Polytopic invariant sets for continuous-time systems

Toni Barjas Blanco and Bart De Moor

***Abstract*— In Model Predictive Control stability can be guaranteed by the use of an invariant terminal set. In this paper a numerical method is described concerning the computation of a low-complexity polytopic-invariant set for linear and nonlinear continuous time-systems subject to state, input and rate constraints. The method determines an (sub)optimal linear feedback gain w.r.t. the volume of the invariant polytope. A trade-off between optimality of the feedback gain and the volume of the invariant polytope is made by use of a tuning parameter.**

Keywords: *Polytopic invariant sets, Optimization, Constrained control, Nonlinear systems, Linear systems*

I. INTRODUCTION

Any locally stable time-invariant dynamical system admits some domains in its state space from which any state-vector trajectory cannot escape. These domains are called positively invariant sets and are widely used in MPC for designing of terminal constraint sets, also called target sets, as a tool for the guarantee of system closed-loop stability. These target sets are mainly used in dual mode MPC strategies. The main idea is to determine a set in the state space invariant for a certain feedback (usually a linear feedback) with the property that no constraints violation occurs as long as the state remains inside this set.

In literature two types of convex sets are essentialy used as candidate invariant sets, ellipsoidal and polyhedral sets [2]. In this paper the focus will be on the computation of (symmetrical) polyhedral sets. Invariant ellipsoidal sets for the continuous-time case can be found in ([14], [15]). In the discrete-time case numeruous techniques are available for the calculation of polytopic invariant sets ([2], [7], [8], [10], [11], [16]). In the continuous-time case there are also several contributions concerning this type of sets ([3], [6], [12]), however this literature is not as elaborated. Furthermore, the existing techniques suffer from some drawbacks. In [12] a method is described for the calculation of a polyhedral invariant set. However, the method only applies to linear systems and the linearizing feedback gain has to be given in advance, restricting the size of the invariant polytope. In [3] the linear continuous-time system is approximated by an Euler approximating system (EAS). It is shown that for a sufficiently small sampling time a set that is invariant for the EAS is also invariant for the continuous-time system. But no upperbound for the sampling time is obtained. In [6] a method is described to determine an invariant set satisfying input and rate constraints. This method, however, only holds for linear systems and trial and error is needed to determine the invariant set. Methods based on nonquadratic Lyapunov functions, as the ones described in ([1], [5]), yield polytopic invariant sets. However, these methods do not take input and state constraints into account.

In this paper a method is described that circumvents most of these drawbacks. The proposed method can be used to determine symmetrical polyhedral invariant sets for linear and nonlinear continuous-time systems subject to state and/or rate constraints. The obtained linear feedback is optimal, possibly suboptimal, w.r.t. the size of the invariant set. The method also allows to make a trade-off between size of the invariant set and optimality of the linear feedback by use of a tuning parameter. Furthermore, a new invariance condition is stated that is only valid for the type of polytopic sets considered in this paper. This condition differs significantly from the more commonly used invariance conditions stated in ([2], [4]) and is the basis for the optimization procedure elaborated in this work.

The remainder of the paper is organized as follows: In section II the studied problem is stated. In section III some preliminary results are stated. An important invariance condition forms the main result of this section. Section IV is devoted to state the main results which consist of an iterative procedure to find an optimal linear feedback gain w.r.t. the size of the invariant polytope. In section V some examples show how the proposed algorithm determines invariants sets for nominal and uncertain continuous-time linear systems.

II. PROBLEM STATEMENT

In this work the following continuous-time linear system,

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

and continuous-time nonlinear system are assumed:

$$\dot{x}(t) = f(x(t), u(t)). \tag{2}$$

It is assumed that both systems are subject to the following constraints:

$$x(t) \in X \tag{3}$$

$$u(t) \in U \tag{4}$$

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Preprint submitted to European Control Conference 2007.
Received October 18, 2006.

