On an Alternative Parameterization of the Solutions of the Partial Realization Problem *

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Abstract. The solutions of the partial realization problem have to satisfy a finite number of interpolation conditions at ∞. The minimal degree of an interpolating deterministic system is called the algebraic degree or McMillan degree of the partial covariance sequence and is easy to compute. The solutions of the partial stochastic realization problem have to satisfy the same interpolation conditions and have to fulfill a positive realness constraint. The minimal degree of a stochastic realization is called the positive degree. In the literature, solutions of the partial realization problem are parameterized by the Kimura-Georgiou parameterization. Solutions of the partial stochastic realization problem are then obtained by checking the positive realness constraint for the interpolating solutions of the corresponding partial realization problem. In this paper, an alternative parameterization is developed for the solutions of the partial realization problems. Both the solutions of the partial and partial stochastic realization problem are analyzed in this parameterization, while the main concerns are the minimality and the uniqueness of the solutions. Based on the structure of the parameterization, a lower bound for the positive degree is derived.

Keywords: rational covariance extension, minimal partial realization, partial stochastic realization, stochastic modeling

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1. Introduction

In signal processing, speech processing and system identification, signals can often be modeled as a stationary random sequence generated by passing white noise through a filter or system with a stable transfer function and letting the system come to a statistical steady state [5, 7, 17, 24, 25]. However, before using the filter and the corresponding spectral density e.g. in design processes, the basic inverse problem has to be solved, that is estimating a filter from given data. In practice, only a finite number of samples of the signal is available and a finite covariance sequence can be computed.

For the systems that are solutions of the *partial realization problem*, the first correlation coefficients are these of the given covariance sequence. This problem of matching the first correlation coefficients is solved in deterministic realization theory [10, 11, 14, 16] and corresponds to interpolation theory [1, 20, 22, 23] through the interpolation conditions at ∞. The minimal degree of an interpolating deterministic system is called the McMillan or algebraic degree. The solutions of the *partial stochastic realization problem* have to fulfill the interpolation conditions of the partial realization problem and the solutions have to be (stable) stochastic systems, i.e., having a spectral density positive on the unit circle. In other words, the solutions have to be positive real. The minimal degree of an interpolating stochastic system will be called the positive degree. The positive degree is bounded from below by the algebraic degree and from above by the length of the covariance sequence. The methods described in the literature propose to calculate the value of the positive degree by using a trial and error approach.

In this paper, an alternative parameterization for the partial realization problem is developed. This parameterization was derived by one of the authors for interpolation conditions at finite points in [21] and for mixed types of interpolation conditions in [22]. The solutions of the partial stochastic realization problem are then obtained by checking the positive realness constraints for each interpolating solution.

Both the partial realization problem and the partial stochastic realization problem are analyzed in this alternative parameterization. The major concern of this analysis is the "open problem" of determining the value of the positive degree without using trial and error methods, while referring to the literature for more details on different aspects of the partial stochastic realization problem. We also briefly discuss possible advantages of this parameterization compared to parameterizations described in the literature. By analyzing the structure of the parameterization, a lower bound for the positive degree is obtained.

2. Preliminaries

In this Section, some theory of stochastic systems is reviewed in subsection 2.1. In the literature [2, 4, 8, 15], (deterministic) solutions of the partial realization problem are described by the Kimura-Georgiou parameterization by using Szegő polynomials of the first and second kind. In this set of solutions, the solutions of the corresponding partial stochastic interpolation problem are obtained by checking the positive real constraints for each interpolating solution. This is described in subsection 2.2. In subsection 2.3, the minimality of the solutions is discussed and the definitions of algebraic and positive degree are reviewed.

2.1. STOCHASTIC SYSTEMS

Let the observed discrete time signal y_t be generated by passing normalized white noise e_t through a linear filter with stable rational transfer function $w(z) = w_0 + w_1 z^{-1} + w_2 z^{-2} + \ldots$, with $w_k \in \mathbb{R}$. The stationary (real) process $\{y_t\}_{t\in\mathbb{Z}}$ has a rational spectral density

$$\Phi(z) = w(z)w(z^{-1}),$$

which is assumed to be positive on the unit circle. The stable transfer function $w(z)$ is called a stable spectral factor of $\Phi(z)$, which is taken minimum phase, i.e., the rational function $w(z) = \sigma(z)/a(z)$ has all its poles and zeros in the open unit disc and $w_0 = w(\infty) \neq 0$. In system-theoretical language, the output $\{y_t\}_{t\in\mathbb{Z}}$ is the output of a shaping filter $w(z)$ driven by a normalized white noise input e_t (Figure 1). The Fourier transform of the spectral density $\Phi(z)$ is equal to

$$\Phi(z) = c_0 + \sum_{i=1}^{\infty} c_i \left(z^i + z^{-i}\right),$$

where $c_0, c_1, c_2, \ldots$ is the covariance sequence defined as $c_k = \mathbf{E}\{y_{t+k}y_t\}$, $l = 0, 1, 2, \ldots$. The partial covariance sequences $C_n = \{c_0, c_1, \ldots, c_n\}$ have the property that the Toeplitz matrices

$$\mathcal{T}(C_n) = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_n \\ c_1 & c_0 & c_1 & \cdots & c_{n-1} \\ c_2 & c_1 & c_0 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & c_{n-1} & \cdots & \cdots & c_0 \end{bmatrix}$$

