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J. Suykens, J. Vandewalle[2]

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[2]ESAT - Katholieke Universiteit Leuven, Kardinaal Mercierlaan 94, 3001 Leuven (Heverlee), Belgium, tel 32/16/22 09 31, fax 32/16/22 18 55, email: *Johan.Suykens@esat.kuleuven.ac.be.* This research work was carried out at the ESAT laboratory of the Katholieke Universiteit Leuven, in the framework of a Concerted Action Project of the Flemish Community, entitled *Applicable Neural Networks*. The scientific responsibility is assumed by its authors.

Between n-double sinks and n-double scrolls $(n = 1, 2, 3, 4, ...)^{\ddagger}$

Johan A.K. Suykens, Joos Vandewalle

Katholieke Universiteit Leuven
Department of Electrical Engineering, ESAT-SISTA
Kardinaal Mercierlaan 94
B-3001 Leuven (Heverlee), Belgium
tel: 32/16/22 09 31 fax: 32/16/22 18 55
e-mail: Johan.Suykens@esat.kuleuven.ac.be

Abstract

It is well known that Chua's circuit leading to the double scroll attractor shows different kinds of behaviour depending on the parameters of the system ranging from sinks and periodic behaviour to chaotic behaviour. In this paper the characteristic of the nonlinear resistor of the circuit is modified by introducing additional break points, leading to more 'complicated' attractors, that are called n-double scrolls $(n = 1, 2, 3, 4, ...)$. The new circuit can be seen as a generalization of Chua's circuit where the 1-double scroll corresponds to the classical double scroll. Transition from multiple sink portraits to multiple scrolls can be obtained as a function of a bifurcation parameter of the circuit.

Keywords: Chua's circuit, n-double scroll, bifurcations

1 Generalization of Chua's circuit

Chua's circuit with dynamics

$$\left\{ \begin{array}{rcl} C_1 \frac{dv_{C_1}}{dt} & = & G(v_{C_2} - v_{C_1}) - g(v_{C_1}) \\ C_2 \frac{dv_{C_2}}{dt} & = & G(v_{C_1} - v_{C_2}) + i_L \\ L \frac{di_L}{dt} & = & -v_{C_2} \end{array} \right. \tag{1}$$

is shown in Fig.1. Here v_{C_1},v_{C_2},i_L denote respectively the voltage across C_1 and C_2 and the current through L and $g(v_{C_1})$ is the piecewise-linear function of Fig.1 consisting of two breakpoints

$$g(v_{C_1}) = m_0 v_{C_1} + 0.5(m_1 - m_0)|v_{C_1} + B_p| + 0.5(m_0 - m_1)|v_{C_1} - B_p| \tag{2}$$

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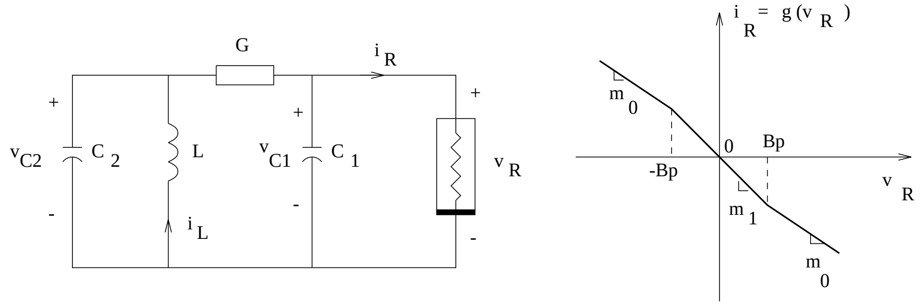

