A Family of n-Scroll Attractors from a Generalized Chua's Circuit

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Abstract

Previously, n-double scroll attractors have been introduced by Suykens & Vandewalle. A generalized Chua's circuit was considered with additional breakpoints in the nonlinear characteristic. A piecewise-linear implementation and experimental confirmation has been given by Arena *et al.* In this paper we present a more complete family of n-scroll attractors generated from the latter circuit. The new family contains both an even and odd number of scrolls, while the previous work considered only an even number. A Lur'e representation of the generalized Chua's circuit is also given.

1 Introduction

Chua's circuit is well-known to be a paradigm for chaos [16], being a simple nonlinear electrical circuit that reveals a rich variety of phenomena [3, 5, 6, 8]. It is the real life existing proof of chaos, as experimentally confirmed first in [27]. After its introduction around 1983, many generalizations have been proposed [8]. From a system theoretical point of view Chua's circuit can be considered as a particular case of a Lur'e system [14, 24, 11, 26]. Lur'e systems consist of a linear dynamical system, interconnected by feedback to a static nonlinearity that satisfies a sector condition. In this sense the generalizations that have been made are twofold. With respect to the linear part, higher dimensional Chua's circuits have been investigated in [10, 15, 26]. For the nonlinear part a cubic smooth nonlinearity instead of the piecewise linear characteristic has been studied in [16, 12]. Another generalization is related to the introduction of additional breakpoints in the nonlinearity, leading to the n-double scroll attractors, first proposed in 1991 by Suykens & Vandewalle [18, 19]. An experimental confirmation of n-double scrolls has been given by Arena *et al.* in 1996 [1, 2]. The latter work follows the ideas of [19] but implements a piecewise linear characteristic.

In this paper a more complete family of n-scroll attractors is presented, based on the work of Arena *et al.* The new family contains attractors with an even as well as an odd number of scrolls. The n-double scrolls ($2n$-scroll attractors in the new family) correspond to the even case with Chua's double scroll being a 2-scroll attractor. According to the CNN paradigm [7, 9], Chua's circuits can be taken as cells within a one- or two-dimensional cellular array. Using weak linear coupling between chaotic cells, hyperchaos is obtained in a CNN array. This has been demonstrated in [13] and [21] with the double-double scroll attractor and the n-double scroll hypercube CNN, using double scrolls and n-double scrolls as cells respectively. Other phenomena that have been obtained from CNNs with Chua's circuits as cells are travelling waves, spiral waves, trigger and target waves and Turing patterns [17]. New phenomena can be expected by taking the generalized Chua's circuit with multiple breakpoints as cells within CNNs. Furthermore Chua's circuit has been used for secure communication applications [25]. For Lur'e systems, master-slave synchronization schemes have been investigated in [22, 23], with illustrations on Chua's circuit, 2-double scroll circuits and coupled cells. A Lur'e representation of the generalized Chua's circuit will be given in this paper, such that the n-scroll attractors can be used for such applications.

This paper is organized as follows. In Section 2 we present the generalized Chua's circuit and the family of n-scroll attractors. In Section 3 we discuss the equilibrium points and Jacobian matrix of the circuit. Finally, in Section 4 a Lur'e representation of the circuit is given.

2 Generalized Chua's Circuit and n-Scroll Attractors

Consider the following generalized Chua's circuit

$$\left\{ \begin{array}{rcl} \dot{x} & = & \alpha\,[y - h(x)] \\ \dot{y} & = & x - y + z \\ \dot{z} & = & -\beta y \end{array} \right. \tag{1}$$

with piecewise linear characteristic

$$h(x) = m_{2q-1}x + \frac{1}{2}\sum_{i=1}^{2q-1}(m_{i-1} - m_i)\left(|x + c_i| - |x - c_i|\right) \tag{2}$$

where q denotes a natural number. This circuit has been implemented by Arena *et al.* in [1, 2] in order to confirm the n-double scroll attractors, introduced by Suykens & Vandewalle in [18, 19]. Taking $q = n$ one obtains the n-double scroll attractors for the parameters $\alpha = 9$, $\beta = 14.286$ with the vectors $m = [m_0; m_1; ...; m_{2q-1}]$, $c = [c_1; c_2; ...; c_{2q-1}]$ chosen as:

- $q = 1$: double scroll (2-scroll) [3, 4, 5, 6, 8, 16]

$$\begin{aligned} m &= [\,-1/7\,;\, +2/7\,] \\ c &= 1 \end{aligned} \tag{3}$$

- $q = 2$: 2-double scroll (4-scroll) [1, 2]

$$\begin{aligned} m &= [\,-1/7\,;\, +2/7\,;\, -4/7\,;\, +2/7\,] \\ c &= [\,1\,;\, 2.15\,;\, 3.6\,] \end{aligned} \tag{4}$$

- $q = 3$: 3-double scroll (6-scroll) [1, 2]

$$\begin{aligned} m &= [\,-1/7\,;\, +2/7\,;\, -4/7\,;\, +2/7\,;\, -4/7\,;\, +2/7\,] \\ c &= [\,1\,;\, 2.15\,;\, 3.6\,;\, 8.2\,;\, 13\,]. \end{aligned} \tag{5}$$

The components of vector m have alternating signs and the sign of m_0 is negative in the case of n-double scrolls. Here we want to propose a more complete family of n-scroll attractors, that contains both an even and odd number of scrolls, with n being a natural number. n-Scroll attractors with an odd number of scrolls are generated from (1) by taking the same values for the α, β parameters but the opposite sign for the vector m in the nonlinearity. For the resulting attractors shown on Figs. 3-6 the following nonlinearities have been taken:

- $q = 1$: 1-scroll

$$\begin{aligned} m &= [\,+1/7\,;\, -2/7\,] \\ c &= 1 \end{aligned} \tag{6}$$

- $q = 2$: 3-scroll

$$\begin{aligned} m &= [+0.9/7\,;\ -3/7\,;\ +3.5/7\,;\ -2.4/7\,] \\ c &= [\,1\,;\ 2.15\,;\ 4\,] \end{aligned} \tag{7}$$

- $q = 3$: 5-scroll

$$\begin{aligned} m &= [+0.9/7\,;\ -3/7\,;\ +3.5/7\,;\ -2.7/7\,;\ +4/7\,;\ -2.4/7\,] \\ c &= [\,1\,;\ 2.15\,;\ 3.6\,;\ 6.2\,;\ 9\,] \end{aligned} \tag{8}$$

- $q = 4$: 7-scroll

$$\begin{aligned} m &= [+0.9/7\,;\ -3/7\,;\ +3.5/7\,;\ -2.4/7\,;\ +2.52/7\,;\ -1.68/7\,;+2.52/7\,;\ -1.68/7\,] \\ c &= [\,1\,;\ 2.15\,;\ 3.6\,;\ 6.2\,;\ 9\,;\ 14\,;\ 25\,]. \end{aligned} \tag{9}$$

Hence for a given value q, an even or odd number of scrolls can be obtained from the circuit (1)-(2), depending on the sign of the vector m. In the odd case a scroll is generated at the origin. The circuits are simulated by means of a Runge-Kutta integration rule with adaptive step size (*ode23* in Matlab) on a Sun-Ultrasparc workstation. The Matlab to C compiler *mcc* has been used.

3 Equilibrium Points and Jacobians

The equilibrium points of the circuit (1)-(2) are given by:

$$\begin{cases} h(x) = 0 \\ x + z = 0 \\ y = 0. \end{cases} \tag{10}$$

Defining $w_{2q} = \alpha m_{2q-1}$ and $w_i = \alpha(m_{i-1} - m_i)$ $(i = 1, 2, ..., 2q-1)$ the condition $h(x) = 0$ corresponds to

$$w_{2q} x + \varphi(x) = 0 \tag{11}$$

where $\varphi(x) = \frac{1}{2} \sum_{i=1}^{2q-1} w_i \left(|x + c_i| - |x - c_i|\right)$. The nonlinearities and the corresponding equilibrium points for the circuit are shown on Figs. 1-2 for the case of an even and odd number of scrolls. The nonlinearity $\varphi(x)$ has $4q-2$ breakpoints, which yields $4q-1$ equilibrium points in the circuit. The origin is an equilibrium point. The other equilibrium points appear in pairs, which we denote here as $eq_j^{\pm} = [x_{eq_j^{\pm}}; 0; -x_{eq_j^{\pm}}]$, for $j = 1, 2, ..., 2q-1$ (the index j orders the points in terms of increasing x coordinates). In Table 1 an overview is given on the relation between the value of q, the number of scrolls, the number of breakpoints in the nonlinearity and the number of equilibrium points.