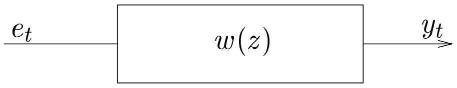

Figure 1. Shaping filter.

are positive definite[1] for $0 \leq n$. If the Toeplitz matrix is not positive definite, the partial covariance sequence is not generated by a stochastic system and there exists no solution of the stochastic realization problem. The partial stochastic realization problem can be stated as follows: *Given* a partial covariance sequence[2] $C_N = \{1, c_1, \ldots, c_N\}$. *Find* an infinite rational extension $\{c_{N+1}, c_{N+2}, \ldots\}$ s.t. $\mathcal{T}(C_n) \geq 0$ for all $n \geq 0$ or find a positive real part $v(z)$ of $\Phi(z) = v(z) + v(z^{-1})$,

$$v(z) = \frac{1}{2}\frac{b(z)}{a(z)} = \frac{1}{2} + \sum_{k=1}^{\infty} c_k z^{-k},$$

which is rational with $a(z) = z^n + a_1 z^{n-1} + \ldots + a_n$ and $b(z) = z^n + b_1 z^{n-1} + \ldots + b_n$. This means that $a(z)$ and $b(z)$ have to satisfy the following conditions:

1. Interpolation conditions at ∞:

$$\Pi_{-N}^{0} \frac{b(z)}{a(z)} = \Pi_{-N}^{0} F(z), \text{ with } F(z) = \frac{1}{2} + c_1 z^{-1} + \ldots + c_N z^{-N} \quad (1)$$

   and the projection operator $\Pi_{\alpha}^{\beta} \sum_k f_k z^k = \sum_{k=\alpha}^{\beta} f_k z^k$.

2. Positive real conditions: if $a(z)$ and $b(z)$ are relatively co-prime, then the rational function $v(z)$ is positive real iff

   (i) $v(z)$ is analytic on $|z| > 1$ or $a(z)$ has all its zeros in $|z| < 1$.

   (ii) $v(z) + v(z^{-1}) > 0$ on $|z| = 1$ or $a(z)b(z^{-1}) + a(z^{-1})b(z) > 0$ on $|z| = 1$.

Condition (i) corresponds to the stability of the modeling filter, whereas condition (ii) is equivalent to $v(z)$ being pseudo-positive real $Re(v(z)) > 0$ on the unit circle [18].

The solutions of the partial deterministic realization problem only have to satisfy the interpolation conditions at ∞.

[1] When $\mathcal{T}(C_n) \geq 0$ and $\det \mathcal{T}(C_n) = 0$, there exists a unique positive real solution. In this case, C_n is called singularly nonnegative.

[2] In the sequel of this paper, we will work with normalized sequences, i.e., with $c_0 = 1$.

2.2. Kimura-Georgiou parameterization

Based on the covariance sequence C_N, the Szegő polynomials of the first and second kind can be calculated:

$$\varphi_t(z) = z^t + \varphi_{t1} z^{t-1} + \ldots + \varphi_{tt}, \tag{2}$$

$$\psi_t(z) = z^t + \psi_{t1} z^{t-1} + \ldots + \psi_{tt}. \tag{3}$$

These polynomials are calculated via a recursion formula as explained in appendix 6.1. The Nth order polynomials satisfy the interpolation condition:

$$\frac{1}{2}\frac{\psi_N(z)}{\varphi_N(z)} = \frac{1}{2} + c_1 z^{-1} + c_2 z^{-2} + c_3 z^{-3} + \ldots + c_N z^{-N} + \ldots$$

This realization is known as the Maximum Entropy solution [26] and corresponds to an all pole realization: $w(z) = \frac{\sqrt{r_N} z^N}{\varphi_N(z)}$. In many applications, finite spectral zeros are desired [5, 7]. Based on the Szegő polynomials, all solutions (up to order N) of the interpolation problem (1) are described in the Kimura-Georgiou parameterization [8, 15]:

$$v(z) = \frac{1}{2}\frac{\psi_N(z) + \alpha_1 \psi_{N-1}(z) + \ldots + \alpha_N \psi_0(z)}{\varphi_N(z) + \alpha_1 \varphi_{N-1}(z) + \ldots + \alpha_N \varphi_0(z)}. \tag{4}$$

However, the elements of this parameterization only satisfy the interpolation conditions (1). The set of all positive real solutions still has to be searched for in this set, i.e., for each choice of parameters $\{\alpha_i\}_{i=1}^N$, one has to check for conditions (i) and (ii). Condition (i) can be checked by some well known stability tests (e.g. the Jury and Schur-Cohn Criterion [1]). In [2] condition (ii) was translated to conditions on the signature of a Hankel matrix. In [9] condition (ii) is checked by repeatedly applying the Schur-Cohn algorithm. In this parameterization, it is not guaranteed that $a(z)$ and $b(z)$ are relatively prime and an additional check is needed. This can e.g. be done by checking whether the resultant of $a(z)$ and $b(z)$ vanishes [12].