Figure 1. *Chua's circuit for generation of the double scroll attractor.*

By setting the parameters $1/C_1 = 9$, $1/C_2 = 1$, $1/L = 7$, $G = 0.7$, $m_0 = -0.5$, $m_1 = -0.8$, $B_p = 1$ chaotic behaviour is obtained (double scroll attractor) [1][2][3]. Eqn (1) can be written in the form

$$\dot{\mathbf{x}} = A(\mathbf{x})\mathbf{x} \tag{3}$$

as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -a - k(x) & a & 0 \\ b & -b & 1 \\ 0 & -c & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{4}$$

or $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ where $\mathbf{x}(t) = [x(t)\ y(t)\ z(t)]^t$, $\mathbf{f} : \mathbb{R}^3 \mapsto \mathbb{R}^3$, $x = v_{C_1}$, $y = v_{C_2}$, $z = i_L$, $a = G/C_1$, $b = G/C_2$, $c = 1/L$, $k(x) = (1/C_1)g(x)/x$ $(x \neq 0)$ and $C_2 = 1$.

A new circuit will be defined now in terms of the state space description (4) where the nonlinear function $k(x)$ will be *parametrized* as $k_q(x)$ $(q \in \mathbb{N}, q \geq 1)$. The nonlinear function $g(x)$ is adapted accordingly into $g_q(x) = C_1 k_q(x) x$. The circuit is described by

$$\boxed{\begin{array}{rcl} \dot{x} & = & (-a - k_q(x))x + ay \\ \dot{y} & = & bx - by + z \\ \dot{z} & = & -cy \end{array}} \tag{5}$$

with the following algorithmic description for $k_q(x)$

$$\boxed{\begin{array}{ll} \text{if } 0 < |x| \leq \delta_1 : & k_q(x) = \alpha_1 \\ \text{for } i = 2 \text{ } to \text{ } q & \\ \quad j = 2 + 3(i-2) & \\ \quad \text{if } \delta_{j-1} < |x| \leq \delta_j : & k_q(x) = \alpha_2 \frac{|x| - \delta_{j-1}}{\delta_j - \delta_{j-1}} + \alpha_1 \\ \quad \text{if } \delta_j < |x| \leq \delta_{j+1} : & k_q(x) = -\alpha_2 \frac{|x| - \delta_j}{\delta_{j+1} - \delta_j} + \alpha_1 + \alpha_2 \\ \quad \text{if } \delta_{j+1} < |x| \leq \delta_{j+2} : & k_q(x) = \alpha_1 \\ \text{end} & \\ \text{if } \delta_{4+3(q-2)} < |x| : & k_q(x) = (\alpha_3/|x|)(\beta_1|x| + \beta_2||x| + \delta_{4+3(q-2)}| + \ldots \\ & \beta_3||x| - \delta_{4+3(q-2)}|) \end{array}} \tag{6}$$

and by definition

$$\Delta_q = [\delta_1\ \delta_2\ \ldots\ \delta_{4+3(q-2)}]^t \tag{7}$$

The graphical description of some $k_q(x)$ and $g_q(x)$ is given in Fig.2. It is easily verified that for $q = 1$ the circuit (5)(6) corresponds to Chua's circuit (4) and $k_1(x) = k(x)$. Throughout the text we will only consider bifurcations with respect to the parameter a and take $b = 0.7$, $c = 7$, $\alpha_1 = -8a/7$, $\alpha_2 = 2a/7$, $\alpha_3 = a/0.7$, $\beta_1 = -0.5$, $\beta_2 = -0.15$, $\beta_3 = 0.15$.