The Jacobian matrix of the circuit is given by

$$J(x) = \begin{bmatrix} -w_{2q} & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix} + \sum_{i=1}^{2q-1} M_i \tag{12}$$

where

$$M_i = \begin{cases} \begin{bmatrix} -w_i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & (-c_i < x < c_i) \\ \\ 0, & (x < -c_i \text{ or } c_i < x). \end{cases}$$

Evaluated at the equilibrium points this becomes

$$J(0) = \begin{bmatrix} -\sum_{i=1}^{2q} w_i & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix}, \; J(eq_j^{\pm}) = \begin{bmatrix} -\sum_{i=j+1}^{2q} w_i & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix} \tag{13}$$

for $j = 1, ..., 2q-1$. Denoting the element J_{11} of the Jacobian matrices as k, the eigenvalue configurations of J with respect to the parameter k are shown on Fig. 7. For $\alpha = 9$, $\beta = 14.286$ four regions can be defined with $k_1 = -7.2740$, $k_2 = -0.7267$, $k_3 = 0$ (see Fig. 7). For the design of n-scroll attractors, the vectors m and c are chosen such that the eigenvalue configurations belong to regions II-IV, i.e. $k_1 < k < k_2$ and $k_3 < k$ respectively. The nonlinearities (3)-(9) lead to the following eigenvalue configurations of the Jacobians, evaluated at the equilibrium points:

1. *Even case:*

   - $q = 1$: double scroll

$$\mathrm{II} - \boxed{\mathrm{IV}} - \mathrm{II}$$

   - $q = 2$: 2-double scroll

$$\mathrm{II} - \mathrm{IV} - \mathrm{II} - \boxed{\mathrm{IV}} - \mathrm{II} - \mathrm{IV} - \mathrm{II}$$

   - $q = 3$: 3-double scroll

$$\mathrm{II} - \mathrm{IV} - \mathrm{II} - \mathrm{IV} - \mathrm{II} - \boxed{\mathrm{IV}} - \mathrm{II} - \mathrm{IV} - \mathrm{II} - \mathrm{IV} - \mathrm{II}$$

2. *Odd case:*

   - $q = 1$: 1-scroll

$$\mathrm{IV} - \boxed{\mathrm{II}} - \mathrm{IV}$$

   - $q = 2$: 3-scroll

$$\mathrm{IV} - \mathrm{II} - \mathrm{IV} - \boxed{\mathrm{II}} - \mathrm{IV} - \mathrm{II} - \mathrm{IV}$$

   - $q = 3$: 5-scroll

$$\mathrm{IV} - \mathrm{II} - \mathrm{IV} - \mathrm{II} - \mathrm{IV} - \boxed{\mathrm{II}} - \mathrm{IV} - \mathrm{II} - \mathrm{IV} - \mathrm{II} - \mathrm{IV}$$

   - $q = 4$: 7-scroll

$$\mathrm{IV} - \mathrm{II} - \mathrm{IV} - \mathrm{II} - \mathrm{IV} - \mathrm{II} - \mathrm{IV} - \boxed{\mathrm{II}} - \mathrm{IV} - \mathrm{II} - \mathrm{IV} - \mathrm{II} - \mathrm{IV} - \mathrm{II} - \mathrm{IV}\ .$$

These structures for the eigenvalue configurations are a necessary condition in order to obtain n-scroll attractors, but are not sufficient due to the local stability analysis. Assuming an n-scroll attractor has been obtained for a certain value of n, a constructive but very qualitative approach to obtain an $(n+2)$-scroll attractor, is to create the possibility for having two additional scrolls by taking into account the structure of the eigenvalue configurations and modifying that part of the nonlinearity which is responsible for generating the outer scrolls, such that a transition can be made to the additional scrolls. In this way the vectors m and c have been determined in (3)-(9).

4 Lur'e Representation

In this Section we give a Lur'e representation of the generalized Chua's circuit. Lur'e systems [14, 24] are of the form

$$\left\{ \begin{array}{rcl} \dot{\xi} & = & A\xi + Bu \\ y & = & C\xi \\ u & = & \sigma(y) \end{array} \right. \tag{14}$$

with $\xi \in \mathbb{R}^p$, $u, y \in \mathbb{R}^{n_h}$, $\sigma(.) : \mathbb{R}^{n_h} \mapsto \mathbb{R}^{n_h}$ a diagonal nonlinearity belonging to sector $[0, K]$ and $A \in \mathbb{R}^{p \times p}$, $B \in \mathbb{R}^{p \times n_h}$, $C \in \mathbb{R}^{n_h \times p}$. Lur'e systems might be interpreted also as a class of recurrent neural networks of the form

$$\dot{\xi} = A\xi + B\sigma(C\xi) \tag{15}$$

with interconnection matrices A, B, C, having one hidden layer with a number of n_h hidden neurons and zero bias terms [20].

The Lur'e representation for (1)-(2) is given by

$$A = \left[\begin{array}{ccc} -\alpha w_{2q} & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{array} \right], B = \left[\begin{array}{cccc} -\alpha w_1 & -\alpha w_2 & ... & -\alpha w_{2q-1} \\ 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \end{array} \right], C = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 0 \end{array} \right]. \tag{16}$$

The vector valued nonlinearity $\sigma = [\sigma_1; \sigma_2; ...; \sigma_{n_h}]$ with $n_h = 2q-1$ consists of components $\sigma_i(x; c_i) = \frac{1}{2}(|x + c_i| - |x - c_i|)$, for $i = 1, 2, ..., n_h$. Hence $\sigma(.)$ belongs to sector $[0, 1]$. This representation is illustrated on Fig.8 for a 3-scroll attractor.

5 Conclusion

In this paper we introduced a family of n-scroll attractors, which contains attractors both with an even and odd number of scrolls. The even case corresponds to n-double scrolls. The n-scroll attractors are obtained from a generalized Chua's circuit, having a piecewise linear nonlinear characteristic with multiple breakpoints. The work is related to the implementation made by Arena *et al.* for n-double scrolls. The generalized Chua's circuit can be used as a new type of cell within cellular neural networks, in order to obtain and model new kinds of phenomena. Furthermore a Lur'e representation has been given, such that n-scrolls can be used for synchronization applications.

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Captions of Figures

Fig.1. Nonlinearities for the generalized Chua's circuit. (Top-left) double scroll; (Top-right) 1-scroll; (Bottom-left) 2-double scroll; (Top-right) 3-scroll. The equilibrium points of the circuits are located at the intersection between the curves in full lines $(-\varphi(x))$ and dashed lines $(w_{2q}x)$. The dotted lines locate the breakpoints.

Fig.2. Nonlinearity of the generalized Chua's circuit for the case of 5-scroll (Top) and 7-scroll (Bottom).

Fig.3. Three dimensional view on the double scroll (Top) and the 2-double scroll (Bottom).

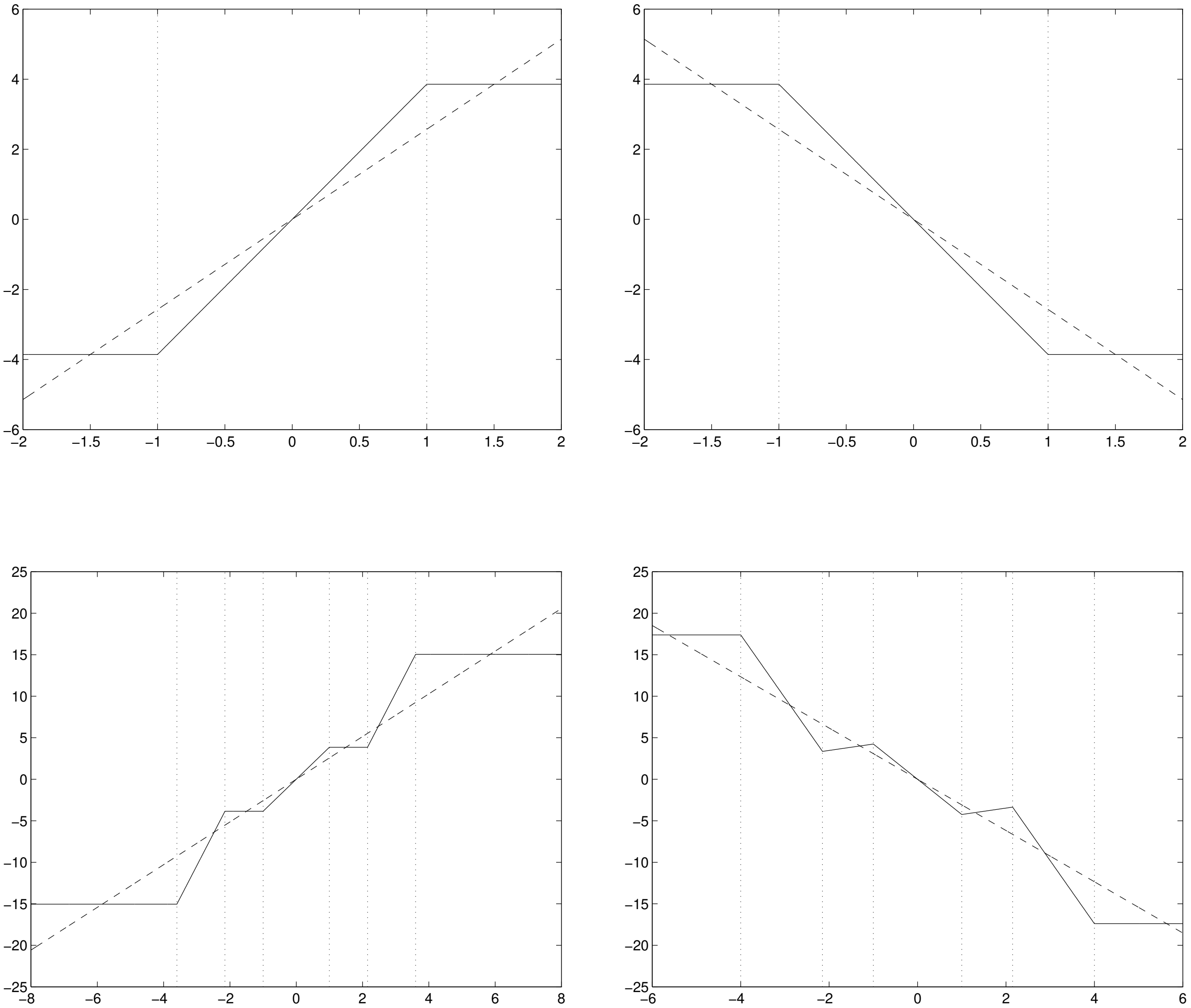
Fig.4. Three dimensional view on the 1-scroll (Top) and the 3-double scroll (Bottom).

