Robust MPC based on symmetric low-complexity polytopes

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***Abstract*—This paper deals with linear discrete-time systems with bounded disturbances and polytopic input and state constraints. The paper discusses a method to determine an initial disturbance invariant set by using the concept of β-contractiveness. In a second step an optimization procedure is outlined in order to increase the volume of the initial set. The proposed optimization scheme scales well with the state dimension. The resulting set is then used as a terminal set for a new robust MPC (RMPC) scheme based on symmetrical tubes. This new RMPC scheme consists of a quadratic program with more degrees of freedom then the existing tube-based RMPC schemes and therefore gives raise to a more optimal control strategy.**

I. Introduction

Model predictive control (MPC) represents a control strategy able to handle hard constraints and performance issues. The predictive control law is obtained by minimizing a performance index based on the prediction of the future states of the system taking both input and state constraints into account. An important issue in MPC is the stability of the obtained control law. The most popular way to guarantee stability is the implementation of MPC as a dual mode controller. In this setting the MPC controller steers the state of the system into a set for which it is known that there exists a control policy such that the state trajectory stays inside the set while satisfying all the state and input constraints.

In the case of uncertainties, stability and recursive feasibility of the robust control problem are ensured by steering the final state into a disturbance invariant terminal set. In this paper the terminal set is chosen to be a symmetric low-complexity polytope. Based on the concept of β-contractiveness an initial feasible disturbance invariant low-complexity polytope is determined. Afterwards, an optimization scheme is outlined in order to determine the feasible disturbance invariant polytope with maximal volume. The proposed optimization scheme has good scalability properties w.r.t. the state dimension. This set is then used as terminal set in a robust dual MPC.

The uncertainty in the model can be caused by model uncertainties or disturbance inputs. In this paper the focus will go to uncertainties caused by disturbance inputs. These uncertainties can be tackled in many ways. A possible approach is to carry out an *open-loop* MPC optimization as outlined in [1]. However, this approach has an important drawback that it results in conservative control behavior leading to infeasibility and instability of the control problem. The reason for this is that the *open-loop* controller doesn't take into account that in the future it will have knowledge about the disturbance that has taken place leading to predicted trajectories with a big 'spread'. In order to tackle this problem *closed-loop* MPC was introduced. In this approach the input is a policy π which is a sequence $\{\mu_0(.),\ldots,\mu_{N-1}(.)\}$ of control laws parametrised as a function of the states of the system. The most popular control feedback law $\mu_i(.)$ is the one in which it is parametrised as $\mu_i(x_k) = Kx_k + v_k$ with $k = 0,\ldots,N-1$. This approach reduces the 'spread' of the trajectories through the feedback matrix K leading to less conservative results.

A further improvement was made in [2] in the form of a tube MPC. A tube MPC consists of a tube $\mathbf{X} = \{X_1,\ldots,X_N\}$ with $X_i = z_i + \alpha_i Z$ a polytope defined by its center z_i, $Z = Co\{v_1,\ldots,v_J\}$ a (random) chosen polytope and α_i a scaling factor that permits the size of X_i to vary. By ensuring all possible trajectories are inside the tube $\mathbf{X}$ and ensuring the tube $\mathbf{X}$ lies inside the feasible region of the state space robust feasibility is ensured. The benefits of this approach are : (i) its complexity is linear in N rather than exponential as in the methods proposed in [3] and [4]; (ii) the policy π is time-varying and piecewise affine which gives raise to less conservative results than the ones proposed in [5]. However, one drawback of the tube MPC as outlined in [2] is that the size of the polytopes X_i defining the tube $\mathbf{X}$ only varies as a scaling α_i of the chosen polytope Z without losing convexity of the control problem. To overcome this problem in this paper the results discussed in [2] are modified by increasing the number of degrees of freedom that influence the shape of the sets X_i. It will be shown that this can be done without loss of convexity of the *on-line* control problem if the sets X_i of the tube $\mathbf{X}$ are assumed to be symmetric low-complexity polytopes.

This paper is organised as follows. Section II explains the used notation and defines the robust control problem. In section III the main results of this paper are discussed in detail. First a method is proposed to determine an initial disturbance invariant set. Then an optimization scheme is outlined that increases the volume of this initial set. This feasible disturbance invariant set is then used as a terminal set in a new robust MPC scheme based on symmetrical tubes. In section IV the proposed methods are applied on a numerical example and the symmetrical tube MPC is compared with the standard tube defined in [2].