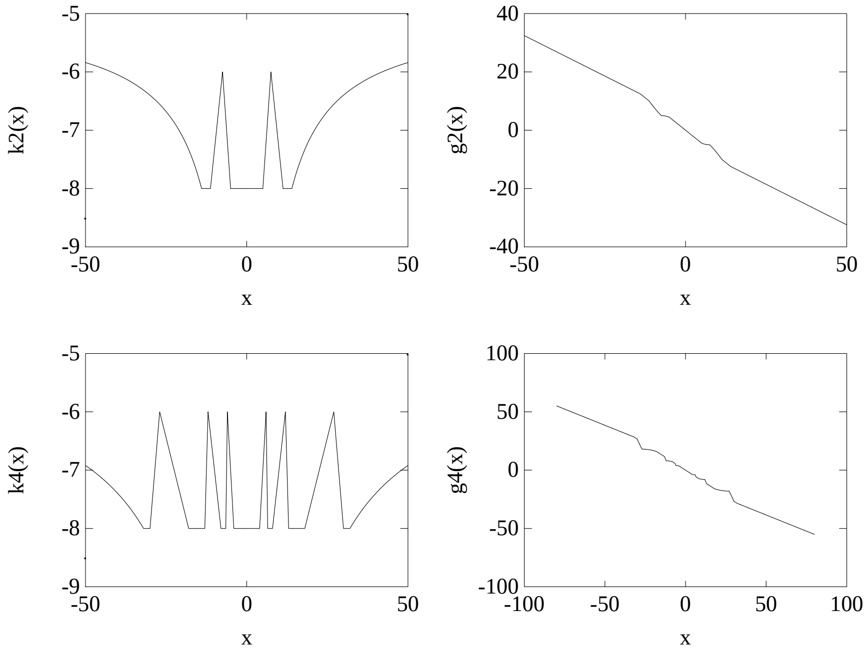

Figure 2. *Nonlinearities* $k_2(x)$,$k_4(x)$ *and corresponding* $g_2(x) = C_1 k_2(x) x$, $g_4(x) = C_1 k_4(x) x$.

2 Generation of n-double scrolls

In order to generate the n-double scroll attractors ($n = 1, 2, 3, 4, ...$) the parameter a is fixed at $a = 7$. The attractors are obtained by setting $\gamma_{j,j-1} = \frac{\delta_j + \delta_{j-1}}{\delta_j - \delta_{j-1}} = 5$ (for $j = 2 + 3(i-2)$, $i = 2, ..., q$). For example a 2-double scroll ($q = 2$) is obtained for $\Delta_2 = [5\ 7.5\ 11.25\ 14]^t$ and a 4-double scroll ($q = 4$) for $\Delta_4 = [4\ 6\ 6.5\ 8\ 12\ 13\ 18\ 27\ 30\ 32]^t$ (see Fig.3).
The functions $k_q(x)$ and $g_q(x) = C_1 k_q(x) x$ are plotted in Fig.2 for $q = 2$ and $q = 4$. Only the 2-double and 4-double scroll will be discussed further on. For a 3-double and a 6-double scroll see [7].

3 Local stability analysis

The equilibrium points of (5)(6) are identified by setting $\dot{x} = \dot{y} = \dot{z} = 0$ which yields the conditions

$$\begin{cases} y = 0 \\ bx + z = 0 \\ x = 0 \ \text{ or } \ k_q(x) = -a \end{cases} \tag{8}$$

This results into $4q - 1$ equilibrium points: $\mathbf{eq}_0 = \mathbf{0}$ and $\mathbf{eq}^{\pm}_{j,j-1}$, $\mathbf{eq}^{\pm}_{j+1,j}$, $\mathbf{eq}^{\pm}_{4+3(q-2)}$ with $j = 2 + 3(i-2)$ and $i = 2, ..., q$ with $q \geq 1$. The latter appear in pairs ($\pm$ refers to the sign of the x component).