Fig.5. 5-Scroll attractor. a/ three dimensional view; b/ (x, y); c/ (x, z); d/ (z, y); e/ $x(t)$.

Fig.6. 7-Scroll attractor. a/ three dimensional view; b/ (x, y); c/ (x, z); d/ (z, y); e/ $x(t)$.

Fig.7. Eigenvalue configurations of the Jacobian matrix, with respect to parameter k. In order to obtain n-scrolls, the nonlinearities are designed such that the eigenvalue configurations of the Jacobians, evaluated at the equilibrium points, belong alternating to the regions II and IV.

Fig.8. Lur'e representation of the generalized Chua's circuit. The nonlinearities $\sigma_1(x, c_1)$ (-), $\sigma_2(x, c_2)$ (- -), $\sigma_3(x, c_3)$ (-.), that belong to sector $[0, 1]$, are shown for a 3-scroll attractor.



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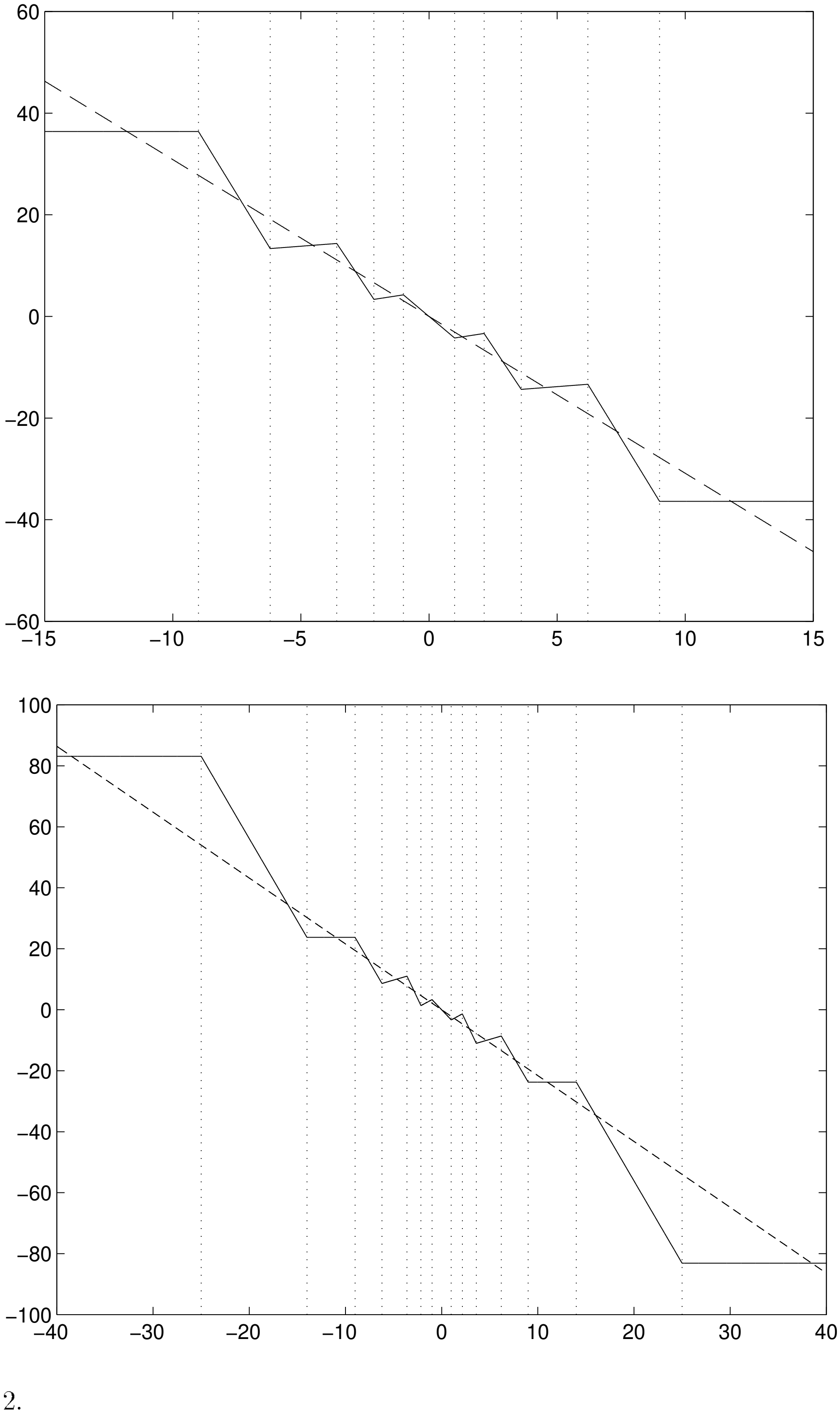
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Fig.2.

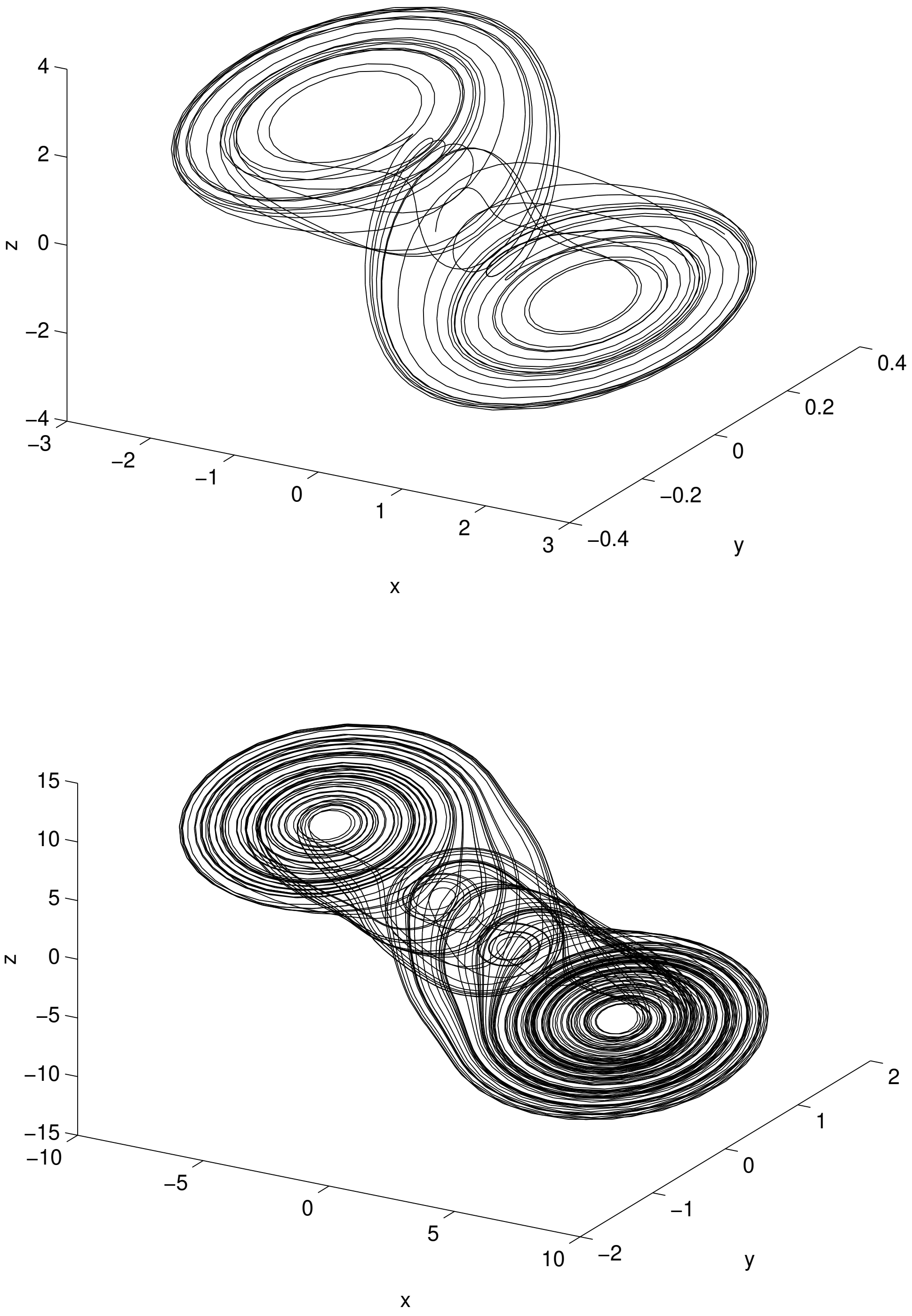

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Fig.3.