II. Problem formulation and preliminaries

A. Notation

The sets of non-negative integers is defined as $\mathbb{N}$ i.e. $\mathbb{N} = \{0,1,2,\ldots\}$. For $n_1 \in \mathbb{N}$ we denote $\mathbb{N}_{n_1} = \{1,\ldots,n_1\}$. The set $\mathbb{R}^n$ defines the set of all n-dimensional vectors x with $x(i) \in \mathbb{R}, i = 1,\ldots,n$, where $x(i)$ corresponds to the i-th element of the vector x. For two sets X_1 and X_2 the Minkowski set addition is defined by $X_1 \oplus$

$X_2 := \{x_1 + x_2 : x_1 \in X_1, x_2 \in X_2\}$. The Minkowski set subtraction is defined by $X_1 \ominus X_2 := \{z : z \oplus X_2 \subseteq X_1\}$. For a set X and $n_1 \in \mathbb{N}$, X^{n_1} is defined by $X^{n_1} = \underbrace{X \times \ldots \times X}_{n_1}$.

B. Robust optimal control problem

The paper addresses a robust MPC scheme for the linear, time-invariant, discrete-time system

$$x_{k+1} = Ax_k + Bu_k + d_k \tag{1}$$

with $x_k \in \mathbb{R}^n$ the state of the system and $u_k \in \mathbb{R}^m$ the control vector at sample time k. The system is subject to the disturbance input

$$d_k \in \mathbb{W} = Co\{d_1, \ldots, d_{n_d}\} \tag{2}$$

and to the state and input constraints

$$x_{k+1} \in \mathbb{X}, u_k \in \mathbb{U}, \forall k \geq 0. \tag{3}$$

with $\mathbb{X}$ and $\mathbb{U}$ convex polytopes containing the origin in their interior. It will be assumed throughout the paper that the pair (A,B) is stabilisable.

A feasible trajectory $t\left(x_0, \mathbf{u}^r, \mathbf{d}^r\right)$ is defined as a sequence of feasible states $\{x_0, x_1, \ldots, x_\infty\} \in \mathbb{X}^\infty$ resulting from a sequence of feasible inputs $\mathbf{u}^r = \{u_0, u_1, \ldots, u_\infty\} \in \mathbb{U}^\infty$ and a disturbance realization $\mathbf{d}^r = \{d_0, d_1, \ldots, d_\infty\} \in \mathbb{W}^\infty$. The set $T\left(x_0\right)$ of all feasible trajectories starting at x_0 is defined as $T\left(x_0\right) = \{t(x_0, \mathbf{u}, \mathbf{d}) | t(x_0, \mathbf{u}, \mathbf{d}) \in \mathbb{X}^\infty, \mathbf{u} \in \mathbb{U}^\infty, \mathbf{d} \in \mathbb{W}^\infty\}$. The robust optimal control problem consists of finding an optimal input sequence $\mathbf{u}^*$ minimizing a cost function $J(x_0, \mathbf{u}, \mathbf{d})$ such that $\forall \mathbf{d} \in \mathbb{W} : t\left(x_0, \mathbf{u}^*, \mathbf{d}\right) \in T\left(x_0\right)$. In this paper a technique based on tubes will be described that efficiently solves the robust optimal control problem.

C. Dual-mode robust optimal control problem

In this paper the considered model predictive controller is a *dual-mode* controller. In a *dual-mode* controller the state of the system at the end of the optimization window is steered into a set X_f that is feasible and disturbance invariant for a local stabilizing controller $u_k = Kx_k$. The general form of the *dual-mode* robust optimal control problem is

$$\min_{\substack{\{x_1, \ldots, x_N\} \\ \{u_0, \ldots, u_{N-1}\}}} J\left(\{x_0, \ldots, x_N\}, \{u_0, \ldots, u_{N-1}\}, \mathbb{W}\right) \tag{4}$$

subject to the constraints (1) and $\forall k = 0, \ldots, N-1$

$$u_k \in \mathbb{U} \tag{5}$$

$$Ax_k + Bu_k + d \in \mathbb{X}, \forall d \in \mathbb{W} \tag{6}$$

$$x_N \in X_f \tag{7}$$

In order to ensure stability and recursive feasibility of the receding horizon strategy it is necessary that the set X_f lies inside the feasible region of the state space and that it is disturbance invariant. Disturbance invariance is defined as follows:

Definition 1: A set X_f is called disturbance invariant for the system (1) under the linear feedback $u = Kx$ if $\forall x \in X_f$ and $\forall d \in \mathbb{W}$ it follows that $(A + BK)x + d \in X_f$.

