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Departement Elektrotechniek

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Time-Domain Weighted Balanced Truncation. [1] [2]

Geert Schelfhout and Bart De Moor[3]

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[2]The following text presents research results obtained within the framework of the Belgian programme on interuniversity attraction poles (IUAP-17 and IUAP-50) initiated by the Belgian State - Prime Minister's Office - Science Policy Programming. The scientific responsibility is assumed by its authors.

[3]ESAT - Katholieke Universiteit Leuven,
Kardinaal Mercierlaan 94, 3001 Leuven (Heverlee), Belgium,
tel. 32/16/32.18.08, fax 32/16/32.19.86,
geert.schelfhout@esat.kuleuven.ac.be *bart.demoor@esat.kuleuven.ac.be*.

TIME-DOMAIN WEIGHTED BALANCED TRUNCATION*

Geert Schelfhout† and Bart De Moor‡
ESAT-SISTA, K.U.Leuven,
Kardinaal Mercierlaan 94, B-3001 Leuven,
Belgium
Tel.: +32-16-32.17.09 Fax.: +32-16-32.19.86
geert.schelfhout@esat.kuleuven.ac.be bart.demoor@esat.kuleuven.ac.be

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Abstract

We propose a type of balanced truncation that aims at reducing an error norm weighted by a piecewise polynomial function in the time-domain. To this end, time-domain weighted gramians are defined both for continuous-time and discrete-time systems, and it is shown how these can be computed and used for model reduction.

1 Introduction

Balanced Truncation [M81] is a widely used model reduction method, with many advantages, e.g. its simplicity, the H_∞ error bound, etc. Many generalizations to this approach have been proposed during the last few years. The first frequency weighted version was put forward by Enns [E84, E84b, AM89]. Wortelboer et al. propose a frequency weighted extension with piecewise polynomial weighting functions [WB92].
Many specifications and robustness results yield natural frequency domain weighting functions, but for other ones, time domain weighting functions are more appropriate. What we wish to provide in this paper is a balanced truncation tool that incorporates this.

2 Time-domain weighted gramians: Continuous-time case

Define the linear time-invariant continuous-time system $H(s) \triangleq [A,B,C,D] \triangleq C(sI-A)^{-1}B+D$.

Recall that for continuous-time systems with state space representation $[A,B,C,D]$, the controllability gramian is defined as $P \triangleq \int_0^\infty e^{At}BB^Te^{A^Tt}dt$ and the observability gramian as $Q \triangleq \int_0^\infty e^{A^Tt}C^TCe^{A^t}dt$. These can be found as the solution of the Lyapunov equations $AP+PA^T+BB^T=0$ and $A^TQ+QA+C^TC=0$.
We wish to generalize the definition of these gramians to expressions of the type $P_f \triangleq \int_0^\infty f(t)e^{At}BB^Te^{A^Tt}dt$ and $Q_f \triangleq \int_0^\infty f(t)e^{A^Tt}C^TCe^{A^t}dt$, where $f(t)$ should be suitable to approximate any desired weighting function. In particular, we define $P_i \triangleq \int_0^\infty t^ie^{At}BB^Te^{A^Tt}dt$ and $Q_i \triangleq \int_0^\infty t^ie^{A^Tt}C^TCe^{A^t}dt$.

Let's first investigate functions of the type $f(t)=t^i$, where i is a positive integer, and define

$$X_i \triangleq \int_0^\infty t^ie^{A_1t}De^{A_2t}dt, \quad X_i(T) \triangleq \int_0^T t^ie^{A_1t}De^{A_2t}dt,$$

then $\frac{dX_i(T)}{dT} = T^ie^{A_1T}De^{A_2T}$ and $\frac{d^2X_i(T)}{d^2T} =$
$iT^{i-1}e^{A_1T}De^{A_2T}+T^iA_1e^{A_1T}De^{A_2T}+T^ie^{A_1T}De^{A_2T}A_2$,