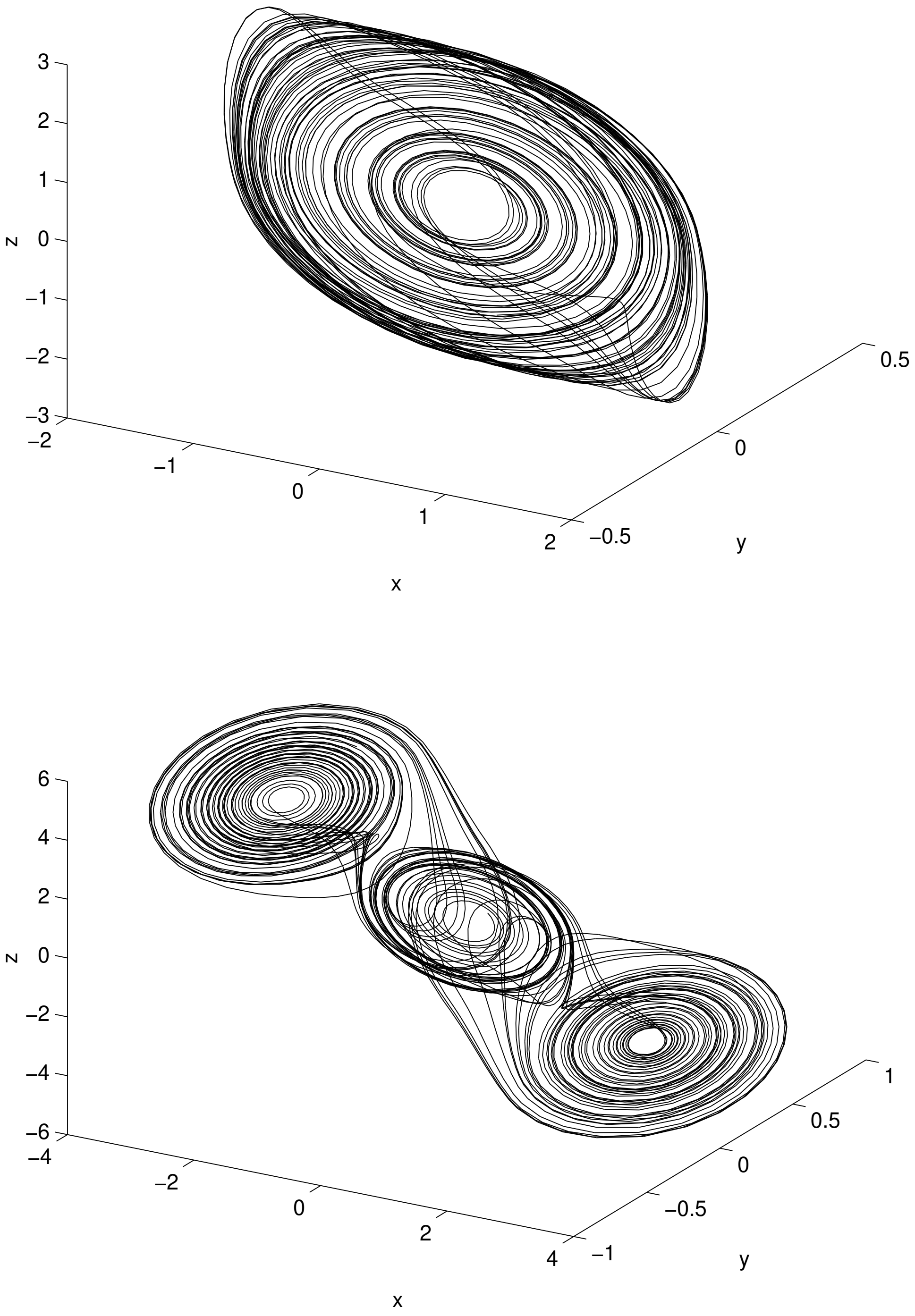

Fig.4.

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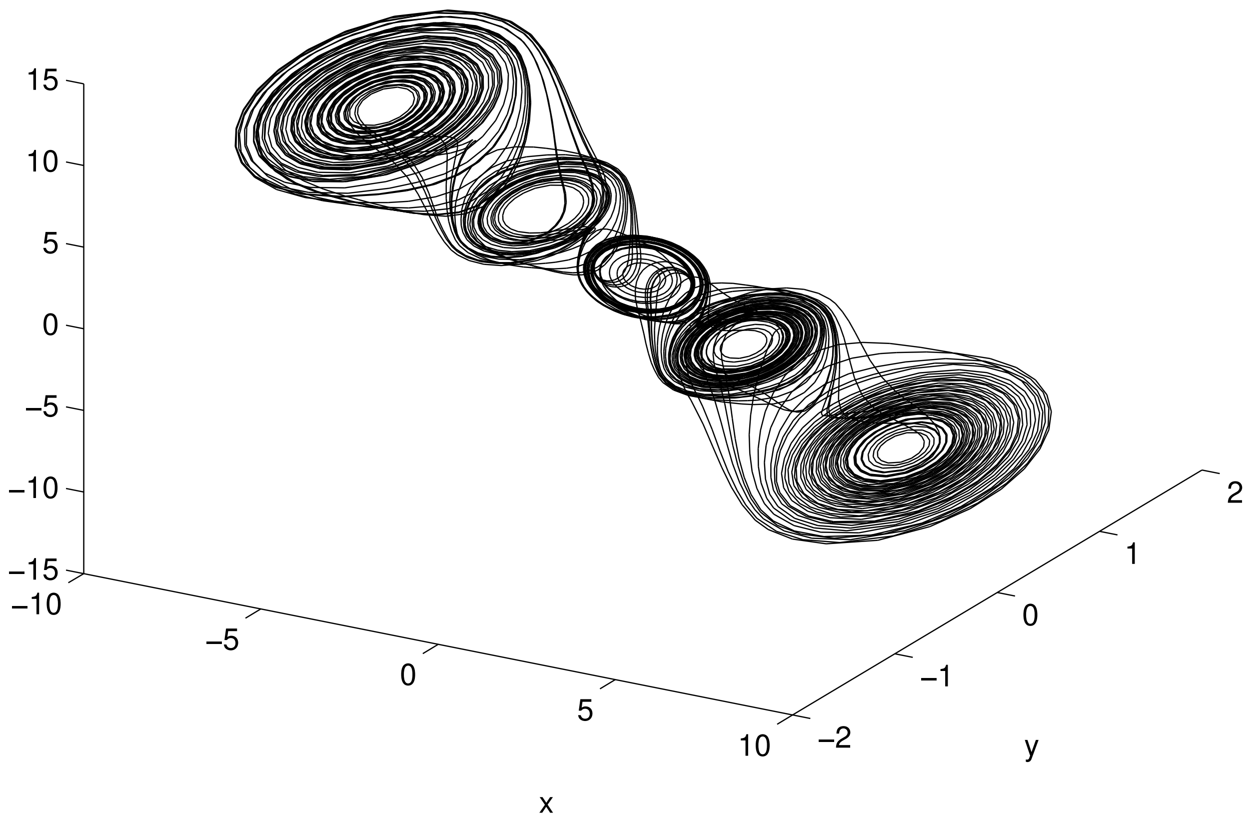

