=1}^{l}\sum_{r=0}^{l-i} \frac{Y_{r+i}}{r!}\frac{d^r}{dz^r}\left(x(z^{-1})\right)_{z=\alpha}\frac{1}{(z-\alpha)^i}
\end{aligned}$$

$\square$