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^l$, $f(0,0) = 0$, X and U symmetrical polyhedral sets. Assume a closed-loop linear state control law

$$u(t) = Kx(t) \tag{5}$$

In the sequel a method will be described to determine a set

$$\varphi = \{x : \|Vx\|_\infty \leq \gamma\} \tag{6}$$

invariant under the control law (5) such that $\forall t \in [0, \infty]$ no violations of the constraints (3), (4) occur. The objective of the method is to determine the linear feedback (5) in such a way that the corresponding invariant set is as large as possible.

In case of the nonlinear system (2), let Π^0 denote a low-complexity polytope:

$$\Pi^0 = \left\{x : \left\|V^0 x\right\|_\infty \leq 1\right\} \tag{7}$$

with full-rank $V^0 \in \mathbb{R}^{n\times n}$. Within this region an LDI $\{[A_i B_i], i = 1, \ldots, p\}$ can be defined such that the system (2) satisfies following condition:

$$f(x,u) \in Co\{A_i x + B_i u\}, \forall (x,u) \in \Pi^0 \times U \tag{8}$$

with Co denoting the convex hull. If (8) holds, every trajectory of the nonlineair system (2) is also a trajectory of the LDI. Techniques for determining an LDI for a given nonlinear system can be found in [9], [13]. In the sequel this property will be intensively used for determining invariant sets for the nonlinear system (2).

III. PRELIMINARY RESULTS

In the following, given a set φ we denote by $\delta\varphi$ its boundary. We assume the set φ is polytopic and denote its vertices by *vert*$\{\varphi\}$. Assume an arbitrary time-dependent variable $v(t)$, than we denote by $\Delta v(t)$ (or $\dot{v}(t)$) the derivative of $v(t)$ in the continuous-time case.

Definition. The set φ is said positively invariant for the system (1) under the feedback law (5) if for all x(0) $\in \varphi$ the solution x(t) $\in \varphi$ for t $\geq$ 0.

This means that each trajectory starting in the set φ stays inside the set under the linear feedback. In case of a polytopic set it is not necessary to check the positive invariance for all x(0) $\in \varphi$. The following theorem provides an easy way to check the positively invariance of a polytopic set.

Lemma 1: If the trajectories of all $v_i \in$ *vert*$\{\varphi\}$ stay inside the set φ for the system (1) under the feedback law (5), the set φ is positively invariant.

Proof: For all x(0) $\in \varphi$ holds that $x(0) = \sum_i^n \lambda_i v_i(0)$, with n the number of vertices of φ, $v_i(0)$ the vertices, $\lambda_i \geq 0$ and $\sum_i^n \lambda_i = 1$. This holds because φ is a convex set. Assuming a linear control feedback law of the form (5) and system (1), the following holds for each vertex v_i:

$v_i(t) = e^{\phi t} \cdot v_i(0)$

with $\phi = A + BK$.

Since $x(0) = \sum_i^n \lambda_i v_i(0)$, the following holds:

$x(t) = e^{\phi t} \cdot x(0) = e^{\phi t} \cdot \sum_i^n \lambda_i v_i(0) = \sum_i^n \lambda_i e^{\phi t} v_i(0) = \sum_i^n \lambda_i v_i(t)$. This means that at each time instant t the trajectory x(t) lies in the convex hull of $v_i(t)$. Since the trajectories of all $v_i(t)$ lie inside the set φ, the trajectory $x(t)$ also lies inside φ.

■

Lemma 1 basically states that in order for a set of the form (6) to be invariant, only invariance conditions on the vertices should be considered. The following theorem provides such invariance conditions.

Theorem 1: A set φ of the form (6) is invariant for system (1) under the linear feedback law (5) if the following condition holds at each vertex v_i of the set:

$$-sign((Vv_i)_j) \cdot \Delta((Vv_i)_j) \geq 0 \tag{9}$$

with $(Vv_i)_j$ the jth component of Vv_i.