$$\begin{aligned}\text{and } \frac{dX_i(T)}{dT}\Big|_\infty - \frac{dX_i(T)}{dT}\Big|_0 &= \int_0^\infty \frac{d^2X_i(T)}{d^2T}dT \\ &= iX_{i-1}+A_1X_i+X_iA_2 \\ &= 0 \quad \text{if} \quad i>0 \\ &= D \quad \text{if} \quad i=0\end{aligned}$$

i.e. the quantities X_i may be found by solving recursively :

$$\begin{aligned}A_1X_0+X_0A_2+D &= 0 \\ A_1X_i+X_iA_2+iX_{i-1} &= 0 \qquad i>0\,.\end{aligned}$$

It is also easily found that

$$\begin{aligned}X_i(\tau_1,\tau_2) &\triangleq \int_{\tau_1}^{\tau_2} t^ie^{A_1t}De^{A_2t}dt \\ &= e^{A_1\tau_1}\left(\int_0^\infty (t+\tau_1)^ie^{A_1t}De^{A_2t}dt\right)e^{A_2\tau_1}\end{aligned}$$

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†Research assistant with the N.F.W.O. (Belgian National Fund for Scientific Research)

‡Senior research associate with the N.F.W.O.

$$
\begin{aligned}
&-e^{A_1\tau_2}\left(\int_0^\infty (t+\tau_2)^i e^{A_1 t} D e^{A_2 t} dt\right) e^{A_2\tau_2} \\
= \; & e^{A_1\tau_1}\left(\sum_{k=0}^{i} \frac{i!}{k!(i-k)!}\tau_1^{i-k} X_k\right) e^{A_2\tau_1} \\
&-e^{A_1\tau_2}\left(\sum_{k=0}^{i} \frac{i!}{k!(i-k)!}\tau_2^{i-k} X_k\right) e^{A_2\tau_2}
\end{aligned}
$$

which enables us to compute the gramians weighted by a piecewise polynomial in the time domain; in this case, $A_1 = A = A_2^T$.

With the techniques provided above, one can now compute P_f and Q_f for every piecewise polynomial function* $f(t)$ (with finitely many discontinuities), i.e. $\exists t_i,\ i = 1,\ldots,n$, s.t. $f(t) = p_i(t),\ t_i < t < t_{i+1},\ i = 0,\ldots,n$, where $t_0 \triangleq 0$, $t_{n+1} \triangleq \infty$ and $p_i(t)$ is a polynomial in t.

If n_f is the order of the piecewise polynomial (defined as the highest order of its polynomial sections, i.e. $n_f = \deg f(t) \triangleq \max_i \deg p_i(t)$), n_f Lyapunov equations of the form $A^T X + XA + (\cdot) = 0$ and n_f of the form $AX + XA^T + (\cdot) = 0$ have to be solved. Solving such a Lyapunov equation classically involves the Schur decomposition of A, requiring about $25n^3$ flops ([GV89]); after that, the solution of the Lyapunov equation can be found after approximately $2n^3$ flops. Of course the Schur decomposition needs only be computed one time, such that all the Lyapunov equations can be solved in $(25+2n_f)n^3$. If n_f is large, one may also consider (block) diagonalizing A; this requires little extra work, and then each Lyapunov equation may be solved with approximately $O(n^2)$ flops. However, as this (block) diagonalization cannot be performed using orthogonal transformations only, this may be numerically unreliable, so caution is advisable.

The gramians can also be computed if $f(t)$ is a rational function in t without poles on the positive real axis (not including infinity), at the cost of much more complicated computations. As is to be expected, these computations can be reduced to integrals of the type $\int_\alpha^\infty \frac{e^{\lambda t}}{t} dt$, $\int_\alpha^\infty \frac{e^{\lambda t}\cos(\gamma t)}{t} dt$ and $\int_\alpha^\infty \frac{e^{\lambda t}\sin(\gamma t)}{t} dt$; these cannot be solved analytically but are tabulated in [AS68].