For the sake of completeness, we mention that an alternative parameterization can be based on the result of [3, 4]. For each choice of a Schur polynomial $\sigma(z)$, there exists just one positive real realization and one $a(z)$ with $w(z) = \sigma(z)/a(z)$. All solutions (up to order N) are then obtained by parameterizing the Schur polynomials $\sigma(z)$ and calculating the corresponding $a(z)$ by solving a convex optimization problem.

2.3. ALGEBRAIC AND POSITIVE DEGREE

An important property of the solutions of the partial (deterministic) and partial stochastic realization problem is the notion of minimality. The minimal degree δ_a of all deterministic realizations is called the McMillan degree of the covariance sequence. In [4, 13] this is also called the *algebraic degree*. The minimal degree δ^+ of all stochastic realizations of a finite covariance sequence is called the *positive degree* [4, 13]. The following property relates the algebraic and positive degree:

$$\delta_a \leq \delta^+ \leq N. \tag{5}$$

The first inequality is obvious, since the positivity constraint cannot reduce the degree of a solution of the interpolation problem. The second inequality holds since the degree of the Maximum Entropy solution is not larger than N.

3. Parameterization for the interpolation problem

In this Section, the parameterization [21, 22] of the solutions of the interpolation problem is derived for interpolation conditions all taken at ∞. In subsection 3.1, some extra notation and the (equivalent) Linearized Rational Interpolation Problem are introduced. The parameterization is described in subsection 3.2. Based on this parameterization, some properties of realization theory are derived and a lower bound for the positive degree δ^+ is obtained. This is done in subsection 3.3.

3.1. NOTATION

The numerator $n_p(z)$ and the denominator $d_p(z)$ of $v(z) = \frac{n_p(z)}{d_p(z)}$ are combined in the ordered *polynomial couple* $p(z) = (n_p(z), d_p(z)) \neq (0,0)$. Two rational expressions or polynomial couples $(n_{p1}(z), d_{p1}(z))$ and $(n_{p2}(z), d_{p2}(z))$ are called *equivalent* if $n_{p1}(z)d_{p2}(z) = n_{p2}(z)d_{p1}(z)$. The degree δ of the rational expression $(n_p(z), d_p(z))$ is defined as $\delta = \max\{\deg\, n_p(z),\ \deg\, d_p(z)\}$.

The polynomial couple $p(z) = (n_p(z), d_p(z))$ satisfies the first l interpolation conditions of the Proper Rational Interpolation Problem (PRIP) (1) iff:

$$\frac{n_p(z)}{d_p(z)} = F(z) + O_{-}(z^{-l}), \quad z \to \infty. \tag{6}$$

The couple $p(z) = (n_p(z), d_p(z))$ with $\delta \geq \deg\, d_p(z)$ satisfies the first l interpolation conditions of the Linearized Rational Interpolation (LRIP) problem iff:

$$n_p(z) - d_p(z)F(z) = O_-(z^{-l+\delta}), \quad z \to \infty. \tag{7}$$

The entity δ chosen in (7) is called the *potential degree* of the polynomial couple $p(z) = (n_p(z), d_p(z))$. The extra flexibility introduced by choosing $\delta \geq \deg d_p(z)$ will be used in subsection 3.2. We call the highest potential degree coefficients of $p(z)$ the coefficients of $n_p(z)$ and $d_p(z)$ with degree equal to the potential degree δ. Also note that for $l > \delta - \deg d_p(z)$, $\deg d_p(z) = \deg n_p(z) = \deg p(z)$. If $\delta = \deg d_p(z)$, then the PRIP (6) follows from the LRIP (7). Given the PRIP (6), the LRIP (7) follows with $\delta \geq \deg d_p(z)$.

The *vector space* of all polynomial couples with degree not greater than δ and satisfying the first l interpolation conditions (7) is denoted by $S_{l,\delta}$. The following important relations hold between various "neighboring" vector spaces $S_{l,\delta}$: $S_{l,\delta} \subseteq S_{l,\delta+1}$ and $S_{l+1,\delta} \subseteq S_{l,\delta}$. Finally we also have $S_{l,\delta} \subseteq S_{l+1,\delta+1}$ and

$$\{p(z) \in S_{l+1,\delta+1} | \text{ highest pot. deg. coeff. of } p(z) \text{ are zero}\} = S_{l,\delta}. \tag{8}$$

The *residual* of the polynomial couple $p(z)$ with potential degree δ satisfying the first l interpolation conditions at the $(l+1)$th interpolation condition is denoted by $\mathcal{R}^{\delta}_{l+1}p(z)$ and is given by:

$$z^{-l}\mathcal{R}^{\delta}_{l+1}p(z) = \Pi^{-l}_{-l}(z^{-\delta}n_p(z) - z^{-\delta}F(z)d_p(z)). \tag{9}$$