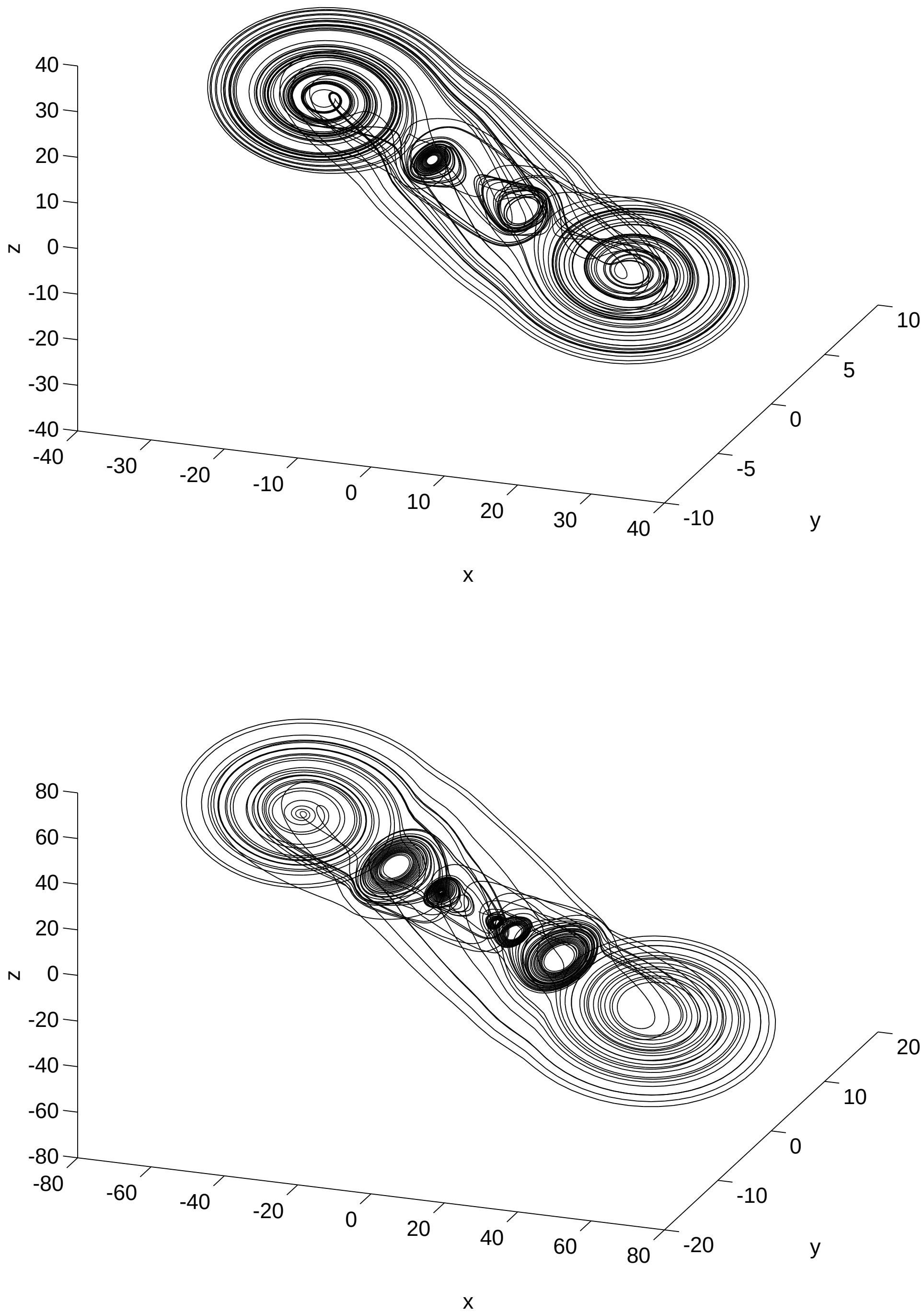

Figure 3. *A three dimensional view on the* 2*-double scroll and* 4*-double scroll.*

For the 2-double we have the following equilibrium points and associated eigenvalues of the Jacobians

$$\begin{array}{llcll} \mathbf{eq}_0 = [0\ 0\ 0]^t & J(\mathbf{eq}_0) & \rightarrow & \lambda_1 = 1.76 & \lambda_{2,3} = -0.73 \pm 1.85j \\ \mathbf{eq}_{2,1}^{\pm} = [\pm 6.25\ 0\ \mp 4.38]^t & J(\mathbf{eq}_{2,1}^{\pm}) & \rightarrow & \lambda_1 = -5.58 & \lambda_{2,3} = 0.04 \pm 2.46j \\ \mathbf{eq}_{3,2}^{\pm} = [\pm 9.38\ 0\ \mp 6.56]^t & J(\mathbf{eq}_{3,2}^{\pm}) & \rightarrow & \lambda_1 = 5.65 & \lambda_{2,3} = -0.67 \pm 2.40j \\ \mathbf{eq}_4^{\pm} = [\pm 21\ 0\ \mp 14.7]^t & J(\mathbf{eq}_4^{\pm}) & \rightarrow & \lambda_1 = -3.05 & \lambda_{2,3} = 0.18 \pm 2.14j \end{array} \quad (9)$$