Proof: 9 can be rewritten into the following conditions:

- if $(Vv_i)_j \geq 0 \Rightarrow \Delta(Vv_i)_j \leq 0$
- if $(Vv_i)_j \leq 0 \Rightarrow \Delta(Vv_i)_j \geq 0$

Since $\|Vv_i\|_\infty = max_j(|(Vv_i)_j|)$ these conditions assure the existence of a $\Delta t_i > 0$ for v_i such that $\forall t \in [0, \Delta t_i]$: $\|Vv_i(t)\|_\infty \leq \gamma \Rightarrow \forall t \in [0, \Delta t_i] : v_i(t) \in \varphi$. Note that if condition (9) is imposed at each vertex, than there exists a Δt_i for each vertex v_i. Remark that each such Δt_i can be different. Now assume $\Delta t = min_i(\Delta t_i)$, than for each point p $\in \delta\varphi$ and $\forall t \in [0, \Delta t]$ the following can be deduced:

$p \in \varphi \Rightarrow p = \sum_i^n \lambda_i v_i$ with $\lambda_i \geq 0$ and $\sum_i^n \lambda_i = 1$

so

$\|Vp(t)\|_\infty = \|V(\sum_i^n \lambda_i v_i(t))\|_\infty \leq \sum_i^n \lambda_i \|Vv_i(t)\|_\infty$

Since $\forall t \in [0, \Delta t]$ and $\forall i : v_i(t) \in \varphi \Rightarrow \|Vv_i(t)\|_\infty \leq \gamma$, it is obvious that $\forall t \in [0, \Delta t] : \|Vp(t)\|_\infty \leq \gamma$.

This means that each trajectory starting on the boundary of the set φ will move on the boundary or will be pushed inside the set. Each trajectory starting inside the set has to pass the boundary in order to leave it. But because at the boundary the system forces the state to follow the boundary or pushes the state inside the set, the trajectory can't leave the set. The set is invariant.

■

This theorem ensures invariance of the whole set by only imposing conditions on the vertices of the set. Notice that this invariance condition differs significantly from the more commonly used invariance conditions ([1], [4]). However, the set not only needs to be invariant but it also has to be feasible, meaning that each point inside the set needs to satisfy the input and state constraints. The following lemma provides conditions that need to be imposed in order to

Preprint submitted to European Control Conference 2007.
Received October 18, 2006.

assure feasibility of the set:

Lemma 2: A set φ of the form (6) is feasible for the constraints (3) and (4) under the linear feedback (5) if the following conditions hold:

$$v_i \in X \tag{10}$$
$$Kv_i \in U \tag{11}$$

Proof: It is obvious that all the points of the set φ satisfy constraint (3) if (10) holds since φ is convex and X is also convex. To proof that constraint (4) is not violated if (11) holds, note that a point p lying in the convex set φ can be written as $p = \sum_i^n \lambda_i v_i$. In order to satisfy constraint (4), Kp must lie inside U. We can rewrite Kp as follows:
$Kp = K \sum_i^n \lambda_i v_i = \sum_i^n \lambda_i K v_i$
This means that Kp lies in the polytope with vertices Kv_i. Constraint (11) assures that the vertices Kv_i lie inside the set U. Because the polytope with vertices Kv_i and U are both convex sets, this leads to the following: $\forall p \in \varphi : Kp \in U$. ■

IV. MAIN RESULTS

Before stating our main results, some theory concerning sets of the form (6) will be described. The set (6) contains 2^n vertices $\{v_j\}$ defined as follows:

$$v_j = \gamma V^{-1} s_j, \ j = 1, \ldots, 2^n \tag{12}$$

with $\{s_j, \ j = 1, \ldots, 2^n\}$ the set of all possible n-dimensional vectors whose elements are ± 1. By choosing n linearly independent $\{s_1, \ldots, s_n\}$, the complete set of 2^n vertices can be parameterized in terms of the first n (primary vertices) alone as follows:

$$v_j = f(n) = \begin{cases} [v_1 \ldots v_n] \, [s_1 \ldots s_n]^{-1}, j = n+1, \ldots, 2^{n-1}, \\ -v_{j-2^{n-1}}, \ j = 2^{n-1}+1, \ldots, 2^n. \end{cases} \tag{13}$$

Also remark that in case of n linearly independent primary vertices the following statement can be derived from (12):

$$V = \gamma \, [s_1 \ldots s_n] \, [v_1 \ldots v_n]^{-1} \tag{14}$$

Taking the results of the previous section into account and combining this with the results for the discrete-time case as described in [10], optimization of φ, K over the primary vertices subject to invariance and feasibility for a continuous-time linear system can be formulated as stated below.