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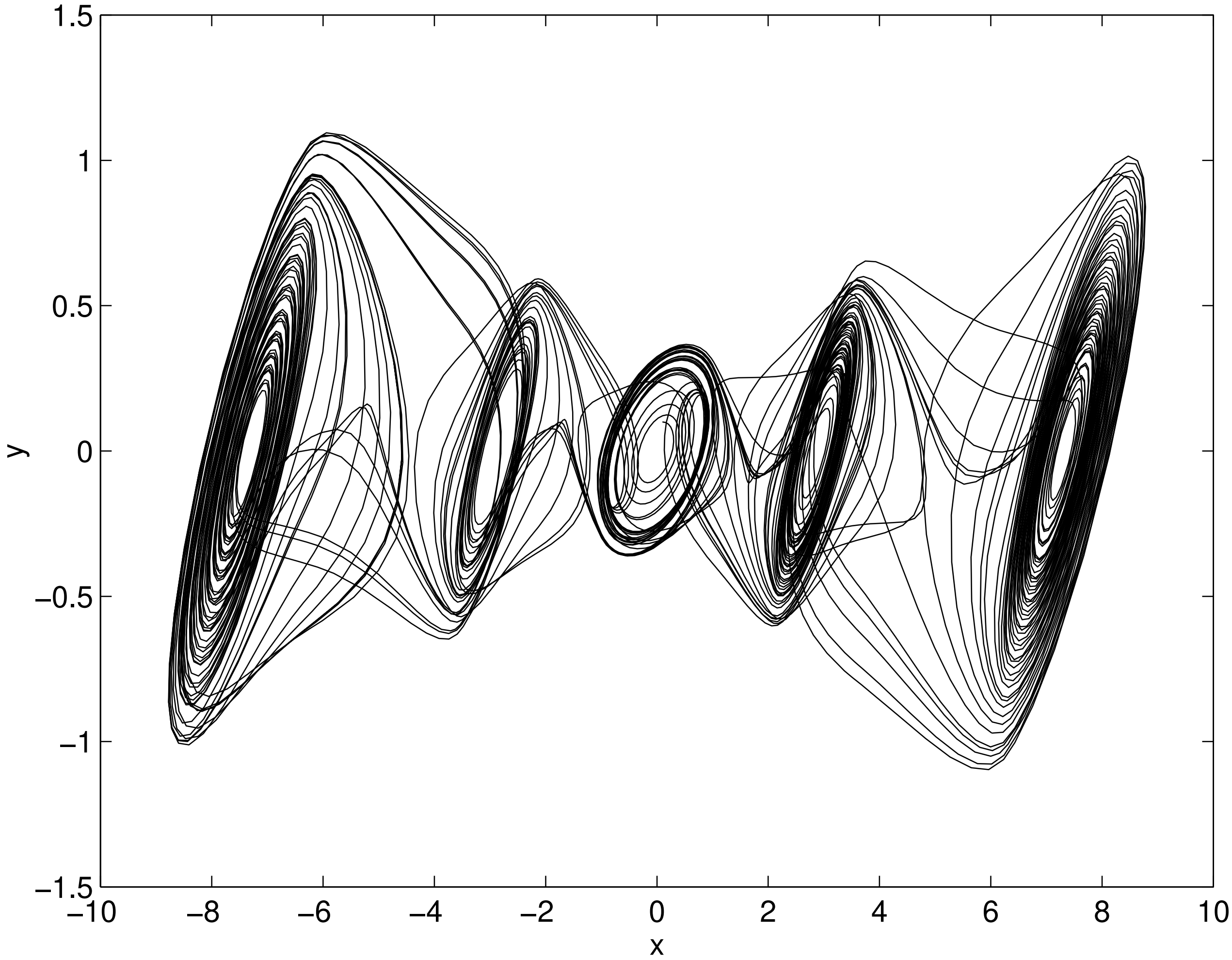

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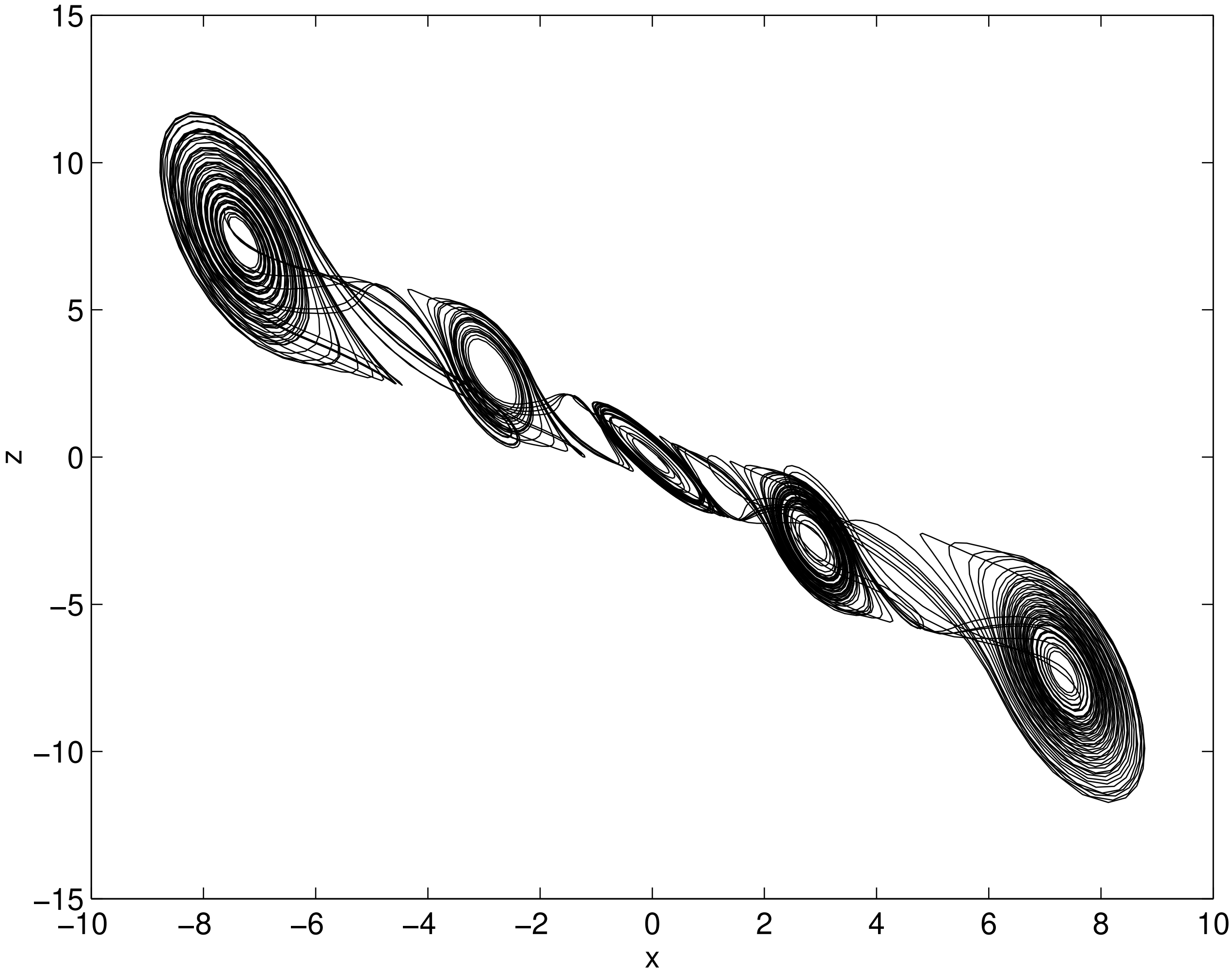

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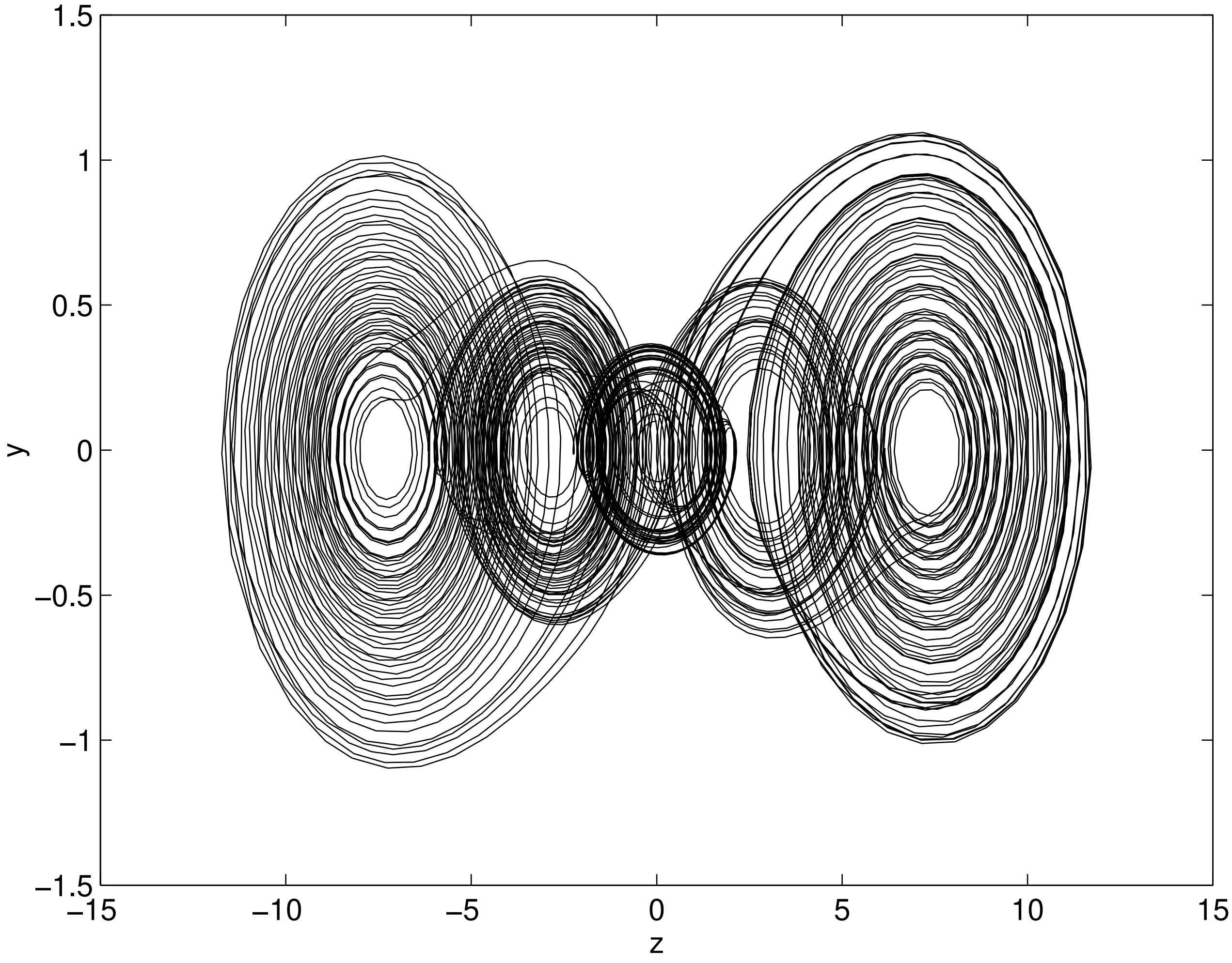