For two polynomial couples $u(z) \in S_{l-1,\delta_u}$, $w(z) \in S_{l-1,\delta_w}$ and two polynomials $x(z)$, $y(z)$, we have that the residual of a linear combination is equal to

$$R^{\delta}_{l}(x(z)u(z) + y(z)w(z)) = x_h R^{\delta_u}_{l} u(z) + y_h R^{\delta_w}_{l} w(z), \tag{10}$$

if the potential degree δ of the linear combination is equal to $\delta = \deg x(z) + \text{potdeg } u(z) = \deg y(z) + \text{potdeg } w(z)$, with x_h and y_h being the highest degree coefficients of the polynomials $x(z)$ and $y(z)$ respectively. From the above definitions it is easy to see that a polynomial couple $p(z) \in S_{l,\delta}$ iff the residuals $R^{\delta}_{i}p$ are equal to zero for $i = 1, \ldots, l$ and the degree of $p(z)$ is smaller than or equal to the potential degree δ.

The last notation that is needed is that of a *set of polynomial couples*: given a $\delta \in \mathbb{N}$ and polynomial couple $p(z)$ with *potential* degree δ_p, the set of polynomial couples $\{p(z)\}^{\delta}$ is defined as follows:

$$\begin{aligned} \{p(z)\}^{\delta} &= \emptyset, & &\text{if} \quad \delta < \delta_p, \\ \{p(z)\}^{\delta} &= \{p(z), zp(z), \ldots, z^{\delta-\delta_p}p(z)\}, & &\text{if} \quad \delta \geq \delta_p. \end{aligned}$$

3.2. PARAMETERIZATION

In this subsection, first a parameterization for all solutions of the LRIP (7) is derived. Property 2 describes the parameterization for the corresponding PRIP (6).

THEOREM 1. *There exist two polynomial couples $u_l(z) = (n_{l,u}(z), d_{l,u}(z))$ and $w_l = (n_{l,w}(z), d_{l,w}(z))$ with potential degree $\delta_{l,u}$ and $\delta_{l,w}$ respectively, such that for each $\delta < \infty$, a basis $BS_{l,\delta}$ for the vector space $S_{l,\delta}$ is given by*

$$BS_{l,\delta} = \{u_l(z)\}^\delta \cup \{w_l(z)\}^\delta.$$

Proof. The Theorem is proven by induction. The Theorem is certainly true for $l = 0$. All polynomial couples $p(z) = (p_n(z), p_d(z))$ with potential degree δ satisfy $p_n(z) - F(z)p_d(z) = O_-(z^{0+\delta})$, since $O_-(F(z)) = 1$, $O_-(p_n(z)) \leq \delta$ and $O_-(p_d(z)) \leq \delta$. Hence, a possible basis for $S_{0,\delta}$ is given by $u_0(z) = (n_{0,u}(z), d_{0,u}(z)) = (1, 0)$ and $w_0(z) = (n_{0,w}(z), d_{0,w}(z)) = (0, 1)$, with potential degrees $\delta_{0,u} = 0$ and $\delta_{0,w} = 0$ respectively.

Suppose now the Theorem to be true for a certain value of l. We assume (without loss of generality) that $u_l(z)$ and $w_l(z)$ are ordered such that $\delta_{l,u} \leq \delta_{l,w}$. Since $S_{l+1,\delta} \in S_{l,\delta}$, a polynomial couple $p(z) = (p_n(z), p_d(z))$ with potential degree δ satisfies the first $l+1$ interpolation conditions iff $p(z) \in S_{l,\delta}$ and $R^\delta_{l+1}p(z) = 0$. Since u_l and w_l with potential degrees $\delta_{u,l}$ and $\delta_{w,l}$ respectively, are a basis for $S_{l,\delta}$, $p(z) \in S_{l,\delta}$ iff $p(z) = u_l(z)x(z) + w_l(z)y(z)$:

$$\begin{bmatrix} n_p(z) \\ d_p(z) \end{bmatrix} = \begin{bmatrix} n_{l,u}(z) & n_{l,w}(z) \\ d_{l,u}(z) & d_{l,w}(z) \end{bmatrix} \begin{bmatrix} x(z) \\ y(z) \end{bmatrix}, \tag{11}$$

with $x(z)$ and $y(z)$ polynomials with respectively degrees $\deg x(z) \leq \delta - \delta_{l,u}$ and $\deg y(z) \leq \delta - \delta_{l,w}$. Let us rewrite $x(z) = x_h z^{\delta-\delta_{u,l}} + x_l(z)$ and $y(z) = y_h z^{\delta-\delta_{u,l}} + y_l(z)$, then the second condition $R^\delta_{l+1}p(z) = 0$ becomes

$$R^\delta_{l+1}p(z) = x_h R^\delta_{l+1}u_l(z) + y_h R^\delta_{l+1}w_l(z). \tag{12}$$

Different situations can occur now: $R^{\delta_{l,u}}_{l+1}u_{l+1}(z) = 0$ or $R^{\delta_{l,u}}_{l+1}u_{l+1}(z) \neq 0$.