Another important concept that will be used throughout this paper is the concept of β-contractiveness.

Definition 2: A set X_f is said to be β-contractive for the system (1) under the linear feedback $u = Kx$ and $0 \leq \beta \leq 1$ if $\forall x \in X_f$ and $\forall d \in \mathbb{W}$ it follows that $(A + BK)x + d \in \beta X_f$.

In this paper the terminal set is assumed to have the following form

$$X_f = \left\{x | \left\|V_f x\right\|_\infty \leq 1\right\}. \tag{8}$$

These types of sets have the advantage that they can be described by a relatively low amount of constraints which speeds up the *on-line* optimization process. In this paper a method will be described to determine an initial feasible disturbance invariant set based on the concept of β-contractiveness. Afterwards an optimization scheme will be described to increase the volume of this set in order to obtain a terminal set X_f with maximal volume. The proposed optimization scheme scales well with the state dimension of the system and can therefore be applied to high dimensional systems. In the final stage of the paper a new robust MPC scheme is introduced based on tubes with sets of the form (8).

III. Main results

A. Feasible disturbance invariant set

In this paper it is assumed that the terminal set X_f is feasible disturbance invariant and has the form (8). This type of set is symmetric and consists of 2^n vertices $v_i, i \in \mathbb{N}_{2^n}$ [6]. In the disturbance-free case with the set $W = \emptyset$ and a stabilizing feedback law $u = Kx$ invariance of the set X_f defined by (8) can be obtained by ensuring the following condition is satisfied:

$$\left\|V_f \phi x\right\|_\infty \leq \left\|V_f x\right\|_\infty \tag{9}$$

with $\phi = A + BK$. In the disturbance-free case, existence of such a set satisfying both the input and state constraints (3) is guaranteed by following theorem:

Theorem 1: Assume the eigenvalues λ_i of the *closed-loop* matrix ϕ are real and $|\lambda_i| \leq 1$, $W = \left[\begin{array}{ccc} w_1 & \ldots & w_n \end{array}\right]$ with w_i the eigenvectors of ϕ corresponding to λ_i and that $V_f = \rho W^{-1}$, it then follows that there always exists a $\bar{\rho}$ such that $\forall \rho \geq \bar{\rho}$ the set of the form (8) is invariant for the system (1) while satisfying the input and state constraints (3).

Proof: The *closed-loop* matrix ϕ can be re-written as $\phi = WDW^{-1}$ with D the diagonal matrix containing the eigenvalues λ_i on the diagonal. Therefore for $V_f = W^{-1}$ it follows that $\|V_f \phi x\|_\infty \leq |\lambda^{max}| \|V_f x\|_\infty$, with $|\lambda^{max}| = \max_i |\lambda_i|$. Because it was assumed that $|\lambda_i| \leq 1$ this implies that the invariance condition (9) is satisfied. Now, assume $V_f = \rho W^{-1}$ with $\rho = 1$ and the smallest possible scaling factors ρ_x, ρ_u for which $X_f \subseteq \rho_x \mathbf{X}$ and $KX_f \subseteq \rho_u \mathbf{U}$. If these scaling factors are smaller or equal to 1, the input and state constraints are satisfied and the obtained invariant set X_f is feasible. If one or both of the scaling factors is/are bigger than 1 the constraints are violated. However, by choosing $\bar{\rho} = \max\{\rho_x, \rho_u\}$ it is ensured that the invariant set ρX_f is also feasible $\forall \rho \geq \bar{\rho}$.

■

Remark 1: The assumption that the eigenvalues λ_i are real and $|\lambda_i| \leq 1$ is not restrictive. Since it is assumed that the system matrices A and B are stabilisable, it is always possible to choose a set of real eigenvalues $\{\lambda_i\}$ and determine a feedback K such that the closed-loop eigenvalues coincide with the chosen set of real eigenvalues. This can be achieved by pole placement.