4 A quasilinear study

In order to get a rough global view onto the state space behaviour of (5)(6) a quasilinear approach is applied, according to [7]. This method is heuristic but may give additional insight in combination with a local stability analysis. However this method has limitations because with respect to its use for stability analysis of nonlinear systems counterexamples to the method exist like to the Aizerman conjecture [6].
The idea is simply to study the eigenvalues of $A(\mathbf{x})$ in $\dot{\mathbf{x}} = A(\mathbf{x})\mathbf{x}$. Suppose

$$\lambda(A(\mathbf{x})) = \sigma(\mathbf{x}) + j\omega(\mathbf{x}) \quad (10)$$

For the system (5)(6) only the following situation occurs

$$\begin{array}{lcl} \lambda_1 & = & \sigma_1 \\ \lambda_{2,3} & = & \sigma_{2,3} \pm j\omega \end{array} \quad (11)$$

Regions $S, U_r, U_i \subset \mathbb{R}^3$ are now defined as

$$\begin{array}{lcl} \text{if } \sigma_1(\mathbf{x}_P) < 0 \text{ and } \sigma_{2,3}(\mathbf{x}_P) < 0 & \rightarrow & x_P \in S \\ \text{if } \sigma_1(\mathbf{x}_P) > 0 \text{ and } \sigma_{2,3}(\mathbf{x}_P) < 0 & \rightarrow & x_P \in U_r \\ \text{if } \sigma_1(\mathbf{x}_P) < 0 \text{ and } \sigma_{2,3}(\mathbf{x}_P) > 0 & \rightarrow & x_P \in U_i \end{array} \quad (12)$$

In Fig.4 these regions are plotted for the cases $q = 2$ and $q = 4$ as a function of the bifurcation parameter a between 3 and 8.

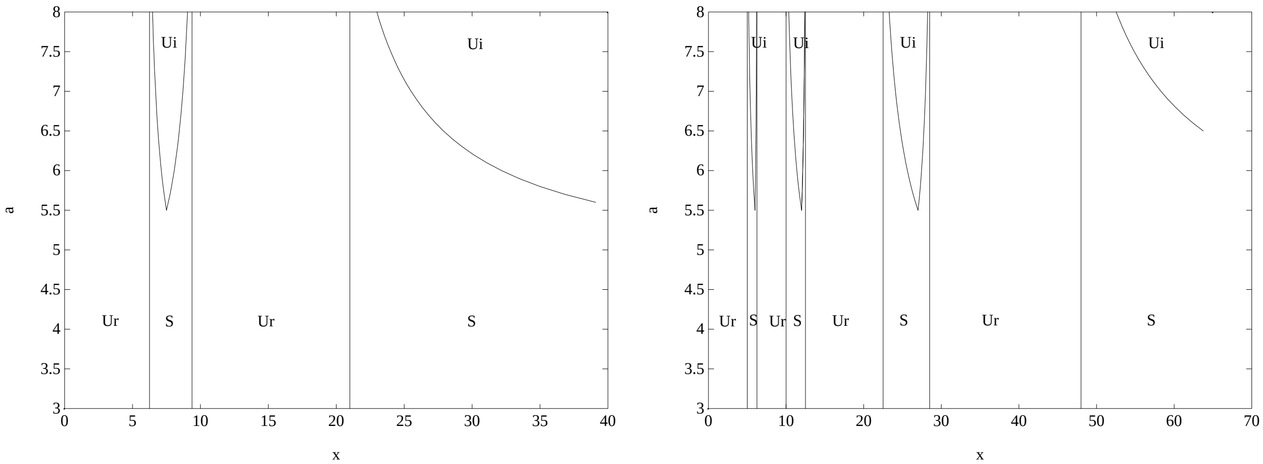

Figure 4. *A quasilinear study of the nonlinear circuit for generation of the 2-double scroll ($q = 2$) and the 4-double scroll ($q = 4$). These n-double scrolls are obtained by setting $a = 7$.*