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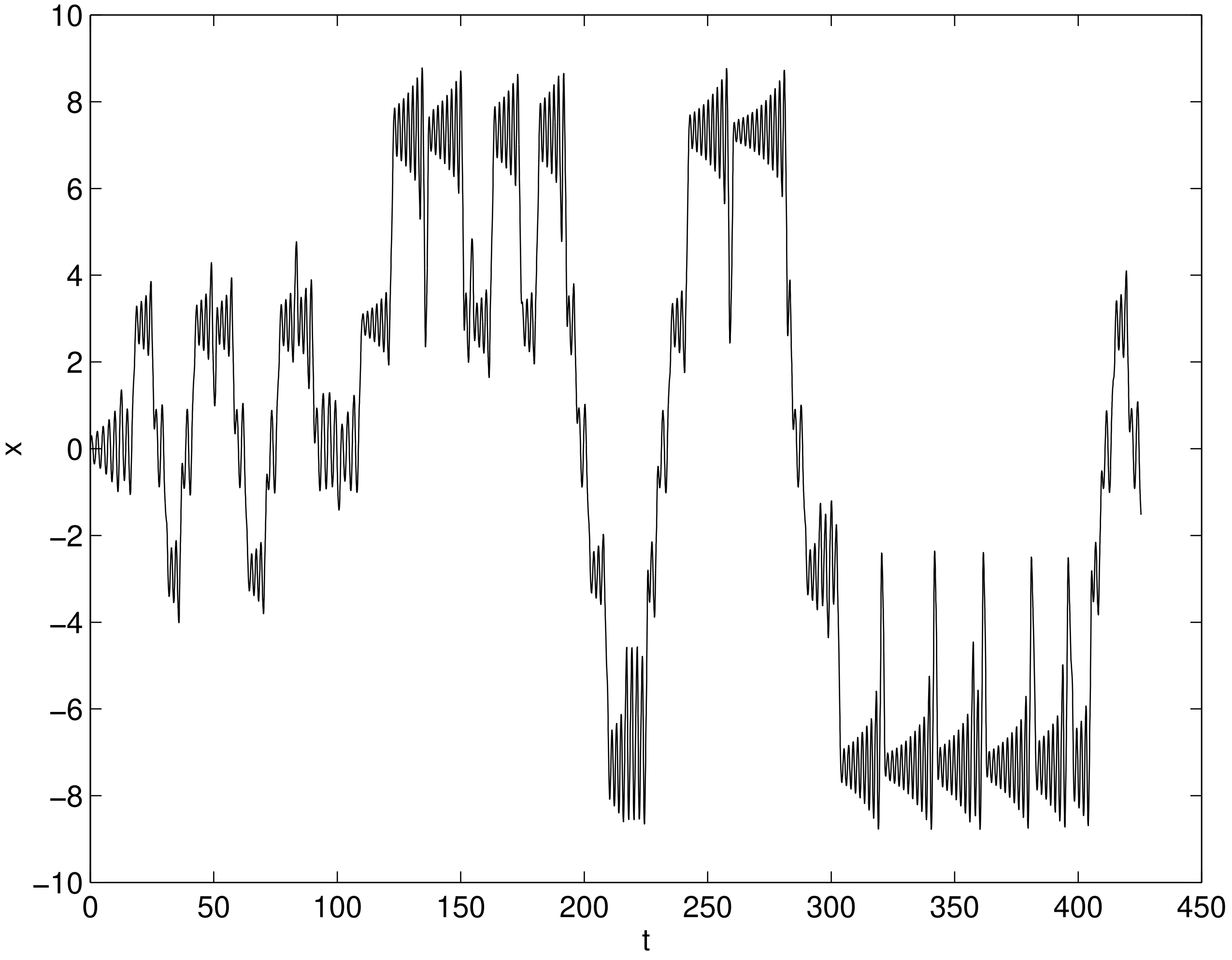

Fig.5.

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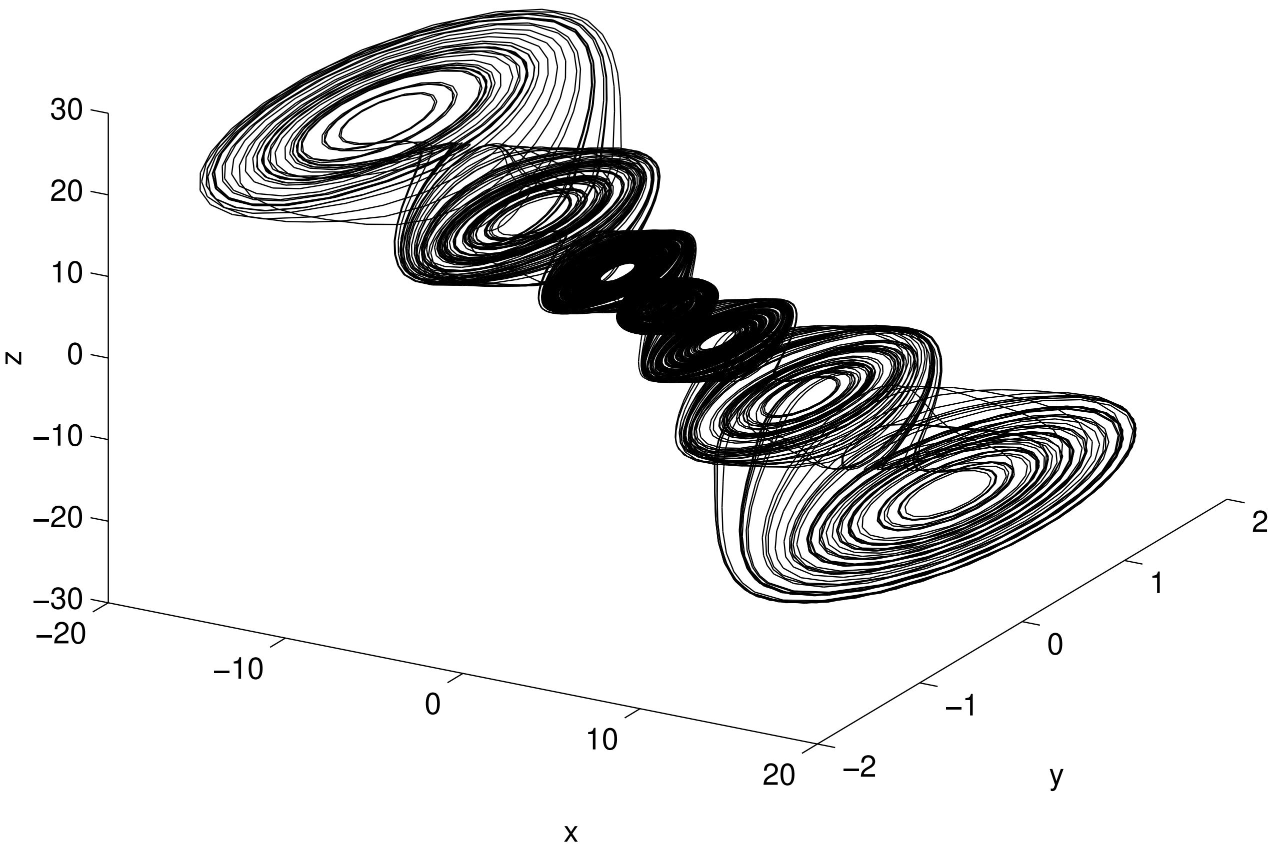