3 Time-domain weighted gramians: Discrete-time case

For discrete-time systems, the controllability gramian is defined as $P \triangleq \sum_0^\infty A^k BB^T A^{kT}$ and the observability gramian as $Q \triangleq \sum_0^\infty A^{kT} C^T C A^k$, and they can be found by solving the Lyapunov equations $APA^T - P + BB^T = 0$ and $A^T QA - Q + C^T C = 0$.

Again we generalize these definitions to expressions of the type $P_f \triangleq \sum_{k=0}^\infty f(k) A^k BB^T A^{kT}$ and $Q_f \triangleq \sum_{k=0}^\infty f(k) A^{kT} C^T C A^k$, where $f(k)$ is a piecewise polynomial function.

Here we do not define the matrices X_i as $\sum_{k=0}^\infty k^i A_1^k D A_2^k$, as one could expect from analogy with the continuous-time case, but as

$$
X_i \triangleq \sum_{k=0}^\infty \frac{(k+i)!}{k!} A_1^k D A_2^k
$$

Accordingly, here we define $P_i \triangleq \sum_{k=0}^\infty \frac{(k+i)!}{k!} A^k BB^T A^{kT}$ and $Q_i \triangleq \sum_{k=0}^\infty \frac{(k+i)!}{k!} A^{kT} C^T C A^k$. This results in simpler recurrence equations for X_i (P_i, Q_i). It is easily seen that $A_1 X_0 A_2 - X_0 + D = 0$, and for $i > 0$ one has :

$$
\begin{aligned}
&A_1 X_i A_2 + iX_{i-1} \\
= \; & \sum_{k=0}^\infty \frac{(k+i)!}{k!} A_1^{k+1} D A_2^{k+1} + \sum_{k=0}^\infty \frac{(k+i-1)!i}{k!} A_1^k D A_2^k \\
= \; & \sum_{k=1}^\infty \frac{(k+i-1)!}{(k-1)!} A_1^k D A_2^k + \sum_{k=0}^\infty \frac{(k+i-1)!i}{k!} A_1^k D A_2^k \\
= \; & \sum_{k=0}^\infty \frac{(k+i-1)!(k+i)}{k!} A_1^k D A_2^k = X_i
\end{aligned}
$$

$$
\text{hence} \qquad A_1 X_i A_2 - X_i + iX_{i-1} = 0. \qquad (1)
$$

Note that $\frac{(k+i)!}{k!} = \sum_{j=0}^i f_{ij} k^j$, where the coefficients f_{ij} are defined recursively with $f_{i0} \triangleq i!$, $f_{ii} \triangleq 1$, and the recursion $f_{ij} \triangleq f_{i-1,j-1} + i f_{i-1,j}$.
Also $k^i = \sum_{j=0}^i g_{ij} \frac{(k+j)!}{k!}$, and hence $\sum_{k=0}^\infty k^i A_1^k D A_2^k = \sum_{j=0}^i g_{ij} X_j$, where again g_{ij} are defined recursively using $g_{i0} \triangleq (-1)^i$, $g_{ii} \triangleq 1$, and $g_{ij} \triangleq g_{i-1,j-1} - (j+1) g_{i-1,j}$.
Some of these coefficients are tabulated in Table 1.,

Thus a segment of a gramian weighted with a piecewise polynomial is computed using the following formula :

$$
\begin{aligned}
&X_i(k_1, k_2) \\
\triangleq \; & \sum_{k=k_1}^{k_2} k^i A_1^k D A_2^k \\
= \; & A_1^{k_1}\left(\sum_{k=0}^\infty (k+k_1)^i A_1^k D A_2^k\right) A_2^{k_1} \\
&-A_1^{k_2+1}\left(\sum_{k=0}^\infty (k+k_2+1)^i A_1^k D A_2^k\right) A_2^{k_2+1} \\
= \; & A_1^{k_1} \sum_{l=0}^i \left(\sum_{j=l}^i C_i^j k_1^{i-j} g_{jl}\right) X_j A_2^{k_1} \\
&-A_1^{k_2+1} \sum_{l=0}^i \left(\sum_{j=l}^i C_i^j (k_2+1)^{i-j} g_{jl}\right) A_2^{k_2+1}
\end{aligned}
$$