5 Bifurcations with respect to a

From the diagram of Fig.4 it is clear that for $3 < a < 5.5$ a sink portrait will be obtained, corresponding with a $\boxed{U_r - S}$ configuration. The n-double scroll attractors are obtained around $a = 7$ with a configuration $\boxed{U_r - S - U_i}$ $-S-$ $\boxed{U_r - S - U_i}$ $-...$ Increasing the parameter a has in fact the effect of having a higher probability to visit each of the $2n$ scrolls and plays a similar role as temperature for a particle in a potential field. In Fig.5 bifurcations in the circuit are shown for $q = 2$

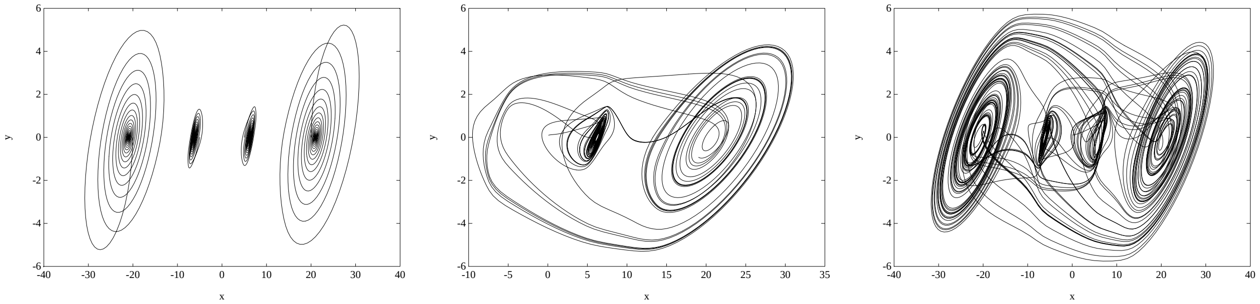

Figure 5. *Bifurcations in the circuit for* $q = 2$*: (left)* $a = 4$*: 2-double sink; (middle)* $a = 6.5$*; (right)* $a = 7$*: 2-double scroll.*

6 Conclusions

It it shown how Chua's circuit can be generalized in order to obtain n-double scroll attractors with as special case Chua's well know (1)-double scroll. A quasilinear study is applied in order to obtain a rough insight into the global behaviour of the system. The method may contribute as a tool towards the construction of dynamics.

References

[1] Chua L.O., 'The genesis of Chua's Circuit', *Archiv für Elektrotechnik und Ubertragungtechnik*, Vol.46, No.4, pp.250-257, 1992.

[2] Chua L.O., 'Global unfolding of Chua's Circuit', *IEICE Trans. Fundamentals*, Vol. E76-A, No.5, pp.704-734, May 1993.

[3] Chua L.O., Komuro M., Matsumoto T., 'The Double Scroll Family', *IEEE Trans. Circuits Syst.*, Vol. CAS-33, No.11, pp.1072-1118, Nov. 1986.

[4] Guckenheimer J., Holmes P., *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, New York: Springer Verlag, 1983.

[5] Hirsch M.W., Smale S., *Differential Equations, Dynamical Systems, and Linear Algebra*, New York: Academic Press, 1974.

[6] Suykens J.A.K., Vandewalle J., 'Quasilinear Approach to Nonlinear Systems and the design of n-Double Scroll ($n = 1, 2, 3, 4, ...$)', *IEE Proceedings-G*, Vol.138, No.5, pp.595-603, Oct. 1991.

[7] Suykens J.A.K., Vandewalle J., 'Generation of n-double scrolls ($n = 1, 2, 3, 4, ...$)', to appear in *Special Issue on Chaos in Nonlinear Electronic Circuits, IEEE Transactions on Circuits and Systems.*