First, we examine the case $R^{\delta_u}_{l+1}u_l(z) = 0$. This can only occur iff $u_l(z)$ minimal potential degree $\delta_{l,u} = \deg u_l(z)$, otherwise $u_l(z) \in S_{l,\delta_{l,u}-1}$. This is contradictory since $S_{l,\delta_{l,u}-1}$ is empty by definition.

The residual $R^{\delta_w}_{l+1} w_l(z) \neq 0$. For $l > 0$, the highest potential degree coefficients of $u_l(z)$ and $w_l(z)$ cannot be linearly independent. If $R^{\delta_w}_{l+1} w_l(z) = 0$, this would imply that by taking a linear combination of $u_l(z)$ and $w_l(z)$, there would exists a couple $q(z) \in S^{\delta_{l,w}}_{l+1}$ with highest potential degree elements zero or with potential degree $\delta_{l,w} - 1$. By use of (8), $q(z) \in S^{\delta_{l-1,w}}_{l}$. This can only occur if $w_l(z)$ is a multiple of $u_l(z)$, which implies that $BS_{l,\delta} = \{u_l(z)\}^\delta$, which is contradictory. Hence, if $R^{\delta_u}_{l+1} u_l(z) = 0 \Rightarrow R^{\delta_w}_{l+1} w_l(z) \neq 0$. It is now easy to see that all solutions in (11) satisfying (12) are described by $y_h = 0$. A possible choice for $u_{l+1}(z)$ and $w_{l+1}(z)$ is $u_{l+1}(z) = u_l(z)$ and $w_{l+1}(z) = w_l(z)$, with pot. degrees $\delta_{l+1,u} = \delta_{l,u}$ and $\delta_{l+1,w} = \delta_{l,w} + 1$ resp.

To examine the case $R^{\delta_u}_{l+1} u_l(z) \neq 0$, we change the basis of (11):

$$\begin{aligned}\begin{bmatrix} p_n(z) \\ p_d(z) \end{bmatrix} &= \begin{bmatrix} n_{l,u}(z) & n_{l,w}(z) \\ d_{l,u}(z) & d_{l,w}(z) \end{bmatrix} \begin{bmatrix} -hz^{\delta_{l,w}-\delta_{l,u}} & 1 \\ 1 & 0 \end{bmatrix} \\ &\qquad \cdot \begin{bmatrix} 0 & 1 \\ 1 & hz^{\delta_{l,w}-\delta_{l,u}} \end{bmatrix} \begin{bmatrix} x(z) \\ y(z) \end{bmatrix}, \\ &= \begin{bmatrix} n_{l,w}(z) - hz^{\delta_{l,w}-\delta_{l,u}} n_{l,u}(z) & n_{l,u}(z) \\ d_{l,w}(z) - hz^{\delta_{l,w}-\delta_{l,u}} d_{l,u}(z) & d_{l,u}(z) \end{bmatrix} \begin{bmatrix} x'(z) \\ y'(z) \end{bmatrix}, \end{aligned} \tag{13}$$

with $h = R^{\delta_{l,w}}_{l+1} w_l(z) / R^{\delta_{l,u}}_{l+1} u_l(z)$, deg $x'(z) \leq \delta - \delta_{l,w}$ and deg $y'(z) \leq \delta - \delta_{l,v}$. In this new basis (13) for $S_{l,\delta}$, condition (12) is satisfied for $y'_h = 0$. Up to a reordering for the lowest potential degree, a possible choice for $u_{l+1}(z)$ and $w_{l+1}(z)$ is $u_{l+1}(z) = w_l(z) - hz^{\delta_{l,w}-\delta_{l,u}} u_l(z)$ and $w_{l+1}(z) = u_l(z)$ respectively, with $\delta_{l+1,u} = \delta_{l,w}$ and $\delta_{l,w} = \delta_{l+1,u} + 1$.

The Algorithm to construct the polynomial couples $u_{N+1}(z)$ and $w_{N+1}(z)$ with corresponding potential degrees $\delta_{N+1,u}$ and $\delta_{N+1,w}$ is given in Algorithm 1. In the sequel of the paper, the ordering of Algorithm 1 is assumed, i.e., $u_l(z)$ is the polynomial couple with lowest potential degree.