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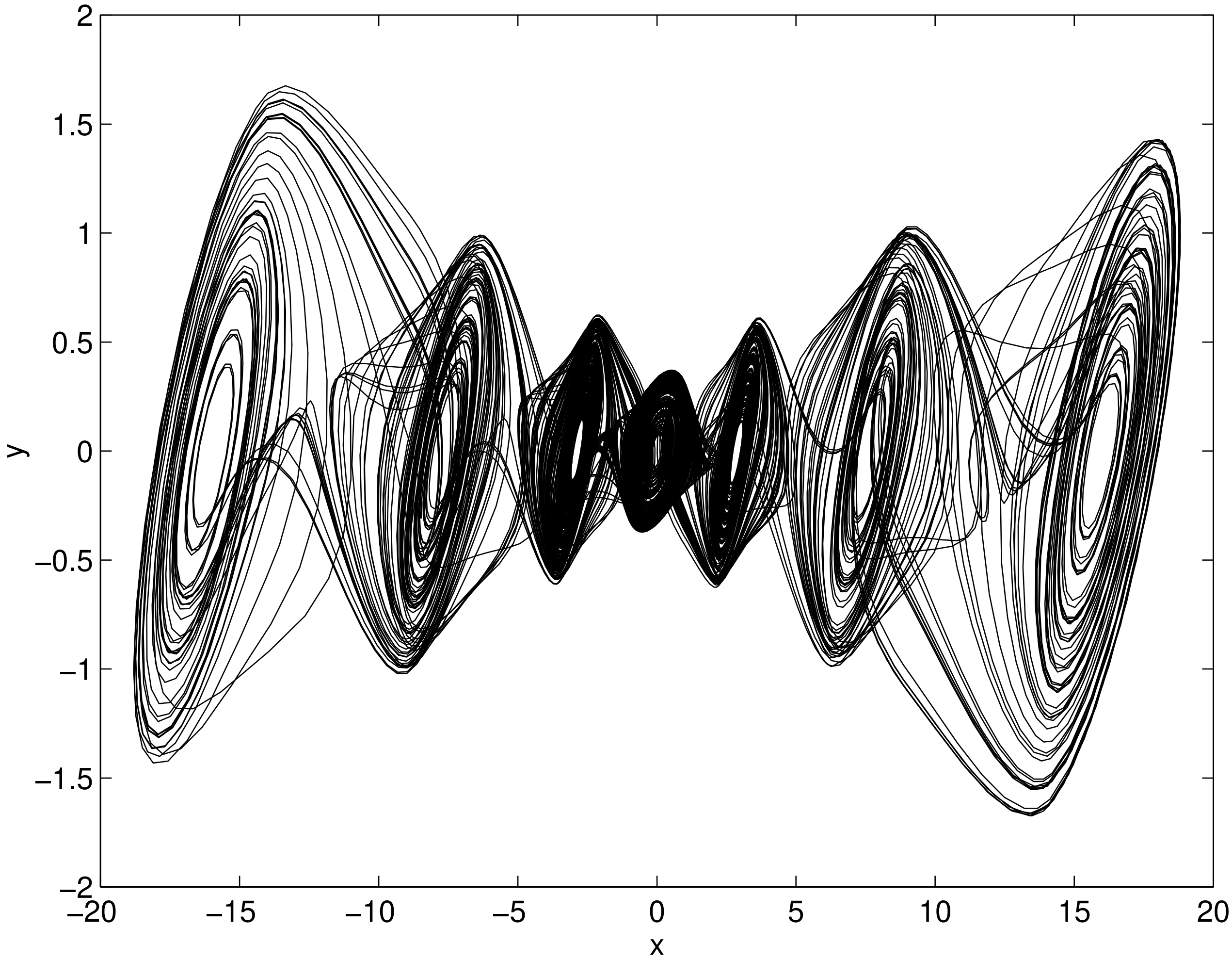

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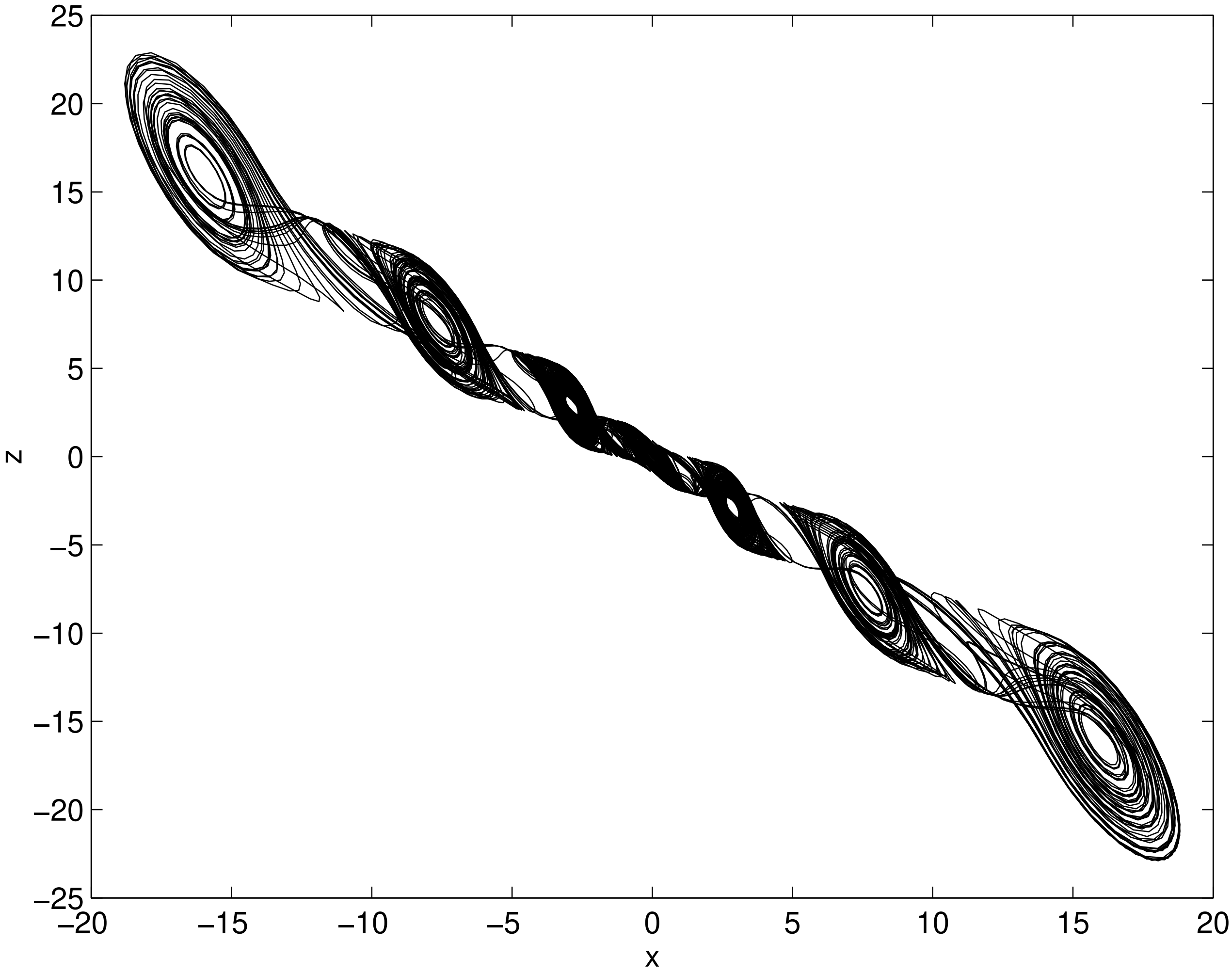

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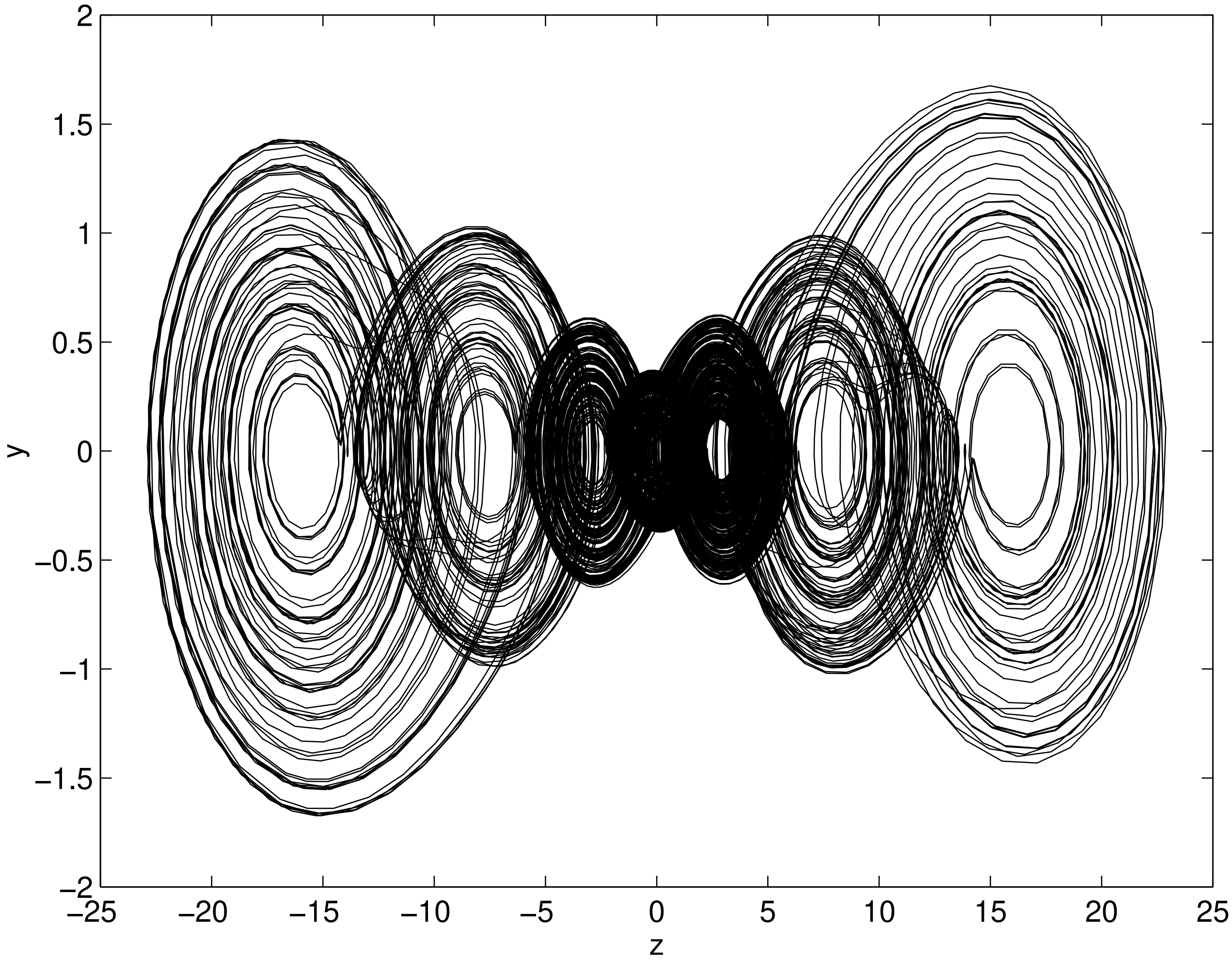