Theorem 2: The following nonlinear program defines $K \in \mathbb{R}^{l \times n}$ with the property that φ is the maximum volume low-complexity invariant polytope which satisfies the feasibility conditions (3) and (4) for (1) under linear feedback (5):

$$\min_{v_i, w_i, i=1,\ldots,n} -log \ det([v_1 \ \ldots \ v_n]) \tag{15}$$

subject to the following constraints invoked for $i = 1, \ldots, 2^{n-1}, j = 1, \ldots, n$:

$$v_i \in X \tag{16}$$
$$-sign((Vv_i)_j) \cdot \Delta((Vv_i)_j) \geq 0 \tag{17}$$

and

$$w_i \in U \tag{18}$$

The associated linear feedback gain can be recovered from the solution for $\{v_i, w_i, \ j = 1, \ldots, n\}$ via
$K = [w_1 \ \ldots \ w_n][v_1 \ \ldots v_n]^{-1}$.

With exception of (17) the inequalities of this nonlinear program are linear. Inequality (17) is clearly nonconvex. However, this constraint can be formulated as a bilinear constraint by use of the following theorem:

Theorem 3: The polytopic set φ (6) with vertices v_i is invariant under the feedback law (5) if the following holds:

$$\dot{v}_i + [v_1 \ \ldots \ v_n] \cdot [s_1 \ \ldots s_n]^{-1} \cdot s_i \in \varphi \tag{19}$$

Proof: Assume $(V \cdot v_i)_j \geq 0$, with $(V \cdot v_i)_j$ the jth entry of $(V \cdot v_i)$. Than, according to Theorem 1 the following must hold for each $v_i$in order for the set φ to be invariant:
$(V \cdot \dot{v}_i)_j \leq 0$
Introducing some conservativeness the following derivation can be made:

$$-2\gamma \leq (V \cdot \dot{v}_i)_j \leq 0 \tag{20}$$

$$-\gamma \leq (V \cdot \dot{v}_i)_j + \gamma \leq \gamma$$

$$-\gamma \leq (V \cdot \dot{v}_i)_j + (\gamma \cdot e_i)_j \leq \gamma$$

$$-\gamma \leq (V \cdot \dot{v}_i + \gamma \cdot e_i)_j \leq \gamma$$

$$-\gamma \leq (V \cdot (\dot{v}_i + V^{-1} \cdot \gamma \cdot e_i))_j \leq \gamma$$

Taking (14) into account this leads to the following condition:

$$-\gamma \leq (V \cdot (\dot{v}_i + [v_1 \ \ldots \ v_n][s_1 \ \ldots \ s_n]^{-1} \cdot e_i))_j \leq \gamma \tag{21}$$

with e_i an arbitrary column vector having 1 as its jth entry. The other elements of e_i can be chosen arbitrarely. For $(V \cdot v_i)_j \leq 0$ it can be shown that the same condition has to hold for invariance with exception that the jth entry of e_i has to be equal to -1. Remark that the vector e_i is different for each j. So in order for the vertex to be invariant, (21) has to hold for n entries with a different e_i for each entry. However, it is easy to show that chosing $e_i = s_i$ for each entry is a valid choice reducing the amount of conditions and leading to condition (19).

■

Remark 4.1 In the proof of the theorem it should be noticed that a condition for each component of v_i is made. However, by making a clever choice and assuming $e_i = s_i$ it is not longer necessary to do a component-wise optimization.