Algorithm 1. Construction of a basis for the Linearized Rational Interpolation Problem

Initialization

$$\begin{bmatrix} u_0(z) & w_0(z) \\ \delta_{0,u} & \delta_{0,u} \end{bmatrix} = \begin{bmatrix} n_{0,u}(z) & n_{0,w}(z) \\ d_{0,u}(z) & d_{0,w}(z) \\ \delta_{0,u} & \delta_{0,w} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

for $l = 0, 1, \ldots, N,$

it $R^{\delta_{l,u}}_{l+1} u_l = 0$

$U_{l+1}(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$[\delta_{l+1,u} \quad \delta_{l+1,w}] = [\delta_{l,u} \quad \delta_{l,w}+1]$
else
$U'_{l+1}(z) = \begin{bmatrix} -hz^{\delta_{l,w}-\delta_{l,u}} & 1 \\ 1 & 0 \end{bmatrix}$
with $h = R_{l+1}^{\delta_{l,w}} / R_{l+1}^{\delta_{l,u}}$
$[\delta_{l+1,u'} \quad \delta_{l+1,w'}] = [\delta_{l,w} \quad \delta_{l,u}+1]$
if $\delta_{l+1,u'} \leq \delta_{l+1,w'}$
$U_{l+1}(z) = U'_{l+1}(z)$
$[\delta_{l+1,u} \quad \delta_{l+1,w}] = [\delta_{l+1,u'} \quad \delta_{l+1,w'}]$
else
$U_{l+1}(z) = U'_{l+1}(z) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
$[\delta_{l+1,u} \quad \delta_{l+1,w}] = [\delta_{l+1,u'} \quad \delta_{l+1,w'}] \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
endif
endif
endfor

$$\begin{bmatrix} n_{N+1,u}(z) & n_{N+1,w}(z) \\ d_{N+1,u}(z) & d_{N+1,w}(z) \end{bmatrix} = U_0(z) \cdot U_1(z) \cdots U_{N+1}(z).$$

Based on the construction of $u_l(z)$ and $w_l(z)$ in Algorithm 1, the following important property concerning the minimality of $u_l(z) = (n_{l,u}(z), d_{l,u}(z))$ and $w_l(z) = (n_{l,w}(z), d_{l,w}(z))$ holds [22]:

PROPERTY 1. *Both polynomial couples are co-prime, i.e.,* $n_{l,u}(z)$ *and* $d_{l,u}(z)$ *have no common factors and* $n_{l,w}(z)$ *and* $d_{l,w}(z)$ *have no common factors.*

PROPERTY 2. *The parameterization of all solutions of the PRIP (6) with degree* δ *satisfying the first* l *interpolation conditions is given by*

$$p_{x,y}(z) = x(z)u_l(z) + y(z)w_l(z), \tag{14}$$

with the following restrictions on the polynomial parameters $x(z)$ *and* $y(z)$*:*

$$\deg\ x(z) \ \leq\ \delta - \delta_{l,u}, \tag{15}$$
$$\deg\ y(z) \ \leq\ \delta - \delta_{l,w}, \tag{16}$$
$$\deg\ \ p_{x,y}(z) \ =\ \delta. \tag{17}$$

The last condition (17) means that the linear combination $(x(z), y(z))$ *may not result in a zero highest degree coefficient. Two solutions of the rational interpolation problem* p_{x_1,y_1} *and* p_{x_2,y_2} *are equivalent iff* $(x_1(z), y_1(z)) = k(x_2(z), y_2(z))$.

The proof follows from the parameterization of the LRIP (7) and the equivalence with the PRIP (6). The two rational expressions $p_{x_1,y_1}(z) = (n_{x_1,y_1}(z), d_{x_1,y_1}(z))$ and $p_{x_2,y_2}(z) = (n_{x_2,y_2}(z), d_{x_2,y_2}(z))$ are equivalent iff

$$\det \begin{bmatrix} n_{x_1,y_1}(z) & n_{x_2,y_2}(z) \\ d_{x_1,y_1}(z) & d_{x_2,y_2}(z) \end{bmatrix} = 0 \Leftrightarrow \det \begin{bmatrix} x_1(z) & x_2(z) \\ y_1(z) & y_2(z) \end{bmatrix} = 0,$$
$$\text{since} \quad \det \begin{bmatrix} n_{l,u}(z) & n_{l,w}(z) \\ d_{l,u}(z) & d_{l,w}(z) \end{bmatrix} \neq 0.$$

From the minimality of $p_{x_1,y_1}(z)$ and $p_{x_2,y_2}(z)$, both $x_1(z)$ and $y_1(z)$ have no nontrivial common factors and neither have $x_2(z)$ and $y_2(z)$. Hence, $(x_1(z), y_1(z)) = k(x_2(z), y_2(z))$, with k a constant and not a polynomial.

3.3. PROPERTIES

With respect to deterministic realization theory, the two following properties are now easy to prove.

PROPERTY 3. *If the degree of $u_{N+1}(z)$ is equal to the potential degree, then the McMillan degree or algebraic degree is this potential degree $\delta_a = \delta_{N+1,u}$. In the other case, $\delta_a = \delta_{N+1,w}$.*

The question when the solution with algebraic degree δ_a is unique was solved in [19] by imposing a rank condition on a finite Hankel matrix, formed with the elements of the covariance sequence. In the above parameterization, the condition becomes:

PROPERTY 4. *There exists only one solution with the algebraic degree δ_a iff $\delta_a = \delta_{N+1,u} < \delta_{N+1,w}$.*

The McMillan or algebraic degree of a polynomial couple satisfying the interpolation conditions of the covariance sequence C_N is given by Property 3. However this realization is not guaranteed to be positive real. Therefore, the McMillan degree δ_a is not necessarily equal to the positive degree δ^+. The following relations hold:

PROPERTY 5. *Based on the properties of the interpolation conditions, the following relations can be derived with respect to the algebraic and positive degree:*

(a) The algebraic degree δ_a of all solutions is given by Property 3. Iff the corresponding solution is positive real, the positive degree is equal to this algebraic degree.