In case of disturbances the invariance condition (9) can never be satisfied. To see this observe that in the origin this condition can never hold. Therefore, in order to ensure disturbance invariance the new invariance condition becomes

$$\|V_f(\phi v_i + d_j)\|_\infty \leq \|V_f v_i\|_\infty, \forall i \in \mathbb{N}_{2^n}, j = 1, \ldots, n_d \tag{10}$$

with v_i the vertices of the set X_f and d_j the vertices of the disturbance set W. Assume the set X_f is invariant in the disturbance-free case and

$$\|V_f d_j\|_\infty \leq \beta \|V_f v_i\|_\infty = \beta \tag{11}$$

with $0 \leq \beta \leq 1$, then because $\|V_f(\phi v_i + d_j)\|_\infty \leq \|V_f \phi v_i\|_\infty + \|V_f d_j\|_\infty$ and imposing $\|V_f \phi v_i\|_\infty + \|V_f d_j\|_\infty \leq \|V_f v_i\|_\infty$ the invariance condition (10) can be re-written as

$$\|V_f \phi v_i\|_\infty \leq (1-\beta) \|V_f v_i\|_\infty. \tag{12}$$

In other words, the set X_f obtained from theorem 1 needs to be $(1-\beta)$-contractive in the disturbance-free case in order for the set to be disturbance invariant for disturbances d_j satisfying (11). This is achieved if the eigenvalues λ_i of ϕ satisfy

$$\max_i |\lambda_i| \leq (1-\beta). \tag{13}$$

Suppose a set X_f with a certain contraction rate α is obtained using theorem 1, then this set is disturbance invariant for $d_j \in \mathbb{W}$ if $\|V_f d_j\|_\infty \leq 1-\alpha$. However, if this condition is violated it can be easily shown that by parameterizing V_f as $V_f = \rho W^{-1}$, the set $X_f = \left\{x | \left\|\rho W^{-1} x\right\|_\infty \leq 1\right\}$ is feasible and disturbance invariant for each ρ satisfying the following condition

$$\bar{\rho} \leq \rho \leq \frac{\beta}{\max_j \|W^{-1} d_j\|_\infty}. \tag{14}$$

B. Volume maximization

In subsection A a method is described to determine an initial feasible disturbance invariant set. However, the volume of the obtained set is not optimal. In [6] methods are described that allow to find the maximal volume feasible invariant set for the disturbance-free case. The following algorithm is an extension of the method described in [6] and defines the maximal volume feasible disturbance invariant set

Algorithm 1: The following nonlinear program defines X_f and K such that X_f is the maximal volume low-complexity polytope (8) satisfying the feasibility and invariance constraints (3) and (10) for (1) under the disturbances defined by (2) and linear feedback $u = Kx$:

$$\min_{v_j, w_j, j=1,\ldots,n} -\log\det\left(\begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix}\right) \tag{15}$$

subject to the following constraints invoked for $i = 1, \ldots, 2^n, p = 1, \ldots, n_d$:

$$\begin{aligned} \|V_f(Av_i + Bw_i + d_p)\|_\infty &\leq \|V_f v_i\|_\infty & (16)\\ w_i &\in \mathbb{U} & (17)\\ v_i &\in \mathbb{X} & (18)\\ d_p &\in \mathbb{W} & (19) \end{aligned}$$

with $w_i = Kv_i$. The optimal linear feedback K can be recovered as

$$K = \begin{bmatrix} s_1 & \ldots & s_n \end{bmatrix} \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix}^{-1}. \tag{20}$$

with $s_1, \ldots, s_n$ defining n linearly independent vectors in $\mathbb{R}^{n \times 1}$ consisting of values equal to ± 1 only.

Similarly as in [6] this nonlinear program can be solved as a sequence of linear programs. However, the drawback of this formulation is that there are $O(2^n)$ variables to be determined which makes the method only applicable to systems with low state dimension. This observation was already stated in [7] for the case of uncertain linear systems. A new method was proposed in [7] to overcome this drawback which resulted in an algorithm scalable with the state dimension. Basically, the reduction in optimization variables was obtained by restating the feasibility and invariance constraints using the Farkas' lemma. A similar approach based on the Farkas' lemma leads to the following optimization problem:

Theorem 2: The maximum feasible disturbance invariant set for system (1) subject to disturbances (2) and under linear feedback $u = Kx$ is obtained as the solution of the following nonlinear program:

$$\min_{H_{1,\ldots,6}, W, \beta, \lambda^j_{1,\ldots,2^n}} -\log\det(W) \tag{21}$$

subject to the following constraints for $j = 1, \ldots, n_d$

$$\begin{aligned} W(H_1 - H_2) &= AW + BQ && (22) \\ H_3 - H_4 &= V_x W && (23) \\ H_5 - H_6 &= V_u Q && (24) \\ d_j &= \beta W \sum_{i=1}^{2^n} \lambda_i^j s_i && (25) \\ H_{1,\ldots,5} &\geq 0 && (26) \\ \begin{bmatrix} H_1 & H_2 \end{bmatrix} \mathbf{1} &\leq \mathbf{1}(1-\beta) && (27) \\ \begin{bmatrix} H_3 & H_4 \\ H_5 & H_6 \end{bmatrix} \mathbf{1} &\leq \mathbf{1} && (28) \end{aligned}$$

with $\mathbf{1} = [1 \ldots 1]^T \in \mathbb{R}^{n \times 1}$. The linear feedback K can be recovered as $K = QW^{-1}$.