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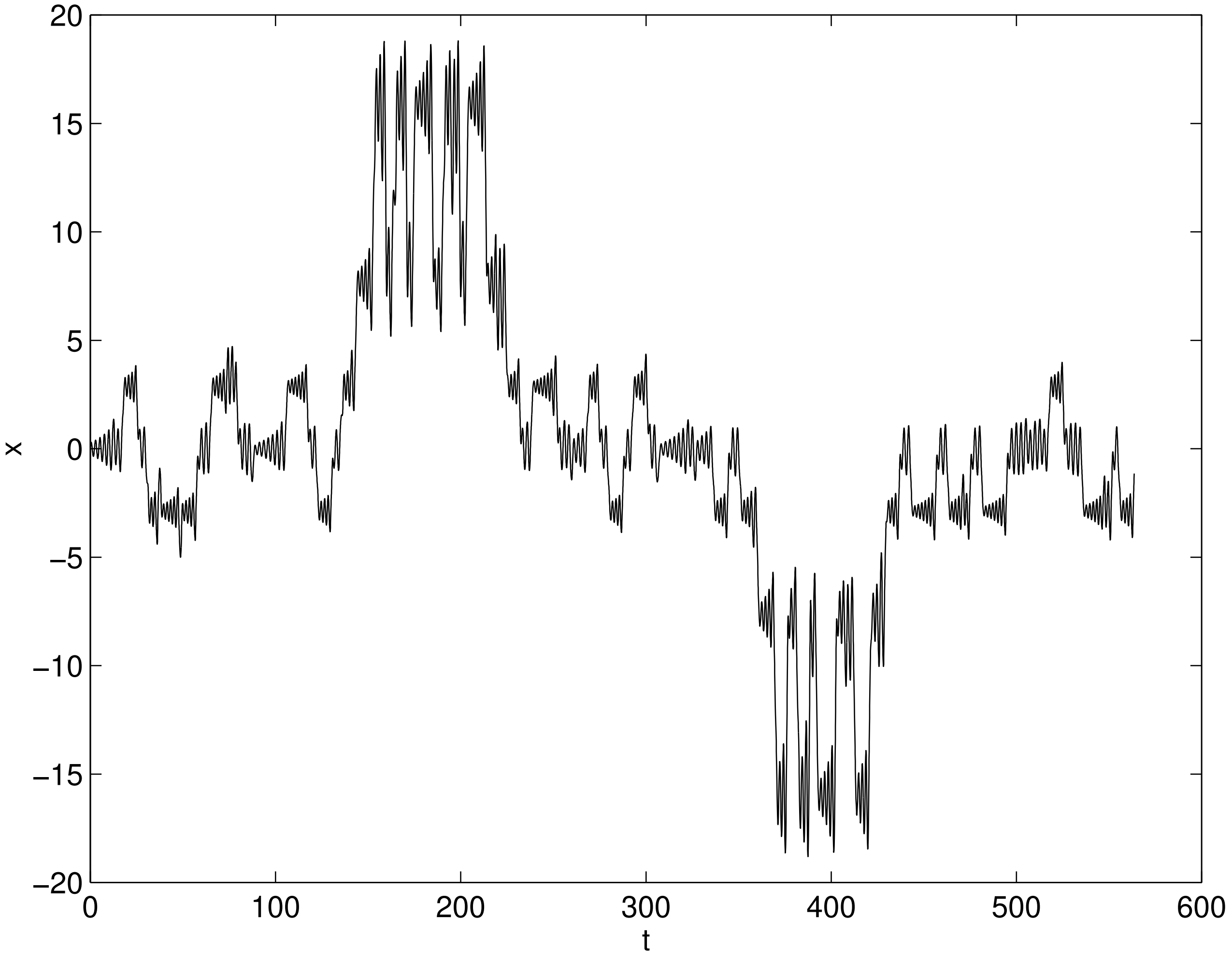

Fig.6.

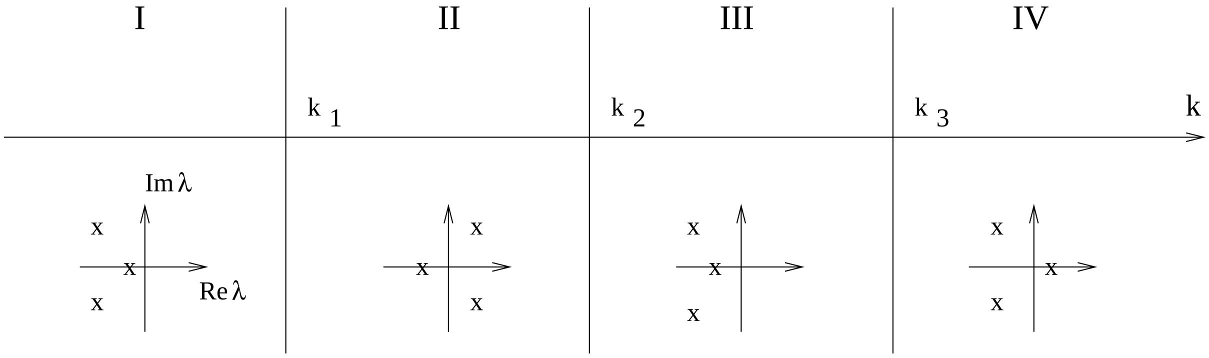

Fig.7.

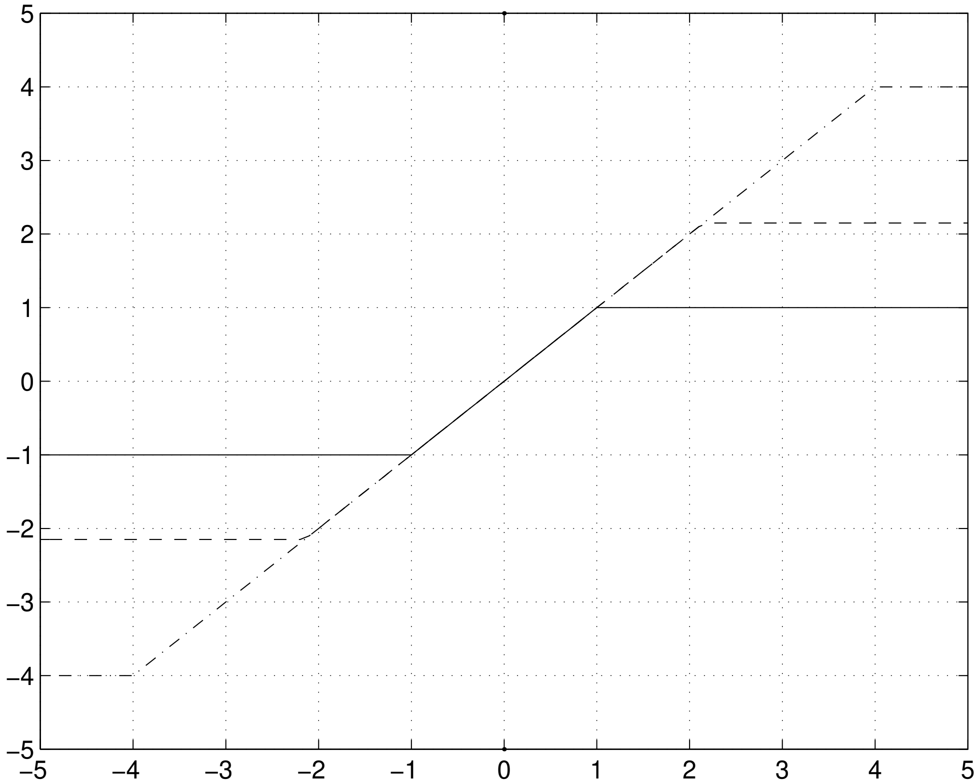

Fig.8.

Caption of Table

Table 1. This Table gives an overview of the relationship between the value of q in the generalized Chua's circuit, the number of hidden units n_h in the Lur'e representation, the number of scrolls for the even and odd case, the number of equilibrium points of the circuit and the number of breakpoints in the nonlinear characteristic. The even or odd case is determined by the sign of the vector m in the nonlinearity.

q	n_h	$n = \#$ scrolls		# eq. points	# breakpoints
		even	odd		
1	1	2	1	3	2
2	3	4	3	7	6
3	5	6	5	11	10
4	7	8	7	15	14
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
q	$2q-1$	$2q$	$2q-1$	$4q-1$	$4q-2$

Table 1.