(b) If $\delta_a = \deg u(z)$ and $u(z)$ is not positive real, a lower bound for δ^+ is $\delta_{N+1,w}$.

(c) If $\delta_a = \deg w(z)$, the lower bound for δ^+ remains δ_a.

Part (a) is trivial. Part (b) follows from the parameterization for the partial realization problem given by Property 2. From inequalities (15)-(16) it follows that the minimal degree for a solution different from $u(z)$ is $\delta_{N+1,w}$. Hence, since $u(z)$ is not positive real, a lower bound for δ^+ is $\delta_{N+1,w}$. Part (c) is easily proven by similar reasoning.

Combination of parts (b) and (c) with Property 4 leads to the following interesting insight. When there exists only one solution (Property 4) for the interpolation problem, the next degree that is possible is equal to $\delta_{N+1,w}$. Hence, when the solution with the McMillan degree is not positive real, the minimal solution of the partial stochastic realization problem should at least have degree $\delta_{N+1,w}$. The number of free parameters is equal to $\delta_{N+1,w} - \delta_a + 1$.

Besides the lower bound for the positive degree, an advantage of the parameterization of Property 2 is that the search for the positive degree can be started from the lower bound. Especially for large covariance sequences, this has substantial benefits with respect to the Kimura-Georgiou parameterization (4). Another advantage is that for non-trivial choices of $x(z)$ and $y(z)$, $n_{x,y}(z)$ and $d_{x,y}(z)$ are relatively prime by Property 2, which has to checked in the Kimura-Georgiou parameterization. Finally we mention that this parameterization also allows for solutions with degree higher than N. An algebraic description of all solutions for the partial stochastic realization problem is given in Algorithm 2.

Algorithm 2. Algebraic description of the rational solutions of the covariance extension problem

Step 0	Check if $\mathcal{T}(C_N) \geq 0$, iff not *Stop*.
Step 1	Calculate the polynomial couples $u_{N+1}(z)$, $w_{N+1}(z)$ with their potential degrees $\delta_{N+1,u}$, $\delta_{N+1,w}$ (Algorithm 1). Calculate the algebraic degree δ_a based on Property 3.
Step 2	For each δ, $\delta_a \leq \delta$, for each possible choice of the polynomials $x(z)$ and $y(z)$ in (14) of Property 2:
Step 2a	Check if $a(z) = p_d(z)$ is stable (e.g. Schur-Cohn Criterion).
Step 2b	Check if $p(z)$ is pseudo-positive real, [2, 9].

4. Examples

Example 1. Let us start with the following covariance sequence: $C_2 = \{1, 0.5, -0.25\}$, derived from the Laurent series [4, 6] of $v(z) = \frac{1}{2}\frac{z+1+\epsilon}{z+1-\epsilon}$, with $\epsilon = 0.5$. For $N > 3$, the covariance sequences C_N are not valid, $\mathcal{T}(C_N) \not> 0$. The solutions of Algorithm 1 are:

$$\begin{bmatrix} n_u(z) & n_w(z) \\ d_u(z) & d_w(z) \\ \delta_u & \delta_w \end{bmatrix} = \begin{bmatrix} z+1.5 & z+1 \\ 2z+1 & 2z \\ 1 & 2 \end{bmatrix}.$$

The parameterization of all solutions of the interpolation problem is given by (14), with the constraints (15) and (16). There is only one solution $u(z) = (n_u(z), d_u(z))$ with McMillan degree 1 (Properties 3, 4). The solution with degree 1 is equal to the deterministic solution $u(z)$ and is not positive real. The lower bound for the positive degree of Property 5 is equal to 2, allowing for $y(z) = k \in \mathcal{R}$. Since the length $N = 2$, this is also the exact positive degree. For instance, the Maximum Entropy solution has degree 2. This means that in terms of the Kimura-Georgiou parameterization, it is not necessary to look for solutions with lower degree. It also shows that it is not necessary to look for Schur polynomials $\sigma(z)$ s.t. the degree of $a(z)$ could become 1. An example of the solutions in terms of the coefficients of the parameter polynomials $x(z)$ and $y(z)$ is given in Figure 2.