Remark 4.2 With a minor modification of the algorithm it is possible to make a trade-off between optimality of the linear feedback and the volume of the invariant set. To see this, note that (20) can be modified into the following condition:

$$-(a+1)\cdot\gamma \le (V\cdot\dot{v}_i)_j \le -(a-1)\cdot\gamma, \qquad (22)$$

which can be rewritten into the following:

$$-\gamma \le (V\cdot\dot{v}_i)_j + a\cdot\gamma \le \gamma, \qquad (23)$$

with tuning parameter $a \ge 1$ determining the trade-off. This results into the following invariance condition:

$$\dot{v}_i + [v_1\ v_2\ \ldots\ v_n]\cdot[s_1\ s_2\ \ldots s_n]^{-1}\cdot a\cdot s_i \in \varphi \qquad (24)$$

Constraint (19) is a bilinear constraint and can be solved using an approach similar to the one described in [10]. In [10] the optimization problem is broken into a sequence of simpler problems, each of which is concerned with optimizing only a single vertex v_k. In this work a modified algorithm will be used that also optimizes a single vertex but compared to the algorithm in [10] reduces the number of variables to be optimised. Let c_k denote the vector that is orthogonal to all exepth the kth primary vertex v_k:

$$c_k^T v_j = 0,\ \forall j \ne k, \qquad (25)$$

than the optimization of Theorem (2) can be performed by solving a sequence of linear problems (LPs) as indicated below.

Theorem 4: The nonlinear program of Theorem (2) can be computed by solving the following LP successively for the individual vertices $v_k, k = 1,\ldots,n$:

$$\min_{v_k, w_k, k=1,\ldots,n} \quad c_k^T v_k \qquad (26)$$

subject to the following linear constraints:

$$Av_i + Bw_i + [v_1\ \ldots\ v_n]\,[s_1\ \ldots\ s_n]^{-1}\, s_i \in \varphi^0,\ \forall i = 1,\ldots,n \qquad (27)$$

$$sign((s_k)_j)\cdot(V\cdot v_k)_j \ge 1 \qquad (28)$$

$$v_k \in X \qquad (29)$$

$$w_k \in U \qquad (30)$$

with φ^0 the invariant set computed by the previously LP.

Proof: As already stated in [10], (26) is proportional to the volume of the set φ. Conditions (27) and (28) ensure invariance condition (19). To see this remark that condition (28) corresponds to the violation of the constraints active at primary vertex v_k. Therefore this condition limits the feasible space in such a way that the set $\varphi^0 \in Co\{v_1,\ldots,v_n\}$, since it ensures $|(V\cdot v_k)_j| \ge 1$. Combining this with (27) ensures (19). (29) and (30) ensure the state and input constraints are satisfied. Also note that condition (27) is a linear constraint. Since $([s_1\ \ldots\ s_n]^{-1}\, s_i)$ is a columnvector, it has the following form:$[a_1\ \ldots\ a_n]^T$. Using some elementary matrix algebra, condition (27) can be rewritten as follows:

$$Av_i + Bw_i + a_1 v_1 + \ldots + a_n v_n \in \varphi, \qquad (31)$$

which is a linear constraint.

■

Remark 4.3 Condition (27) can be imposed as follows:

$$\mathbf{-1} \le V^0\cdot x \le \mathbf{1} \qquad (32)$$

with $x = Av_k + Bw_k + [v_1\ \ldots\ v_n]\,[s_1\ \ldots\ s_n]^{-1}\, s_k$, $\mathbf{1} = [1\ \ldots\ 1]^T$ and V^0 the matrix determining φ^0. This can be done because V^0 is known at the start of each optimization procedure. So in contrary to the technique applied in [10], this method avoids the need of additional variables in order to express x as a convex combination of the primary vertices of φ^0.

Remark 4.4 In case of a nonlinear system (2) invariance condition (27) should hold for each $[A_i\ B_i]$ of the LDI (8).

Remark 4.5 The algorithm is also able to cope with rate constraints in case the linear feedback gain is known in advance. To see this notice that the rate constraint $\dot{u}_{min} \le \dot{u} \le \dot{u}_{max}$ for a linear feedback gain K can be formulated as follows:

$$\dot{u}_{min} \le K(A+BK)x \le \dot{u}_{max} \qquad (33)$$

If K is known in advance, (33) reduces to a linear constraint in the state variable.