Proof: Assume $W = V_f^{-1}$, then it is straightforward to show that the objective function (21) maximizes the volume of the set $X_f = \{x | \|V_f x\|_\infty \leq 1\}$. In order for the set X_f to satisfy the state constraints according to Farkas' lemma [7] it is necessary there exist positive matrices $H_2, H_3 \in \mathbb{R}^{n \times n}$ satisfying $(H_2 - H_3)\mathbf{1} \leq \mathbf{1}$ such that $(H_2 - H_3)V_f = V_x$ which coincides with the constraints (22) and (28). The same reasoning w.r.t. to the input constraints and taking $Q = KV_f$ leads to condition (24). Conditions (22) and (27) can also be derived using Farkas' lemma and ensure the set X_f is $(1-\beta)$-contractive in absence of disturbances. Condition (25) ensures $\|V_f d_j\|_\infty \leq \beta$ which combined with the fact that X_f is $(1-\beta)$-contractive ensures the set X_f is disturbance invariant. ■

Note that the optimization outlined in theorem 2 still scales exponentially w.r.t. the state dimension due to the introduction of the $2^n \times n_d$ variables λ_i^j in condition (25). Condition (25) is a bilinear constraint needed to ensure $\|V_f d_j\|_\infty \leq \beta$. The following theorem replaces constraint (25) by new constraints leading to a significant reduction in the number of variables of the optimization problem:

Theorem 3: Assume a random set $X_0 = \{x | \|V_0 x\|_\infty \leq 1\}$ such that $\max_j \|V_0 d_j\|_\infty \leq \beta_0 \leq 1$ then $\max_j \|V_f d_j\|_\infty \leq \beta_0$ can be enforced by following set of constraints

$$\begin{aligned} W(H_7 - H_8) &= V_0^{-1} && (29) \\ (H_7 - H_8)\mathbf{1} &\leq \mathbf{1} && (30) \\ H_{7,8} &\geq 0 && (31) \end{aligned}$$

with $H_{7,8} \in \mathbb{R}^{n \times n}$ and $W = V_f^{-1}$.

Proof: Conditions (29)-(31) can be derived using Farkas' lemma and state $X_0 \subseteq X_f$; therefore, $\forall d \in W$: $\|V_f d\|_\infty \leq \|V_0 d\|_\infty \leq \beta_0$, meaning that as long as $X_0 \subseteq X_f$ it follows that $\max_j \|V_f d_j\|_\infty \leq \beta_0$. ■

Combining theorems 2 and 3 for a random chosen set $X_0 = \{x | \|V_0 x\|_\infty \leq 1\}$ with $\max_j \|V_f d_j\|_\infty \leq \beta_0 \leq 1$ an approximate solution for the maximal volume feasible disturbance invariant set can be obtained by following algorithm:

Algorithm 2:

$$\min_{H_{1,\ldots,8}, W} -\log\det(W) \quad (32)$$

subject to the following constraints

$$\begin{aligned} W(H_1 - H_2) &= AW + BQ && (33) \\ H_3 - H_4 &= V_x W && (34) \\ H_5 - H_6 &= V_u Q && (35) \\ W(H_7 - H_8) &= V_0^{-1} && (36) \\ H_{1,\ldots,8} &\geq 0 && (37) \\ (H_1 - H_2)\mathbf{1} &\leq \mathbf{1} - \beta_0 && (38) \\ \begin{bmatrix} H_3 & H_4 \\ H_5 & H_6 \\ H_7 & H_8 \end{bmatrix} \mathbf{1} &\leq \mathbf{1} && (39) \end{aligned}$$

Remark 2: A logical choice for V_0 is to choose $V_0 = V_f^0$ with V_f^0 corresponding to the initial feasible disturbance invariant set obtained with the method discussed in subsection A. Note that the number of variables in this optimization problem is of order $O(n^2)$.