By an appropriate (small) choice of $\epsilon > 0$, N can be made arbitrarily large such that $\mathcal{T}(C_N) > 0$ and $\mathcal{T}(C_{N+1}) \not> 0$. For example, for $\epsilon = 0.1$, $C_{15} = \{1, 0.1, -0.09, 0.081, -0.0729, 0.0656, -0.0590, 0.0531, -0.0478, 0.0430, -0.0387, 0.0349, -0.0314, 0.0282, -0.0254, 0.0229\}$ with $\mathcal{T}(C_{15}) > 0$, while $\mathcal{T}(C_{16}) \not> 0$. The solutions of Algorithm 1 for C_{15} are:

$$\begin{bmatrix} n_u(z) & n_w(z) \\ d_u(z) & d_w(z) \\ \delta_u & \delta_w \end{bmatrix} = \begin{bmatrix} 10z+11 & 5z+1 \\ 2(10z+9) & 10z \\ 1 & 15 \end{bmatrix}. \tag{18}$$

Hence, since $u(z)$ is not positive real, the positive degree $\delta^+ = 15$. Also note that as an intermediate result, $u_3(z) = (n_u(z), d_u(z))$ and $w_3(z) = (n_w(z), d_w(z))$ with potential degrees $\delta_{3,u} = 1$ and $\delta_{3,w} = 2$ respectively, are obtained. This solution corresponds to the "deterministic" system that generated the valid partial covariance sequence. For all next interpolation points $R_{l+1}^{\delta_{l,u}} u_l = 0$ for $l = 3, 4, \ldots$ and the potential degree δ_w is increased by one, resulting into the solution (19). For this example, the lower bound of Property 5 is strict. A positive real solution with minimal degree is e.g. the Maximum Entropy solution.

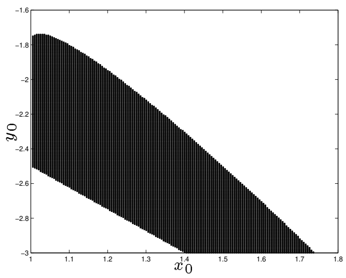

Figure 2. Solutions of the parameterization for Example 1 with $\epsilon = 0.5$ satisfying the positive real conditions for some values of the coefficients of the parameter polynomials. These parameter polynomials are $x(z) = z + x_0$ and $y(z) = y_0$.

Example 2. Given the following covariance sequence: $C_4 = \{1, 0, 0, 0, 0.5\}$, which has known algebraic and positive degree $\delta_a = \delta^+ = 4$. The solutions of Algorithm 1 are:

$$\begin{bmatrix} n_u(z) & n_w(z) \\ d_u(z) & d_w(z) \\ \delta_u & \delta_w \end{bmatrix} = \begin{bmatrix} 1/2 & -z^4 + 1 \\ 1 & 2z^4 \\ 1 & 4 \end{bmatrix}.$$

Since $\delta_{4,u} = 1 \neq \deg u_4(z) = 0$, the algebraic degree $\delta_a = 4$ and equals the positive degree. There exists infinitely many solutions, by means of Property 4.

5. Conclusions

In this paper, an alternative parameterization [21, 22] is derived to parameterize the (deterministic) solutions of the partial realization problem. The minimal degree of an interpolating solution is called the McMillan or algebraic degree and is easily calculated. A description of all solutions of the corresponding partial stochastic realization problem is obtained by checking the positive real constraints for all deterministic solutions. This description of all solutions of the partial stochastic realization problem is compared with results in the literature. The minimal degree of an interpolating stochastic solution is called the positive degree. This degree is bounded from below by the algebraic degree and from above by the length of the covariance sequence. The positive degree was the main concern of this paper, while referring to the literature for other results on the partial stochastic realization problem. Based on the structure of the parameterization, a lower bound for the positive degree is obtained.

6. Appendix

6.1. Szegő polynomials

The Szegő polynomials (2) and (3) of the first and second kind defined on $\mathcal{C}_N$ are calculated by the recursions:

$$\begin{cases} \varphi_{t+1} = z\varphi_t(z) - \gamma_t\varphi_t^*(z), & t=0,1,\ldots,N-1, \ \varphi_0(z)=1, \\ \varphi_{t+1}^* = z\varphi_t^*(z) - \gamma_t\varphi_t(z), & t=0,1,\ldots,N-1, \ \varphi_0^*(z)=1, \end{cases} \tag{19}$$

and

$$\begin{cases} \psi_{t+1} = z\psi_t(z) + \gamma_t\psi_t^*(z), & t=0,1,\ldots,N-1, \ \psi_0(z)=1, \\ \psi_{t+1}^* = z\psi_t^*(z) + \gamma_t\psi_t(z), & t=0,1,\ldots,N-1, \ \psi_0^*(z)=1, \end{cases} \tag{20}$$

with the reversed polynomials $\psi_t^*(z) = z^t\psi_t(1/z)$ and $\varphi_t^*(z) = z^t\varphi_t(1/z)$. The Schur parameters are defined by:

$$\gamma_t = \frac{1}{r_t}\sum_{i=0}^{t}\varphi_{t,t-i}c_{i+1}, \qquad t=0,1,2,\ldots,N-1, \tag{21}$$

$$r_{t+1} = (1-\gamma_t^2)r_t, \qquad t=0,1,2,\ldots,N-1; \ r_0=1.$$

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