In order for the algorithm of Theorem 4 to ensure it returns a feasible invariant set it is necessary to find an initial feasible set satisfying invariance condition (19). For a nonlinear system it is very difficult to determine such a set. However, for a linear system an initial set can be determined by the procedure as indicated below.

Preprint submitted to European Control Conference 2007.
Received October 18, 2006.

Theorem 5: In order for an initial set to satisfy the conditions (10), (18) and (19), the feedback K and γ must be chosen in such a way that following condition holds for each eigenvalue λ_i of A+BK:

$$-2 \leq \lambda_i \leq 0 \tag{34}$$

The primary vertices of the initial set can than be found as the eigenvectors of the matrix $(A+BK)$ multiplied with some scalingfactor δ in order to fullfill the state and input constraints.

Proof: Assume K is chosen in such a way that $\phi(=A+BK)$ has real negative eigenvalues and let the eigenvectors of ϕ coincide with primary vertices of φ. Since $V\cdot\phi v_i = \lambda_i \cdot V v_i$ and $\lambda_i \leq 0$ this means that the vertices v_i form an invariant set, because $sign(V\dot{v_i}) = -sign(Vv_i)$. In order for the set to also satisfy constraint (19) the following deduction can be made:

$$-\gamma\mathbf{1} \leq V(\phi v_i + [v_1 \ldots v_n][s_1 \ldots s_n]^{-1} s_i) \leq \gamma\mathbf{1}$$

$$-\gamma\mathbf{1} \leq \lambda_i \cdot Vv_i + \gamma s_i \leq \gamma\mathbf{1}$$

$$-\gamma\mathbf{1} \leq (\lambda_i + 1)\cdot\gamma s_i \leq \gamma\mathbf{1}$$

$$-\mathbf{1} \leq (\lambda_i + 1)\cdot s_i \leq \mathbf{1}$$

Since $-\mathbf{1} \leq s_i \leq \mathbf{1}$, this leads to $|\lambda_i + 1| \leq 1$, which is equivalent to condition (34). However, imposing condition (34) is not always sufficient. In order to satisfy the state and input constraints it can be necessary to scale the obtained primary vectors with some constant δ smaller than 1. Remark that the eigenvalues don't change by scaling the primary vectors, so condition (34) can't get violated by the scaling. ■

Remark 4.6 In order to use this theorem to obtain an initial feasible invariant set a suitable K can be found by the use of pole placement after the determination of suitable poles.

Remark 4.7 In case of introducing the tuning parameter a a similar approach results into following condition:

$$-1 - a \leq \lambda_i \leq 1 - a \tag{35}$$

Notice that $\lambda \leq 0$ since $a \geq 1$. A high value for the tuning parameter a results in very negative eigenvalues which results in a smaller invariant set but a more optimal linear feedback.

Remark 4.8 For the nonlinear system it is not straightforward to find a feasible initial invariant set. So for a nonlinear system the algorithm starts the optimisation procedure from a random set. However, in practice this doesn't pose many problems, as will be demonstrated in the examples.

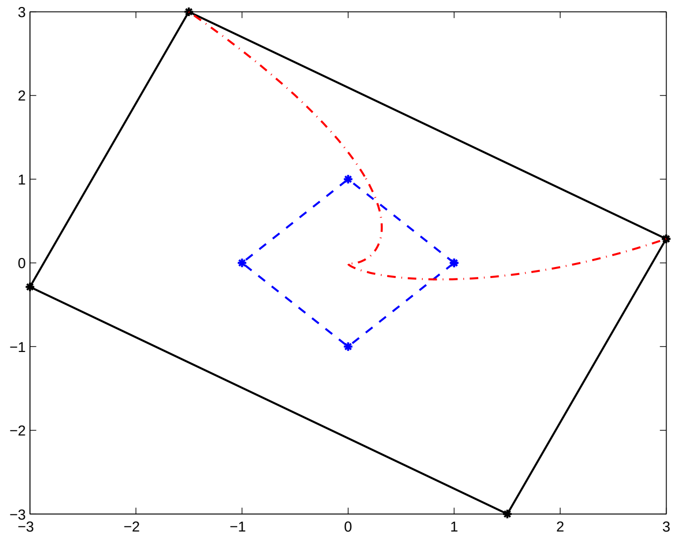

Fig. 1. Simulation Results: Invariant Set

V. EXAMPLE

Our illustrative example concerns the following linear continuous-time system:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} u$$

with state constraint **-3** $\leq x \leq$ **3**.