Remark 3: Algorithm 2 is a nonlinear program due to the bilinear constraints (33),(36) and the nonlinear cost function (32). Similar to the approaches in [6] and [7] it is possible to find convex relaxations for solving this optimization problem. However, in [7] it was shown that solving this type of nonlinear problem can be done very fast due to the specific structure of the cost function and the constraints. Moreover, in [7] solving the nonlinear problem gave raise to invariant sets with much bigger volumes than the sets obtained with the convex relaxations. Therefore, in this paper no convex relaxations are used to solve this algorithm.

C. Symmetric tube MPC

A *tube* is defined as a sequence $\mathbf{X} = \{X_0, X_1, \ldots, X_N\}$ of sets of states and an associated policy $\pi = \{\mu_0, \mu_1, \ldots, \mu_{N-1}\}$ satisfying

$$X_k \subseteq \mathbb{X}, \forall k > 0, \quad (40)$$

$$X_N \subseteq X_f \subseteq \mathbb{X}, \quad (41)$$

$$\mu_k(x) \in \mathbb{U}, \forall x \in X_k, k \geq 0 \quad (42)$$

$$Ax_k + Bu_k + d_k \in X_{k+1}, \forall d_k \in \mathbb{W} \tag{43}$$

In the tube controller described in [2] the sets X_i are defined as

$$X_i = z_i + \alpha_i Z \tag{44}$$

with the sequence $\{z_i\}$ representing the centers of the polytopes X_i, the sequence $\{\alpha_i\}$ representing scaling factors allowing the size of the polytopes X_i to vary and Z a (random) chosen polytope. In this paper the sets X_i are assumed to have the following form:

$$X_i = x_i + Z_i \tag{45}$$

with $Z_i = \left\{x | \left\|G_i^{-1}V^0 x\right\|_\infty \le 1\right\}$, G_i a diagonal matrix defined by

$$G_i = \begin{bmatrix} \gamma_i^1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \gamma_i^n \end{bmatrix} \tag{46}$$

and $\gamma_i^j \ge 0, \forall j \in \mathbb{N}_n, \forall i \in \mathbb{N}_N$. From this it follows that $X_i = \{x | \|V_i(x - z_i)\|_\infty \le 1\}$, with $V_i = G_i^{-1}V^0$. The resulting tube $\mathbf{X}$ is called a symmetric tube. Note that the size of the sets X_i is determined by n degrees of freedom $\gamma_i^{1,\dots,n}$ in contrast with the 1 degree of freedom α_i of the sets X_i defined in (44). Therefore, the tube based on (45) will lead to a more optimal control strategy. The set Z defines a symmetric polytope defined by the vertices $\left\{v^1, \dots, v^{2^n}\right\}$. For each $i, X_i = co\left\{v_i^1, \dots, v_i^{2^n}\right\}$ with for each $j, v_i^j = z_i + \alpha_i v^j$. With each tube $\mathbf{X}$ a control sequence $\mathbf{U} = \{U_0, \dots, U_{N-1}\}$ is defined with $U_0 = \{x_0\}$ and $U_i = co\left\{u_i^1, \dots, u_i^{2^n}\right\}$ for $i = 1, \dots, N$; for each j, control u_i^j is associated with vertex v_i^j. Each pair $(\mathbf{X}, \mathbf{U})$ defines a control policy $\pi = \{\mu_1(.), \dots, \mu_{N-1}(.)\}$ with $\mu_i(.) = \mu_{X_i, U_i}$. The control law $\mu_{X,U} : X \to U$ is defined as follows: for each $x \in X = Co\left\{v^1, \dots, v^J\right\}$ defined as $x = \sum_{i=1}^J \lambda^i v^i$ with $\lambda^i \ge 0$ and $\sum_{i=1}^J \lambda^i = 1$, the control law $\mu_{X,U}(x)$ is defined as

$$\mu_{X,U}(x) = \sum_{j=1}^{J} \lambda^i u_i, \forall x \in X. \tag{47}$$

The decision variable for the new robust MPC scheme based on symmetric sets is defined by