*$f(t)$ may in fact be replaced by a piecewise polynomial symmetric matrix-valued function $F(t)$ without considerable complications.

i \ j	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	1	1	0	0	0	0	0
2	2	3	1	0	0	0	0
3	6	11	6	1	0	0	0
4	24	50	35	10	1	0	0
5	120	274	225	85	15	1	0
6	720	1764	1624	735	175	21	1

i \ j	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	-1	1	0	0	0	0	0
2	1	-3	1	0	0	0	0
3	-1	7	-6	1	0	0	0
4	1	-15	25	-10	1	0	0
5	-1	31	-90	65	-15	1	0
6	1	-63	301	-350	140	-21	1

Table 1: Coefficients f_{ij} (top) and g_{ij} (bottom)

Hence, if $f(t)$ (or $f(k)$ for a discrete-time system) is a piecewise polynomial function, P_f and Q_f are readily computed using the techniques given above.

4 Time-domain weighted norms

These gramians P_f and Q_f can be used for computing time-domain weighted H_2 norms, defined, for continuous-time systems, by :

$$\|H(s)\|_f^2 = \text{Trace} \int_0^\infty f(t) H^T(t) H(t) dt$$

where $H(s) \triangleq C(sI-A)^{-1}B$, $H(t) \triangleq Ce^{At}B$, and for discrete-time systems, by :

$$\|H(z)\|_f^2 = \text{Trace} \sum_0^\infty f(k) H_k^T H_k$$

where $H(z) \triangleq C(zI-A)^{-1}B$, $H_k \triangleq CA^kB$. Then $\|H(s)\|_f^2$ ($\|H(z)\|_f^2$) $= \text{Trace}(CP_fC^T) = \text{Trace}(B^TQ_fB)$.

5 Model reduction

It is easily verified that if a similarity transformation $[A,B,C,D] \to [TAT^{-1}, TB, CT^{-1}, D]$ is applied to the realization of a system, the gramians P_i and Q_i transform into TP_iT^T and $T^{-T}Q_iT^{-1}$, respectively. It is standard to find a transformation such that these two matrices are balanced; partitioning and truncating the transformed state space matrices accordingly, a state space model for a reduced order system is found.
This may also be accomplished using Safonov's Schur method [SC88] :

- Compute matrices V_r and V_l, whose r columns form bases for the right and left eigenspaces of P_fQ_f associated with the r biggest eigenvalues.
- Compute a singular value decomposition $U_e\Sigma_eV_e^T = V_l^TV_r$
- Defining $L \triangleq \Sigma_e^{-\frac{1}{2}}U_e^TV_l^T$ and $T \triangleq V_rV_e\Sigma_e^{-\frac{1}{2}}$, $A_r \triangleq LAT$, $B_r \triangleq LB_e$, $C_r \triangleq CT$, the reduced order system is $G_r(s) \triangleq [A_r, B_r, C_r, D]$

6 Relation to other model reduction methods

The method and techniques proposed in this paper bear some affinity to the weighted impulse response gramian model reduction of Sreeram and Agathoklis [SA91]. Indeed, in [SA91], Lyapunov equations of the same type as (1) appear; however, there these equations are solved for the system in controllability canonical form, and the controllability gramians P_i are then system invariants. The model reduction proposed by Sreeram and Agathoklis is achieved by retaining the states corresponding to the largest eigenvalues of P_i, which is equivalent to balanced truncation using P_i and the identity matrix I for gramians. This identity matrix is of course in general not equal to the observability gramian of the system in controllability canonical form, and therefore the method is not equivalent with the one proposed in this paper; but for discrete-time systems with a rapidly damped impulse response, I is a good approximation for Q_i, and hence in those cases the two methods are very similar.