In this case, 2 eigenvalues between -2 and 0 must be chosen in order to determine the initial feasible invariant set. We choose $\lambda_1 = -1.5$ and $\lambda_2 = -1$. Remark both eigenvalues satisfy condition (34). Using pole placement the following value for the linear feedback gain is obtained:

$$K = \begin{bmatrix} 0 & -1 \\ -1.5 & -1 \end{bmatrix},$$

$$\phi = \begin{bmatrix} -1.5 & 0 \\ 0 & -1 \end{bmatrix}.$$

The eigenvectors of ϕ are $v_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and $v_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$. Both primary vertices satisfy the input constraints so there is no need to scale the vertices. The resulting set is depicted in figure 1 as the set with the dashed contours. Applying the algorithm of Theorem 4 gives the set with the solid contours in figure 1. Also note that the path of the primary vertices under the optimal linear feedback stays inside the solid set, which means that the set is invariant. The optimal linear feedback corresponding with the solid set is the following:

$$K^{opt} = \begin{bmatrix} -0.1430 & -0.6316 \\ -0.5101 & -0.6175 \end{bmatrix}$$

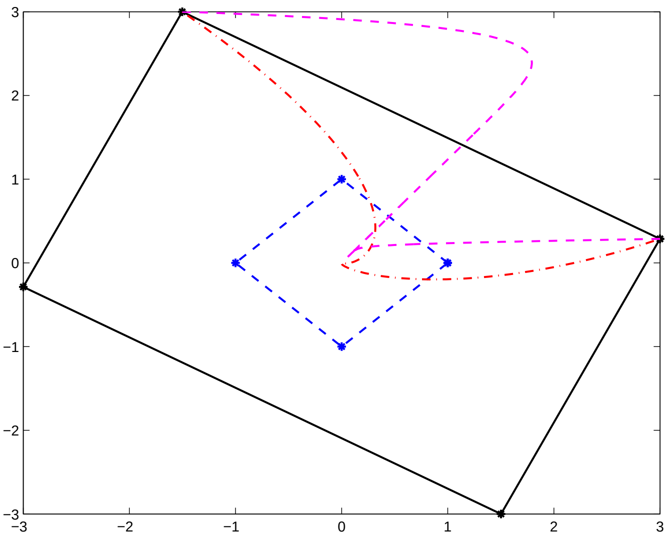

Fig. 2. Simulation Results: Uncertainty and Invariance

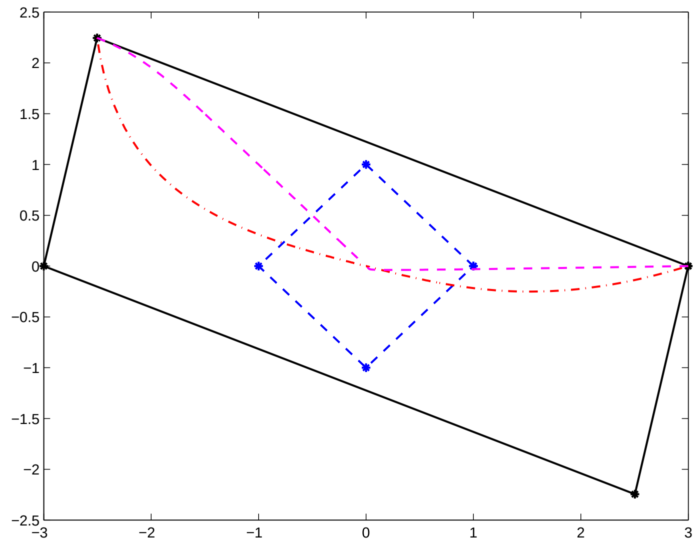

Fig. 3. Simulation Results: Invariant Set Uncertain System

Now, lets assume some uncertainty on the matrix B of the system:

$$B \in Co\left\{\left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right], \left[\begin{array}{cc} 0 & 1 \\ 0.05 & 0 \end{array}\right]\right\}$$