$$\vartheta = \{\mathbf{g}, \mathbf{z}, \mathbf{U}, \mathbf{V}\}, \tag{48}$$

with $\mathbf{g} = \left\{\gamma_i^j\right\}, i = 1, \dots, N, j = 1, \dots, n, \mathbf{z} = \{z_1, \dots, z_N\}$, $\mathbf{U} = \{U_0, \dots, U_{N-1}\}$ and $\mathbf{V} = \left\{v_i^1, \dots, v_i^{2^n}\right\}, i = 1, \dots, N$ with $v_i^{1,\dots,2^n}$ the vertices of the set X_i. For each x let $\theta(x)$ denote the set of ϑ

$$\begin{aligned} \theta(x) = \{&\vartheta | \mathbf{g} \ge 0, X_i \subset \mathbb{X}, U_i \subset \mathbb{U}, X_N \subset X_f \subset \mathbb{X}, \\ &Av_i^j + Bu_i^j \in X_{i+1} \ominus W, \forall (i,j) \in \mathbb{N}_{N-1} \times \mathbb{N}_{2^n}\Big\}. \end{aligned} \tag{49}$$

It can be easily shown that the set $\theta(x)$ defines the set of all robust feasible solutions. The purpose of the optimal tube control problem is to find ϑ^* that minimizes the cost function $J_0(x, \vartheta)$. In analogy with [2] in this paper the cost function $J_0(x, \vartheta)$ is defined as

$$J(x, \vartheta) = l(x, u_0) + \sum_{i=1}^{N-1} l(X_i, U_i) \tag{50}$$

with $l(X_i, U_i) = \sum_{j=1}^{2^n} l\left(v_i^j, u_i^j\right)$ and $l(x, u) = |x|_Q^2 + |u|_R^2$. The following theorem shows that the optimal solution ϑ^* can be found by solving a quadratic program.

Theorem 4:

$$P_N(x) : \min_{\vartheta} \{J(x, \vartheta) | \vartheta \in \theta(x)\} \tag{51}$$

with $\theta(x)$ defined by (49), is a quadratic program.

Proof:

The vertices $v_i^1, \dots, v_i^{2^n}$ of the set X_i are handled as optimization variables of the optimization problem (51). In order to ensure $v_i^1, \dots, v_i^{2^n}$ to be the vertices of the set X_i the following constraint is necessary for $i = 1, \dots, N-1$ and $j = 1, \dots, n$

$$\begin{gathered} \begin{bmatrix} v_i^1 - z_i & \dots & v_i^{2^n} - z_i \end{bmatrix} = \\ \left(V^0\right)^{-1} G_i(\gamma_i^j) \begin{bmatrix} s_1 & \dots & s_{2^n} \end{bmatrix}. \end{gathered} \tag{52}$$

Note that this constraint is linear in the decision variables. The state feasibility constraint $X_i \subseteq \mathbb{X}$ can be imposed by the linear constraint

$$v_i^j \in \mathbb{X}, i = 1, \dots, N, j = 1, \dots, 2^n. \tag{53}$$

It is straightforward to show that the constraints $\mathbf{g} \ge 0, U_i \subseteq U$ and $X_N \subseteq X_f$ are linear with respect to the decision variables. In order to show that the constraint

$$Av_i^j + Bu_i^j \in X_{i+1} \ominus W \tag{54}$$

can be expressed as a linear combination of the decision variables, note that it is sufficient the following constraint is satisfied for $p = 1, \dots, n_d$

$$Av_i^j + Bu_i^j + d_p \in X_{i+1}. \tag{55}$$

Taking this into consideration (55) can be re-written as

$$\left\|V_i \left(Av_i^j + Bu_i^j + d_p - z_i\right)\right\|_\infty \leqslant 1 \tag{56}$$

for $p = 1, \dots, n_d$. Because $V_i = G_i^{-1}V^0$ and G_i is a diagonal matrix with strictly positive elements on the diagonal this can be re-written into the following linear constraint

$$G_i \begin{bmatrix} -1 \\ \vdots \\ -1 \end{bmatrix} \le V^0 \left(Av_i^j + Bu_i^j + d_p - z_i\right) \le G_i \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}. \tag{57}$$