7 Stability of the reduced-order models

We shall show in this section that if, for a continuous-time system, $f(t) = \sum_{i=0}^m f_it^i$, with all $f_i > 0$, is used as a weighting function, and the original system is controllable and asymptotically stable, then the reduced-order model is asymptotically stable as well. Indeed, for $f(t) = 1$, this is a standard result on balanced truncation [PS82], hence also for $f(t) = f_0$. It is also well known that P_0 is positive definite under the assumptions above. Now the Lyapunov criterion [K80] states that $[A,B,C,D]$ is asymptotically stable if and only if for any positive definite symmetric matrix of suitable dimensions L there is a positive definite symmetric matrix P such that $AP + PA^T + L = 0$. Since $AP_i + P_iA^T + iP_{i-1} = 0$, by the Lyapunov criterion, one recursively proves $P_i > 0$ for all $i > 0$. Also, we have $P_f = \sum_{i=1}^m f_iP_i$, and $-AP_f - P_fA^T = P_f' \triangleq f_0BB^T + \sum_{i=1}^m f_iiP_{i-1} > 0$. Therefore, again by the Lyapunov criterion, $P_f > 0$. By applying a balancing similarity transformation to $[A,B,C]$, P_f is transformed into a diagonal matrix Σ, and P_f' into a matrix P_f'' which is of course still symmetric positive definite. Hence

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} + \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T$$

$$+ \begin{bmatrix} P''_{f11} & P''_{f12} \\ P''^T_{f12} & P''_{f22} \end{bmatrix} = 0$$

and

$$-A_{11}\Sigma_1 - \Sigma_1 A_{11}^T = P''_{f11} > 0,$$

so the Lyapunov criterion once more assures that A_{11} has only eigenvalues with negative real part.
A dual result holds of course if the system is observable and asymptotically stable.

A similar and only slightly more complicated proof may be given for discrete-time systems for weightings $f(k) = \sum_{i=0}^{m} F_i \frac{(k+i)!}{k!}$ with $F_i > 0$.

8 Example

To illustrate the the effectiveness of this method w.r.t. time-domain weighted error norms, consider the following example, where we wish to obtain a reduced order system with a small error in the Hilbert-Schmidt-Hankel norm. This norm is defined by $\|H(s)\|_{HSH}^2 \triangleq \int_{t=0}^{\infty} t\mathrm{Trace}(H(t)^T H(t))$ for continuous-time systems (where $H(t)$ is the impulse response matrix), and $\|H(z)\|_{HSH}^2 \triangleq \sum_{i=0}^{\infty} i\mathrm{Trace}(H_i^T H_i)$ for discrete-time systems (where $H(z) \triangleq \sum_{i=0}^{\infty} H_i z^{-1}$), and has some interesting properties, e.g. for strictly proper systems of a given order, the H_∞ and the HSH norms are equivalent (see [H92]).

Consider the following continuous-time system :

$$G(s) = \frac{s^4 + 2s^3 + 6s^2 + 18s + 31}{s^5 + 7s^4 + 22s^3 + 36s^2 + 34s + 3}$$

Unweighted balanced truncation to order 2 yields the reduced order model

$$G_b(s) = \frac{0.2460z + 0.7430}{z^2 + 0.7321z + 0.0752},$$

with $\|G(s) - G_b(s)\|_{HSH} = 0.835$. Time domain weighted balanced truncation with weighting function $f(t) = t$ yields

$$G_r(s) = \frac{-0.0671z + 1.2649}{z^2 + 1.3661z + 0.1244},$$

with $\|G(s) - G_r(s)\|_{HSH} = 0.434$. For comparison, the optimal approximant w.r.t. Hilbert-Schmidt-Hankel norm yields an error of 0.369.

9 Conclusions

We have introduced time-domain weighted gramians, useful for computing norms $\|\ \|_f$ weighted with a piecewise polynomial function $f(t)$ (f_k) in the time domain, and have presented a model reduction method that, given a system $G(s)$ $(G(z))$, produces a reduced order model $G_r(s)$ $(G_r(z))$ such that $\|G - G_r\|_f$ is made small. For a certain class of weighting functions, the reduction method is proved to preserve stability, like ordinary Balanced Truncation.

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