Figure 2 shows that the invariant set for the nominal linear system isn't invariant for the uncertain system. However, by taking the uncertainty into account in the algorithm a set can be found invariant for the uncertain system (see figure 3). The values corresponding to this optimal feasible invariant set are the following:

$$K^{opt} = \left[\begin{array}{cc} 0.1797 & 1.0911 \\ 0.5565 & 1.5112 \end{array}\right]$$

$$V^{opt} = \left[\begin{array}{cc} 0.3333 & 0.8170 \\ 0.3333 & -0.0738 \end{array}\right]$$

Remark that the initial feasible set used to determine a feasible invariant set for the uncertain system is the same as the initial feasible set for the nominal system without uncertainty. Though not shown on the figure this set is not invariant for the uncertain system. This means that the algorithm of Theorem 4 started the optimization from an infeasible starting point. However, as depicted in figure 3, the algorithm is still capable of determining a invariant set satisfying the given state constraints.

VI. CONCLUSIONS AND FUTURE WORKS

A. Conclusions

In this paper a method was proposed to determine feasible invariant sets for continuous-time linear and nonlinear systems. The method determined an optimal linear feedback gain maximizing the invariant feasible set by solving a sequence of linear programs. A trade-off between optimality and volume of the invariant set was obtained by introducing a tuning parameter a. The method was able to deal with constraints on the input, rate and the state. In the case of a nonlinear system the method was able to determine an invariant set by using an LDI representation of the nonlinear system. Several simulations on linear and uncertain continuous-time systems showed the method was capable of obtaining invariant feasible sets. In case of the linear system the algorithm first determined an initial invariant feasible set satisfying all necessary constraints. As a result all the future determined sets satisfied those same constraints. In the nonlinear case the initial set was not feasible. However, this didn't prevent the algorithm to determine an (sub)optimal feasible invariant set.

B. Future Works

Future work can be done by extending the algorithm in order to deal with input disturbances. Furthermore, the sets used in this work have a limitation due to their symmetry. More general sets without symmetry could significantly increase the volume of the invariant set, especially when asymmetric constraints are considered. Future work is needed in this area.

VII. ACKNOWLEDGMENTS

Toni Barjas Blanco is a research assistant at the Katholieke Universiteit Leuven, Belgium. Bart De Moor is a full professor at the Katholieke Universiteit Leuven, Belgium.

KUL research is supported by Research Council KUL: GOA AMBioRICS, CoE EF/05/006 Optimization in Engineering, CoE EF/05/007 SymBioSys, IDO (Genetic networks), several PhD/postdoc & fellow grants; Flemish Government: FWO: PhD/postdoc grants, projects, G.0407.02, G.0197.02, G.0141.03, G.0491.03, G.0120.03, G.0413.03, G.0388.03, G.0229.03, G.0452.04, G.0499.04, G.0499.04, G.0232.05, G.0318.05, G.0211.05, G.0226.06, G.0321.06, research communities (ICCoS, ANMMM, MLDM); AWI: Bil. Int. Collaboration Hungary/ Poland; IWT: PhD Grants, GBOU-McKnow, GBOU-SQUAD, GBOU-ANA, Eureka-Flite2, TAD-BioScope, Silicos; Belgian Federal Science Policy Office: IUAP P5/22; PODO-II; EU-RTD: FP5-CAGE; ERNSI; FP6-NoE Biopattern; FP6-IP e-Tumours, FP6-MC-EST Bioptriain, FP5-Quprodis; Contract Research/agreements: ISMC/IPCOS, Data4s, TML, Elia, LMS, Mastercard.

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