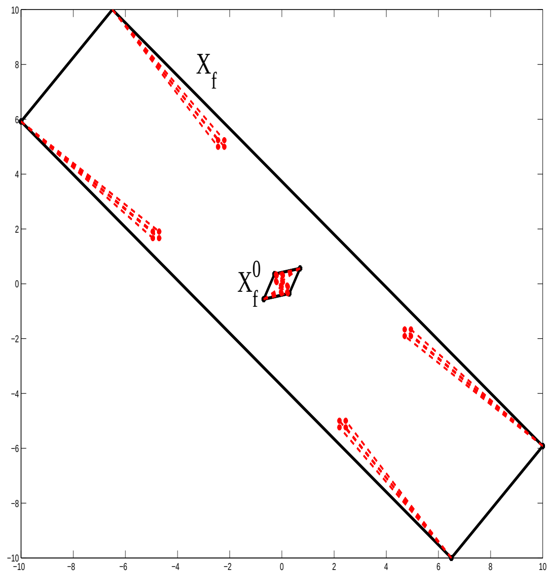

Fig. 1. Robust feasible invariant set

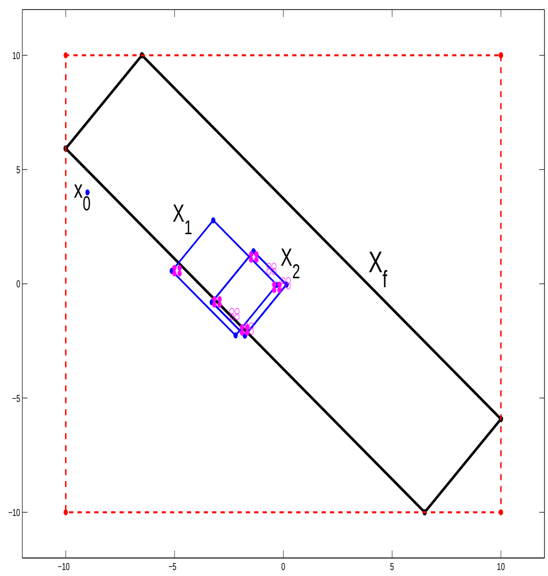

Fig. 2. Symmetric tube MPC

The fact that all the constraints can be expressed as linear combinations of the decision variables and the cost function (50) is quadratic, the tube optimal control problem (51) is a quadratic program. ■

IV. Numerical example

We consider the following system

$$A=\begin{bmatrix} 0.9347 & 0.5194 \\ 0.3835 & 0.8310 \end{bmatrix},\quad B=\begin{bmatrix} -1.4462 \\ -0.7012 \end{bmatrix} \tag{58}$$

subject to disturbances $d \in W = \{x| \|x\|_\infty \leq 0.12\}$ and the input constraint $-1 \leq \bar{u} \leq 1$. Using the method outlined in subsection A a contraction rate $1-\beta = 0.7$ is chosen for the disturbance-free case. In order to achieve this contraction rate, the eigenvalues of the closed-loop system should be real and satisfy $\max_i |\lambda_i| \leq 0.7$. By choosing the set of eigenvalues to be $\begin{bmatrix} 0.7 & -0.2 \end{bmatrix}$ the corresponding feedback K is

$$K=\begin{bmatrix} -0.63 & -0.51 \end{bmatrix} \tag{59}$$

leading to the following 0.7-contractive set

$$X_f^0 = \left\{ x \middle| \left\| \rho \begin{bmatrix} -1.13 & 0.51 \\ 0.26 & -1.21 \end{bmatrix} x \right\|_\infty \leq 1 \right\}. \tag{60}$$

Setting $\rho = 2$ leads to the feasible disturbance invariant set displayed in figure 1. Applying algorithm 2 on the set X_f^0 increases the volume of the set and leads to the set X_f, also depicted in figure 1. The different trajectories of the vertices are depicted in dashed lines and show the set is disturbance invariant.

Applying the symmetric tube MPC on the system (58) with a horizon $N = 3$, initial state $x_0 = \begin{bmatrix} -9 & 4 \end{bmatrix}^T$, terminal set X_f and $V^0 = V_f$, this leads to the tube displayed in figure (2). The crosses in the figure correspond to the next state of each vertex v_i^j of the set X_i for each disturbance realization $d_l \in \{d_1, \ldots, d_{n_d}\}$. It can be seen from the picture that each vertex of the set X_i in the next time step lies in the set X_{i+1}. Therefore, because all the sets X_i lie inside the feasible state space and satisfy the input constraints and the vertices of the set X_{N-1} in the next time step lie inside X_f, the system is robustly stabilized and recursive feasibility is guaranteed. The value of the cost function $J(x, \vartheta)$ associated with this symmetric tube MPC is 78248; the cost function associated with the tube MPC of [2] with the sets X_i a scaled version of the set X_f is 98460. This shows the performance of the symmetric tube MPC is better than the performance of the tube MPC of [2]. Note that a short horizon is chosen to avoid overloading figure 2.

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